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PORTFOLIO INVESTMENT ALLOCATION ALLOWING VOLATILE GROWTH IN CONTINUOUS TIME

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ABSTRACT

The present article builds on the binomial model replication of portfolio selection under uncertainty in continuous time as presented by Dempsey (2005). In the context of observable data, the model reveals an arithmetically-defined equity risk premium for equity stocks over corporate bonds of approximately two percent as consistent with a natural log-wealth utility. An interesting observation is that the calculated premium is a consequence of the mathematics of risk itself (rather than that of investors setting prices at the start of each investment period). The model predicts that investors are unlikely to be attracted to a low return risk-free asset (as either a positive investment or as a leveraging component of their portfolios) when allocating their portfolios across assets. The model thereby challenges the practical application of the separation theorem, which holds that all investors constitute their portfolios as the risky market in combination with a risk-free asset. Importantly and importantly, the model predicts that with changing sensitivity to market risk, investors, while trading at significantly changed prices, may nevertheless maintain their portfolio compositions relatively unadjusted.

1 INTRODUCTION.

In the context of financial markets, the present paper builds on Dempsey's (2005) model for portfolio selection when outcome asset returns are volatile in continuous time. The model illuminates a number of key issues in finance. For example, both the essential nature and practical estimation of the equity risk premium as compensation for volatility (risk). Arnott and Bernstein (2002) - corroborated by authors such as Jones, Wilson and Lundstrum (2002), Ilmanen (2003), Hunt and Hoisington (2003), and Bostock (2004) - argue convincingly for a low equity risk premium. Briefly, while a considerable risk premium for stocks over bonds is supported by US history (stocks having outpaced bonds by some 5 percent throughout 75 years of history), a careful assessment of inflation and economic conditions through this time would not have justified investors holding as a reasonable "expectation" the high equity rates of return actually achieved. Thus it appears that history has been kind to the most successful equity market in the world. On the other hand, the history of inflation has worked, on balance, to be unkind to bonds. In the present paper, we demonstrate that such assessment is consistent with the proposition that portfolio choices are subject to a "natural" log-wealth utility function.

In the model, a log-wealth utility investor is characterized as selecting across assets so as to maximize the portfolio's geometric continuously-compounded growth rate. The selected portfolio is thereby a function of the individual asset variances and covariances, but is independent of the portfolio's overall volatility. In this sense, log-wealth utility is characterized as "risk neutral". Or, we can say, for log-wealth utility, the outcome risk creates its own reward.

A novelty of the model is that the reward for risk is recognized as the result of the mathematics of risk itself, rather than as the result of investors setting prices at the commencement of a one-period model (in accordance with required risk-return tradeoffs).

The proposed model advances a number of further implications for traditional portfolio theory. For example, the model implies that investors allocate their portfolios between equity stocks and bonds with only a marginal regard for a low return risk-free asset. The model thereby challenges a fundamental tenet of portfolio theory - that depending on their individual level of risk aversion, investors allocate their portfolios between a market representation of the risk-free asset and a market representation of all risky assets. Nonetheless, the model's outcome results appear consistent with portfolio allocations as they are reported in practice (Canner, Mankiw, and Weil. 1997). The model proposes a risk-free rate consistent with the CAPM at close to 2¼ percent real per annum, as opposed to a Treasury bill rate at less than 1.0 percent real per annum. The implication here is a much-reduced theoretical dependence of asset returns on beta; which is substantiated by share price performances (Fama and French, 2003, for example).

The model is extended to consider the implications of a utility function other than log-wealth utility. We determine a level of investor risk aversion at somewhat twice that encapsulated by the log-wealth utility function as required to account for investor expectations of return consistent with those actually achieved over US stockmarket history. While generally regarded as reasonable, such level of risk aversion nevertheless represents a significant departure from log-wealth utility. The model predicts that outcome adjustments in investor risk aversion may give rise to investors trading significantly adjusted share prices while maintaining relatively unadjusted the broad composition of their portfolios. A level of stability is thereby imputed to the equity market.

The remainder of the article is arranged as follows. The following section assesses the implications of the Dempsey model for portfolio composition across the asset classes of equity stocks, bonds and a risk-free asset in the contest of log-wealth utility investors; while the section thereafter generalises the analysis to allow the more general risk class of maximum mean return – minimum risk seeking investors. The final section summarises the article.

2 ASSET ALLOCATION ACROSS EQUITY STOCKS, BONDS AND CASH: THE MARKET RISK PREMIUM

Dempsey (2005) has presented a binomial growth model for a portfolio of assets with normally distributed growth rates. With two risky assets - stocks (S) and bonds (B) - and a risk–free asset (in proportions, respectively, ω_S, ω_B, $(1-\omega_S-\omega_B)$), the basic model is as depicted as in Figure 1. In the model, the mean exponential growth rates for stocks and bonds are represented respectively as μ_S, μ_B, with standard deviations σ_S and σ_B; the continuously applied risk-free rate is represented as r_f; and the correlation coefficient between the exponential growth rates for stocks and bonds is represented as C_{SB}.

Consistent with Dempsey (2005), the equations that model a natural log-wealth utility investor's propensity to invest in such a portfolio may be summarised:

(1) The investor's utility (U_P) may be identified as the portfolio's mean exponential growth rate (μ_P):

$$U_P = \mu_P$$

where the portfolio's mean exponential growth rate (μ_P) relates in turn to (*a*) the portfolio's continuously applied return (R_P) that delivers the portfolio's expected wealth outcome and (*b*) the volatility about such portfolio return (σ_P) as:

$$\mu_P = R_P - ½\,\sigma_P^2$$

so that on combining the above two equations we have:

$$U_P = \mu_P = R_P - ½\,\sigma_P^2 \qquad (1)$$

(2) Consistent with expression 1, the portfolio return (R_P) relates to the portfolio's volatility (σ_P) as:

$$[R_P - r_f] = \sigma_P^2 \qquad (2)$$

(3) The investor's vector (W) of optimal weights (ω_i) across portfolio assets i is a function of (*a*) the matrix (Ω) of portfolio asset co-variances ($\sigma_{i,j}$) and (*b*) the vector (R) of expected asset returns (R_i) over and above the risk-free rate (r_f) as:

$$W = \Omega^{-1} R \qquad (3)$$

and,

(4) The CAPM relationship between an individual asset i's expected return (R_i) and the market risk premium ($R_M - r_f$), the risk-free return (r_f), and the asset beta.(β_i) is determined in continuous time as.

$$R_i = r_f + \beta_i * [R_M - r_f] \qquad (4)$$

It should be noted that for normally distributed returns with mean μ_i and standard deviation σ_i, the return R_i that delivers the expected outcome relates to μ_i and σ_i as:

$$R_i = \mu_i + ½\,\sigma_i^2 \quad (i = S, B) \qquad (5)$$

Thus for a "two risky assets (Stocks, Bonds) with one risk-free asset" portfolio, the utility equation 1 expands as:

$$U_P = \mu_P = \omega_S.(\mu_S + ½\sigma_S^2) + \omega_B.(\mu_B + ½\,\sigma_B^2) + (1 - \omega_S - \omega_B).\,r_f - ½c[\omega_S^2.\sigma_S^2 - ½\omega_B^2.\sigma_B^2 - 2.C_{SB}.\omega_S.\omega_B.\sigma_S.\sigma_B] \qquad (6)$$

with subscript S for equity stock and B for bonds. In this case, the variance-covariance matrix Ω in equations 3 represents:

$$\begin{bmatrix} \sigma_S^2 & \sigma_{S,B} \\ \sigma_{B,S} & \sigma_B^2 \end{bmatrix}$$

and R (the vector of expected returns over the risk-free rate) represents: $[R_S - r_f\,, R_B - r_f]$.

In which case, equation 3 may be expressed:

$$\begin{aligned} R_S - r_f - (\omega_S.\,\sigma_S^2 + \omega_B.\,\sigma_{S,B}) &= 0 \\ R_B - r_f - (\omega_S.\,\sigma_{B,S} + \omega_B.\,\sigma_B^2) &= 0 \end{aligned} \qquad (7)$$

or:

$$R_S - r_f = \sigma_{S,P} \qquad (8)$$

$R_B - r_f \quad = \quad \sigma_{B,P}$

The above equations are summarised in Figure 2 with Pratt's measure of investor relative risk aversion c set equal to one (the equations 10–16 reduce to equations 1-4 and 6-8, respectively).

For a "two risky assets (Stocks, Bonds) with one risk-free asset" portfolio, the utility equation 6 may be represented on a spreadsheet as in Figure 3 as a function of equity stocks (ω_S) and bond (ω_B) proportions. In Figure 3, the top horizontal axis denotes the proportion of the portfolio comprised of stocks (from 0 to 100 percent) while the left vertical axis denotes the proportion of the portfolio comprised of bonds (from 0 to 100 percent). Hence the diagonal line in bold has portfolio compositions for which $\omega_S + \omega_B = 1$ (which is to say, these portfolios have zero risk-free asset component). Portfolios which include the risk-free asset (in proportion $1- \omega_S - \omega_B$) are those that lie above the diagonal line. The inclusion of portfolios below the diagonal would imply borrowing of the risk-free asset in proportion ($1- \omega_S - \omega_B$).

In Figure 3, the input values are somewhat rounded out but chosen nevertheless to reflect historical data. The mean exponential rate for corporate bonds (μ_B) is set at 2.5 percent per annum, with standard deviation (σ_B) about such rate at 10 percent. The standard deviation (σ_S) for equity stock exponential rates is set at 20 percent, and the correlation coefficient between stocks and bond returns (C_{SB}) at 0.25 (cf, for example, Ibbotson Associates, 2001). The risk-free rate (r_f) and the mean exponential growth rate (μ_S) for equity stocks have been chosen in accordance with equilibrium considerations. Thus the significance of designating 2.25 percent as the risk-free rate (r_f) is that at any lower rate, investors decline to invest in the risk-free asset, while at a higher rate, investors decline to lever their portfolios with the risk-free asset. The significance of designating the annualised mean exponential growth rate for equity stocks (μ_S) at 3.0 percent in Figure 3 is that such rate is then sufficient to "clear" the market" at 60 - 70 percent equity stocks and 30 – 40 percent corporate bonds (in the sense that expected utility is maximised by such a portfolio).[1]

The insights of this section may be summarised:

1. *Portfolio selection: maximizing mean exponential growth rate*. In Figure 3, portfolio utility is identified as the portfolio's mean exponential growth rate (μ_B)(expression 6). The maximum rate, μ_P = 3.25, is achieved with asset components ω_S, ω_B, respectively, 65 percent and 35 percent (consistent with the solution of equations 3, see footnote 1). Such rate μ_P = 3.25, we note, is greater than the mean exponential growth rate of either of the asset component growth rates individually (μ_S = 3.0, μ_B = 2.5).[2]

2. *The equity risk premium*. Expressed in terms of a mean exponential growth rate as in Figure 3, the equity risk premium for the expected return on equity stocks (μ_S = 3 percent) over the rate on corporate bonds (μ_B = 2.5 percent) appears as little as ½ percent. The premium has been calculated as that which clears the market at about 60-70 percent equity stocks / 30-40 percent corporate bonds. Expressed in terms of the continuously applied growth rate R_i required to generate the asset i's expected wealth outcome, we have (with equation 5): $R_i = \mu_i + ½\sigma_i^2$ (i = S, B), and hence for equity stocks, $R_S = [0.03 + ½\ (0.2)^2]$ = 5 percent annualised, and for corporate binds, $R_B = [0.025 + ½\ (0.1)^2]$ = 3 percent annualised, as clearing the market at approximately 60-70 percent equity stocks / 30-40 percent corporate bonds. In discrete time, with an asset's simple arithmetic return (AR_i) related to the asset's growth rate as :

$$1 + AR_i \;=\; exp(R_i) \;=\; \exp(\mu_i + ½\ \sigma_i^2) \qquad (9)$$

(consistent with footnote 3), the periodic arithmetic returns for stocks (AR_S) and corporate bonds (AR_B) are both adjusted slightly upward from the R_S, R_B values, implying a periodic arithmetic risk premium for equity stocks over corporate bonds at approximately 2.0 percent. Such premium values are obviously much less than what is inferred from a direct measurement from US markets historical returns. The number does, however, support the arguments of such as Arnott, and Bernstein (2002) for a low equity risk premium – based on arguments that actual stock equity returns have *in the event* over-scored rationally anticipated returns (calculated as dividend yield plus dividend yield growth) (due to once-off, largely unanticipated upward revaluations of stocks), while actual corporate bond returns have *in the event* under-scored rationally anticipated returns (due in main part to unanticipated rates of high inflation).

3. *The risk-free rate as input parameter to the CAPM*. The model determines the theoretical rate that equates borrowing and lending of the risk-free asset in equilibrium with the market's risky assets at 2.25 percent. As such, the number represents the appropriate risk-free rate in the context of equilibrium models such as the CAPM. By stipulating such rate as the "equilibrium intercept", the analysis is in the spirit of Black's (1972) rather than the Sharpe (1964)-Lintner (1965) version of the CAPM. Notwithstanding, the estimated geometric return on long-term Government bonds comes in at close to 2.25 percent (Ibbotson Associates, 2001), which suggests that such rate might be taken as representing the *market's* risk-free rate.[4] In which case, the Sharpe-Lintner CAPM also remains supported by the analysis. We should note that tests of the CAPM that fail to incorporate the correct equilibrium value of the risk-free rate are likely to be invalid. For example, the impact of establishing the equilibrium risk-free rate at 2.25 percent - as compared with a lower

rate (0.6 percent as indicated by the historical return for Treasury bills, for example) - is to significantly "flatten" the theoretical dependence of expected risky returns on beta from a gradient of approximately 4 percent additional return per unit of beta to about 2 percent expected return per unit of beta.[5] It is noted that Fama and French's (2003) presentation of CAPM predictions of returns versus beta (using monthly Treasury Bills as the risk-free rate) appear on a gradient that appears approximately twice that as for their empirical findings (their Figure 2).

4. *"Cash" in investor portfolios: the separation theorem*. It appears that the Treasury bill rate (with an historical real rate of return 0.6 percent per annum) does not constitute a risk-free asset in equilibrium with markets. The rate is simply insufficient to attract investment as a component of a diversified portfolio. As to borrowing at such a risk-free rate (0.6 percent real per annum), we note that even allowing perfect capital markets, an investor is unable to borrow at a risk-free rate (guaranteed repayment of principal plus risk-free interest) without additional insurance against a shortfall below risk-free growth for that portion of the risky portfolio financed by the risk-free borrowing. As calculated by Bodie (1995), the cost of the required (Black-Scholes calculated) "put" option per dollar of investment decreases on an annualised basis with the duration of the investment. Nevertheless, Bodie calculates an investment horizon longer than 10 years as required to bring the annualised cost below 2.5 percent. The cost of guaranteeing a risk-free rate is with the borrower – the outcome of which is that a risk-free asset at 0.6 percent may remain unattractive as either a long or a short position

5. *Diversity in equity-bond portfolio compositions*. The "flatness" of the utilities about the 65 percent equity stocks – 35 percent corporate bonds mark in Figure 3 implies an effective range of portfolio compositions across equity and bonds (with little or no risk-free asset component), which appears broadly consistent with observed practice (for example, Canner, Mankiw and Weil, 1997).

In the following section, we present the log-wealth utility function as a special case of the broader class of minimum risk - maximum return utility functions.

3 MEAN-VARIANCE OPTIMIZATION: THE EFFICIENT FRONTIER

Consistent with the previous section, we set the continuously-applied growth rate for equity stocks (R_S) and corporate bonds (R_B) at, respectively, 5.0 and 3.0 percent real per annum, with standard deviations (σ_S and σ_B), respectively, 20 and 10 percent and correlation coefficient (C_{SB}) 0.25. We incorporate a Treasury bill with continuously-applied growth rate (R_{TB}) 0.6 percent real with standard deviation (σ_{TB}) 4.0 percent and correlation coefficients with equity stocks and bonds ($C_{S,TB}$ and $C_{B,TB}$), respectively, at 0.12 and 0.60 (cf, for example, Ibbotson Associates, 2001). Figure 4 then depicts the "efficient frontier" of maximum expected portfolio returns R_P (y-axis) that can be achieved by manipulation of the portfolio proportions ω_S, ω_B, ($1-\omega_S-\omega_B$) for stocks, bonds and Treasury bills for a given volatility (standard deviation) σ_P (x-axis).[6]

The dotted curve in Figure 4 represents the portfolio mean exponential growth rate, μ_p (as equation 6). Consistent with the previous section (Figure 3), such growth rate is maximised at the portfolio 65 percent equity stocks, 35 percent corporate bonds. Consistent also with the previous section, the portfolio is located on the efficient frontier at the point σ_P (x-axis) = $\sqrt{0.02}$ = 14.14 percent; R_P (y-axis) = 4.25 percent (cf footnote 1) and the line from the intercept at 2.25 percent is tangent to the efficient frontier at this point.

The log-wealth utility function of the investor who chooses the portfolio 65 percent equity, 35 percent bonds, along with the essential equations describing the composition and return characteristics of the portfolio are summarised in Figure 2 with Pratt's measure of investor relative risk aversion c set equal to one (equations 10–16 then reduce to equations 1-4 and 6-8, respectively).[7] Investors who are "risk-seeking" (with c less than 1) (but who chose the highest allowable arithmetic return for a given volatility of returns) choose portfolios higher on the efficient frontier curve, while investors who are "risk-averse" (with c greater than 1) choose portfolios lower down on the curve.[8] Equation 12 reproduces Merton's (1969) result (his equation 60) and equation 13 Merton's (1973) derivation of the CAPM.

Alternatively, as the "data to be explained", we may consider stock equity and corporate bond rates at, say, respectively, 8.0 and 3.0 percent real per annum as reflecting historical US rates. Figure 5 displays the efficient frontier in this case. Imposing the requirement that the market clears at close to 65 percent equity stocks, 35 percent bonds (reflecting Canner et al, as above), we then have equations 15 as:

$$0.080 - r_f = c.\,[\omega_S.(0.2)^2 + \omega_B.\,(0.25).(0.2).(0.1)\,]$$
$$0.030 - r_f = c.\,[\omega_B.(0.1)^2 + \omega_S.\,(0.25).(0.2).(0.1)\,]$$

with $\omega_S = 0.65$, $\omega_B = 0.35$. The equations now imply an investor's relative risk aversion c at a little less than 2.5 and a risk-free rate at a little below 1.5 percent real annually. The portfolio is located on Figure 5 where the line from the intercept at 1.5 percent is tangent to the efficient frontier, which occurs at the point σ_P (x-axis) = $\sqrt{0.02}$ = 14.14 percent (as calculated in footnote 1) and R_P (y-axis) = (0.65).(0.08) + (0.35).(0.03) = 6.25 per cent. The portfolio is that which maximises the utility function equation 14 (Figure 2) (with c = 2½), which is represented by the lower dashed curve in Figure 5. The higher dotted curve in Figure 5 represents equation 14 with c = 1, which is to say, the function that is maximised by a log-wealth utility investor.[9]

A level of relative risk aversion c = 2 has generally been regarded as reasonable in the literature (Copeland and Weston, p. 90, for example).[10] Nevertheless, the implications of even such departure from c = 1 [with log-wealth utility *U*(W) = ln(W)] are not at all trivial. To see this, consider that c = 2 may be represented as the power function of wealth, *U*(W) = - W_o /W (the inverse of outcome wealth, W, normalised by initial wealth, W_o, since *U*''(W)/*U*'(W) = 2, see footnote 7). We may now ask, for example: what is the growth factor for an investor's wealth that offers positive utility equal to the negative loss in utility when the investor's wealth is halved? To answer first for a log-wealth utility investor, since:

$$\ln(N) = -\ln[(1/N), \text{ for all } N$$

we have: an increase in wealth by a factor of 2 (or N) provides a positive utility equal to the negative loss in utility when wealth is halved (or multiplied by 1/N). For an investor subject to the power utility function, *U*(W) = -W_o /W, however, a halving of wealth leads to the loss of utility: [-W_o /(½W_o) - (-W_o /W_o)] = -1. The outcome wealth (NW_o) required to provide utility equal to +1, is determined as: [-W_o /(NW_o) - (-W_o /W_o)] = 1, implying the necessity of N = infinite (in other words, such investor if offered a 50 percent probability of losing half of initial wealth, cannot be compensated by *any* upside potential).

The implications of the above observations may be summarised:

1. *"Risk neutrality" of log-wealth utility investors: risk creates its own reward.* On the maximum return - minimum variance frontier, there exists a portfolio for which mean exponential growth rate is maximised. A log-wealth utility investor (with relative risk aversion c = 1) is characterised as choosing this particular portfolio (independent of the portfolio's variance).

2. *Potentially high values for the equity risk premium*. An accommodation of historical US stock performances as indicative of investor expectations requires a degree of risk aversion at somewhat greater then twice that as captured by a log-wealth utility function.

3. *Changing equity prices and stability*. *Changing equity prices and stability*. In response to changes in investor risk aversion (brought about, for example, by investors becoming more "fretful" in response to perceived changing conditions), (*i*) required expectations of return – and thereby prices – are likely to adjust quite significantly (an adjustment of investors' risk aversion parameter c from 1 to 2½ predicts an adjustment in required equity expected returns from 5 to 8 percent), while, simultaneously, (*ii*) investor portfolio compositions may remain relatively unadjusted. The model thereby implies a level of stability with markets continuing to clear.

4 CONCLUSION

The paper has presented a binomial model for volatile asset returns and investor optimal portfolio composition in continuous time. The model is consistent with the framework developed by Merton (1969, 1973) and with Samuelson's arguments (1963, 1989, 1994) that portfolio choices can be *indifferent* to the investor's time-horizon. It is also consistent with Dempsey's (2002) model applied to a single risky asset combined with a risk-free asset.

A log-wealth utility investor (with c = 1) seeks to maximise utility by maximizing the mean (geometric) exponential growth rate of their portfolios. In which case, an expected (arithmetic) annualised real return on stocks at about 5.0 percent (as against a real return on corporate bonds at 3.0 percent) is sufficient to "clear" the market" at 60 - 70 percent stocks and 30 – 40 percent bonds, with investors choosing portfolios within this range *without* resource to a risk-free asset. The model thereby challenges the fundamental "separation theorem" that investors construct efficient portfolios as a combination of the market of all risky assets and a risk-free asset.

A feature of the model is that the equity risk premium is accounted for by the internal mathematics of risk over continuous time - rather than by the concept of investors setting prices at the commencement of a one-period investment time frame as in the CAPM. In which context idiosyncratic risk is recognised as a component of equity returns. Notwithstanding, the formulation is actually consistent with the CAPM in continuous time. It has been observed that the model's predictions fit quite well with Malkiel and Xu's (1997) empirically observed relationship between US equity performances and idiosyncratic risk.

The model predicts that a level of investor risk aversion at somewhat twice that encapsulated by the log-wealth utility function as required to account for investor expectations of return consistent with those actually achieved over US stockmarket history. Additionally, and importantly, the model predicts that with changing sensitivity to market risk, investors may continue to hold their portfolios relatively unadjusted while trading at significantly changed prices. The model thereby implies a level of stability for stock markets.

NOTES

1. We see this in terms of equations 7 (as a simplified form of equations 3) for the portfolio comprising 65 percent stocks, 35 percent bonds. For then (noting that $R_i = \mu_i + \frac{1}{2}\sigma_i^2$, i = S, B, equation 5) we have:

$$\mu_S + ½(0.2)^2 - 0.0225 = (0.65).(0.2)^2 + (0.35).(0.25).(0.2).(0.1)$$

$\mu_B + \frac{1}{2}(0.1)^2 - 0.0225 = (0.35).(0.1)^2 + (0.65).(0.25).(0.2).(0.1)$ which reproduces (to nearest ¼ percent) μ_S = 3.0 percent and μ_B = 2.5 percent.

For this portfolio, we may additionally derive the portfolio's overall arithmetic return, R_P, and its variance, σ_P^2, as:

$$R_P = 0.65{*}R_S + 0.35{*}R_B = 0.65\,(5.0) + 0.35\,(3.0) = 4.25 \text{ percent (to nearest ¼ percent)}$$

and

$$\sigma_P^2 = \omega_S^2.\sigma_S^2 + \omega_B^2.\sigma_B^2 + 2\,\omega_S.\omega_B.C_{S,B}.\sigma_S.\sigma_B = (0.65)^2.(0.2)^2 + (0.35)^2.(0.1)^2 + 2.(0.65).(0.35).(0.25).(0.2).(0.1) = 2.0 \text{ percent}$$

so that we have both $[R_P - r_f]$ = (4.25 – 2.25) = 2.0 percent, and σ_P^2 = 2.0 percent, consistent with equation 2. The portfolio's standard deviation, σ_P, is then $\sqrt{2.0}$, approximately 14.14 percent.

2. In discrete time, the portfolio's mean exponential growth rate (μ_P) may be regarded as identifying most closely with the portfolio's geometric growth rate (GR_P) calculated over N discrete periods defined as:

$$(1 + GR_P)^N = \prod_{i=1}^{N} (1 + r_i) = W_N / W_o$$

where: W_N = [measured value of portfolio at end of period N], W_o = [commencement value of portfolio] (for example, Jacquier, Kane and Marcus, 2003).

3. The portfolio's periodic arithmetic return (AR_P) – calculated for a sequence of N (equally) discrete periodic returns (r_i) as:

$$AR_P = 1/N \sum_{i=1}^{N} r_i$$

[where each $r_i = W_i/W_{i-1} - 1$, W_i = outcome wealth at end of period i] - and the standard deviation about such return (S_P) do not strictly identify the portfolio's mean continuous growth rate (μ_P) and the standard deviation about such rate (σ_P). The theoretical relationships are: $\mu_P = \ln(1+AR_P) - \frac{1}{2}\ln\{1 + [S_P/(1+AR_P)]^2\}$; and $\sigma_P^2 = \ln\{1 + [S_P/(1+AR_P)]^2\}$(for example, de la Grandville, 1998, Ibbotson Associates, 2001, Jacquier, Kane and Marcus, 2003).

4. In which case we are obliged to interpret the short-term Treasury bill rate as operating under considerations disconnected from the otherwise prevailing market. Investor liquidity requirements may be considered as allowing such a possibility, along with such requirements as Treasury interventions to influence interest rates, foreign Treasury purchases or sales of such bills in order to influence domestic exchange rates.

5. To see this, consider the individual component asset betas calculated as:

$$\beta_i = \sigma_{i,M} / \sigma_M^2 \quad (i = S, B)$$

where for the portfolio M [65% stocks, 35% bonds]:

$$\sigma_{S,M} = \omega_S.\sigma_S^2 + \omega_B.C_{S,B}.\sigma_S.\sigma_B = (0.65)(0.2)^2 + (0.35)(0.25)(0.2)(0.1) = 2.75 \text{ percent;}$$

and σ_M^2 = 2.0 percent (footnote 1),

giving: β_S = 0.0275 / 0.02 = 1.4. And similarly, β_B = 0.4. The return R_M for this portfolio is 4.25 percent (footnote 1). We thereby "recover" the above input returns for stocks and bonds [R_S = 5.0 percent, R_B = 3.0 percent] as:

$$R_S = r_f + \beta_S.[R_M - r_f] = 2.25 + 1.4{*}(4.25 - 2.25) = 5.0 \text{ percent}$$

$$R_B = r_f + \beta_B.[R_M - r_f] = 2.25 + 0.4{*}(4.25 - 2.25) = 3.0 \text{ percent}$$

which imply a gradient of (5.0 – 3.0)/(1.4 – 0.4) or 2 percent per unit of beta. If, however, we had set r_f = 0.6 percent (the yield on Treasury bills), we would have *anticipated* the returns as: R_S = 0.06 +1.4.(4.25 - 0.06) = 5.84 percent, and R_B = 0.06 + 0.4 (4.25 - 0.06) = 1.65 percent – which is to say, a gradient of approximately 4 percent per unit of beta.

6. We plotted the efficient frontiers using the Excel Solver routine. The facility minimises the portfolio variance:

$$\sigma_P = \omega_S^2.\sigma_S^2 + \omega_B^2.\sigma_B^2 + (1-\omega_S-\omega_B)^2.\sigma_{TB}^2 + 2\omega_S.\omega_B.C_{S,B}.\sigma_S.\sigma_B + 2\omega_S.(1-\omega_S-\omega_B).C_{S,TB}.\sigma_S.\sigma_{TB} + 2\omega_B.(1-\omega_S-\omega_B).C_{B,TB}.\sigma_B.\sigma_{TB}$$

in terms of the variables ω_S, ω_B for a given portfolio return R_P subject to the condition:

$R_P = \omega_S R_S + \omega_B R_B + (1-\omega_S - \omega_B) R_{TB}$ (where R_S, R_B, R_{TB} represent, respectively, the designated expected returns for stocks, bonds and Treasury bills).

7. The generalisation may also be recognised on identifying Pratt's measure of relative risk aversion, c as: $-W.U''(W)/U'(W)$ (for example, Copeland and Weston, 1988, p. 88), so that in the particular case $U = \ln(W)$, we have $c = -W \ln''(W)/\ln'(W) = 1$.

8. To see how equations 10-16 are consistent with investors who seek to maximise portfolio expected return for a given level of risk of portfolio returns measured as variance, consider that the Lagrangian to be maximised by such an investor is:

$$L = R_P - \lambda\,(\sigma_P^2 - \sigma_X^2) \qquad (*)$$

with individual λ and σ_X^2, where σ_X represents a particular standard deviation for the portfolio (as for example Cuthbertson, 1997, pg. 45). Expanding (*) we have:

$$L = \omega_S R_S + \omega_B R_B + (1-\omega_S - \omega_B)\,r_f + \lambda\,(\omega_S^2.\sigma_S^2 + \omega_B^2.\sigma_B^2 + 2\,\omega_S.\omega_B.\,\sigma_{S,B} - \sigma_X^2)$$

and accordingly differentiating with respect to first ω_S and then ω_B, and equating outcomes to zero, we obtain:

$$R_S - r_f - c\,(\omega_S.\sigma_S^2 + \omega_B.\sigma_{S,B}) = 0$$

$$R_B - r_f \;-\; c\,(\omega_S.\sigma_{B,S} + \omega_B.\sigma_B^2) = 0 \qquad (15)$$

(where we have abbreviated $c = 2\lambda$). With c = 1, the above equations 15 reduce to equations 7. In analogy with equations 3, equations 15 are thereby expressed:

$$W = \Omega^{-1} R / c \qquad (12)$$

Likewise as in the text, we take it that equations 15 identify the return-variance structure for the total market portfolio. Then multiplying the first of these equations by ω_S and the second by ω_B and adding, we obtain:

$$[R_M - r_f] = c.\sigma_M^2 \qquad (11)$$

Eliminating c in equations 15 with equation 11, and with beta defined as $(\sigma_{i,M} / \sigma_M^2)$, we derive:

$$R_i = r_f + \beta_i * [R_M - r_f] \quad (i = S, B) \qquad (13)$$

Having established that an investor seeks to maximize the Lagrangian expression (*), the expression (with constant terms omitted) thereby represents an adequate measure of utility for such investor. That is, we may write:

$$U_P = R_P - ½\, c\, \sigma_P^2 \qquad (10)$$

(as for example Cuthbertson, 1997, pg. 55).

9. If, reflecting historical Treasury bill rates, we consider a risk-free return at say, 0.5 percent per annum, we then have equations 15 as:

$$0.075 = c.\,[\omega_S.(0.2)^2 + \omega_B.(0.25).(0.2).(0.1)]$$
$$0.025 = c.\,[\omega_B.(0.1)^2 + \omega_S.(0.25).(0.2).(0.1)]$$

The equations imply that investors with relative risk aversion c about 3.3 choose their portfolios with proportions 50 percent equity and 50 percent corporate bonds. (The portfolio is located in Figure 5 where the line from the intercept at 0.5 percent is tangent to the efficient frontier, which occurs at the point σ_P (x-axis) = $\sqrt{(\omega_S^2.\sigma_S^2 + \omega_B^2.\sigma_B^2 + 2.\omega_S.\omega_B.C_{S,B}.\sigma_S.\sigma_B)}$ = $\sqrt{[(0.5)^2 .(0.2)^2 + (0.5)^2 .(0.1)^2 + 2.(0.5).(0.5).(0.25).(0.2).(0.1)]}$ = $\sqrt{0.015}$ = 12.25 percent (cf footnote 1) and R_P (y-axis) = (0.5).(0.08) + (0.5).(0.03) = 5.5 percent.)

10. The deferred consumption model of an investor subject to constant relative risk aversion of Mehra and Prescott (1985) and Mehra (2003) argues for a relative risk aversion value closer to 50, leading to the labelling of an “equity premium puzzle”. In contrast, our model appears capable of accommodating historical US equity performances as indicative of investor expectations within much more reasonable levels of investor risk aversion.

APPENDIX A: FIGURES

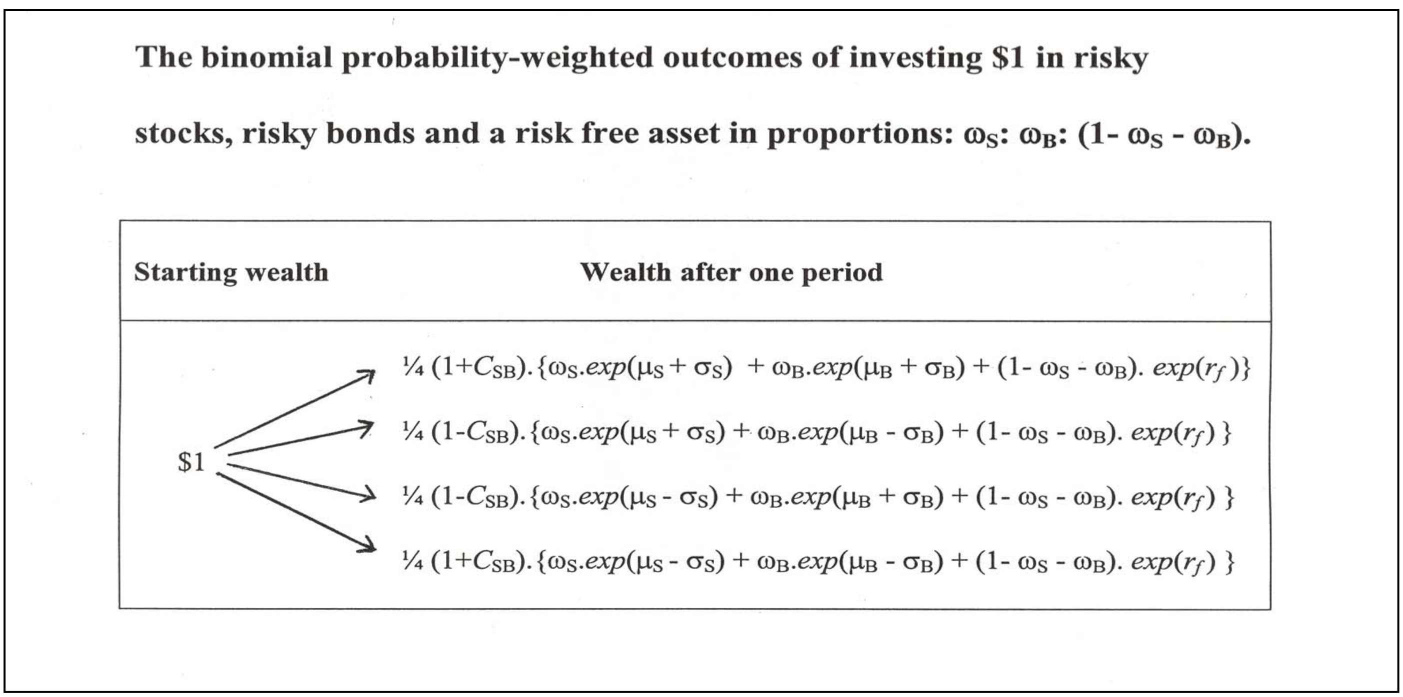


Figure 1:

Figure 2: The Fundamental Equations of Investor Utility and Portfolio Composition

$U_P = R_P - ½ \, c. \, \sigma_P^2$ (10)

$[R_P - r_f] = c.\sigma_P^2$ (11)

$W = \Omega^{-1} R / c$ (12)

$R_i = r_f + \beta_i * [R_M - r_f]$ (13)

For a "two risky assets (Stocks, Bonds) one risk-free asset" portfolio, equation 10 expands as:

$U_P = \omega_S.(\mu_S + ½ \, \sigma_S^2) + \omega_B.(\mu_B + ½ \, \sigma_B^2) + (1 - \omega_S - \omega_B).r_f$

$- ½ \, c \, [\omega_S^2.\sigma_S^2 - ½ \, \omega_B^2.\sigma_B^2 - 2.C_{SB}. \, \omega_S.\omega_B. \, \sigma_S.\sigma_B]$ (14)

and equation 12 expands as:

$R_S - r_f - c \, (\omega_S. \, \sigma_S^2 + \omega_B. \, \sigma_{S,B}) = 0$

$R_B - r_f - c \, (\omega_S. \, \sigma_{B,S} + \omega_B. \, \sigma_B^2) = 0$ (15)

or:

$R_S - r_f = c. \, \sigma_{S,P}$

$R_B - r_f = c. \, \sigma_{B,P}$ (16)

Figure 3 UTILITY AS A FUNCTION OF PORTFOLIO COMPOSITION

EQUITY STOCKS % →

BONDS % ↓	0%	10	20	30	40	50	60	70	80	90	100 %
0%	**2.25**	2.51	2.72	2.90	3.03	3.12	3.18	3.19	3.17	3.10	**3.0**
10	2.32	2.57	2.78	2.95	3.08	3.16	3.21	3.22	3.19	**3.12**	3.01
20	2.38	2.62	2.83	2.99	3.11	3.19	3.24	3.24	**3.20**	3.13	3.01
30	2.43	2.67	2.86	3.02	3.14	3.21	3.24	**3.25**	3.21	3.12	3.00
40	2.47	2.70	2.89	3.04	3.15	3.22	**3.25**	3.24	3.20	3.11	2.98
50	2.50	2.73	2.91	3.06	3.16	**3.22**	3.24	3.23	3.18	3.08	2.95
60	2.52	2.74	2.92	3.06	**3.16**	3.21	3.23	3.21	3.15	3.05	2.91
70	2.53	2.74	2.92	**3.05**	3.14	3.19	3.21	3.18	3.11	3.01	2.86
80	2.53	2.74	**2.90**	3.03	3.12	3.16	3.17	3.14	3.07	2.95	2.80
90	2.52	**2.72**	2.88	3.00	3.08	3.12	3.13	3.09	3.01	2.89	2.73
100%	**2.5**	2.70	2.85	2.97	3.04	3.07	3.07	3.02	2.94	2.81	2.65

mean equity stock exponential return = 3.0 % mean bond exponential return = 2.5 % risk free rate = 2.25 %
standard deviation of equity stock returns = 20 % standard deviation of bond returns = 10 %

correlation coefficient between stock and bond returns = = 0.25

Figure 4: The Market Efficient Frontier

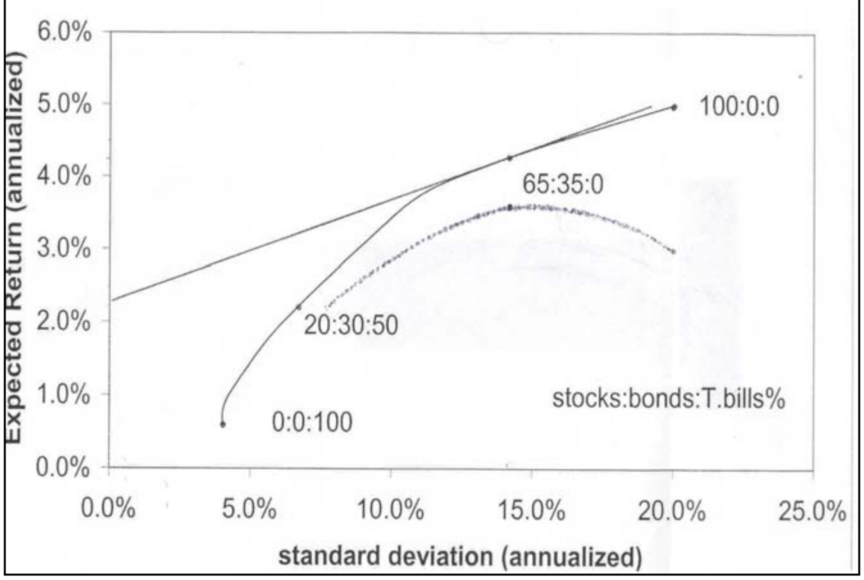
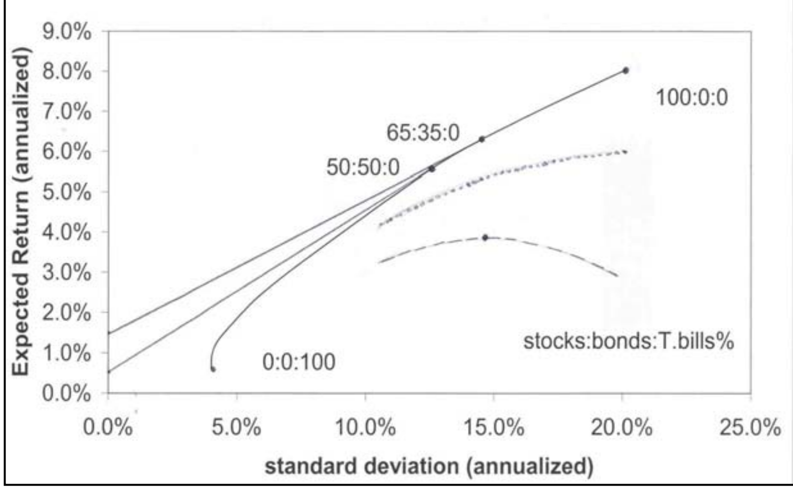


Figure 5: The Market Efficient Frontier



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