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A Modified Multilevel Monte Carlo method and its numerical performance

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Abstract: When performing Monte Carlo simulations, it is important to determine how to perform simulations with the required estimation accuracy at the lowest possible computational cost. For example, in financial practice, simulations may be performed over a long period of time, and even a 10% to 20% reduction in computational cost can be practically significant. In this regard, compared to the traditional standard Monte Carlo method, Multilevel Monte Carlo method offers superior computational cost reduction effects, or in other words, variance reduction effects. Since 2008, numerous applied studies have been conducted, and the original paper, "Multilevel Monte Carlo Path Simulation", by Professor Michael B. Giles has now achieved over 2,000 citations.

In this paper, we propose the Modified Multilevel Monte Carlo method as a technique to further enhance the variance reduction effect (computational cost reduction effect) of the Multilevel Monte Carlo method. Under the Modified Multilevel Monte Carlo framework, many Modified Multilevel Monte Carlo estimators can be constructed. Notably, in special cases, A Modified Multilevel Montecarlo estimator coincides with the Multilevel Monte Carlo estimator. Similar to the Multilevel Monte Carlo method, we mathematically prove that the Modified Multilevel Monte Carlo method can achieve the same estimation accuracy as the standard Monte Carlo method at a lower computational cost. Additionally, we provide a theorem to determine the optimal number of simulation paths for a given computational cost when performing simulations using the Modified Multilevel Monte Carlo method. Furthermore, we attempt to identify the best Modified Multilevel Monte Carlo estimator with the lowest variance through numerical experiments. We demonstrate that many Modified Multilevel estimators exhibit superior performance compared to the original Multilevel Monte Carlo estimator. In the valuation of European vanilla options, Lookback options, and Asian options, the finest Modified Multilevel Monte Carlo estimator achieves a maximum variance reduction effect of 28% compared to the Multilevel Monte Carlo estimator.

Keywords: *Monte Carlo; multilevel Monte Carlo; variance reduction; computational complexity reduction; option pricing*

1 INTRODUCTION

The Monte Carlo method is already a useful computational tool in finance; however, its computational complexity becomes too large for achieving the required accuracy. Giles (2008) proposed a multilevel Monte Carlo (MLMC) method to reduce computational complexity and this study tests the MLMC method for pricing European style options. The MLMC method has been studied in recent years (e.g., variance reduction: see Nobile and Tesei (2015), Quasi Monte Carlo: Dick, Kuo, Gla, and Scwab (2016)). We consider a stochastic differential equation (SDE) of the form $dS_t = \mu(S_t, t)\,dt + \sigma(S_t, t)\,dW_t, 0 \le t < T$, where $S_t \in \mathbb{R}^m$, $S_0 = s$ is given, $T < \infty$, $W_t \in \mathbb{R}^d$ is a standard Brownian motion, and $\mu : \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are drift and volatility coefficients, respectively. Let P be a discounted payoff function. In options pricing, $E[P(S_T)]$means the option price. We denote $t_0 = 0$ and $t_D = T$ and divide the interval $[0, T]$ in D subintervals of equal lengths, that is, $[t_0, t_1], [t_1, t_2], \ldots, [t_D - 1, t_D]$; $\Delta t \equiv t_n - t_{n-1} = T/D$ for any $1 \le n \le D$. Then the discretization of $\{S_t\}_t$ using an Euler scheme with Δt is given by

$$\widehat{S}_{t_{n+1}} - \widehat{S}_{t_n} = \mu\left(\widehat{S}_{t_n}, t_n\right)\Delta t + \sigma\left(\widehat{S}_{t_n}, t_n\right)\Delta W_{t_n}, \quad n = 0, 1, \ldots, D-1,$$

where $\Delta W_{t_n} = W_{t_n} - W_{t_{n-1}}$. Under a standard Monte Carlo (SMC) method, we can compute $Y = N^{-1}\sum_{i=1}^{N} P(\widehat{S}_T^{(i)})$, that is, the approximation of $E[P(S_T)]$, where N is the number of simulation paths. Set $D = M^L$. Under the MLMC method, the option price is constructed by

$$\mathrm{E}[\widehat{P}_L] = \mathrm{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathrm{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], \tag{1}$$

where each P_ℓ is the approximation of $P(S_T)$ on level ℓ; $\widehat{P}_\ell$ means the discretization of $P(S_T)$ with a time step $h_\ell = T/M^\ell$. The coarsest level and the finest level are zero and L, respectively. The MLMC method uses all levels from zero to L. The MLMC estimator is uniquely constructed by

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell, \quad \text{where} \quad \widehat{Y}_\ell = \begin{cases} N_0^{-1}\sum_{i=1}^{N_0} \widehat{P}_0^i, & (\ell = 0), \\ N_\ell^{-1}\sum_{i=1}^{N_\ell} \big(\widehat{P}_\ell^i - \widehat{P}_{\ell-1}^i\big), & (0 < \ell \le L). \end{cases}$$

Note that $\widehat{Y}_0$ is the estimator of $E[P_0]$ using N_0 simulation paths and that each $\widehat{Y}_\ell$ is the estimator of $E[P_\ell - P_{\ell-1}]$ using N_ℓ paths for $\ell = 1, 2, \ldots, L$. We organize this paper as follows. In Sections 2 and 3, we propose a modified multilevel Monte Carlo (MMLMC) method that allows the construction of multiple estimators. In Section 4, we illustrate the numerical performance of the MMLMC method for two European style options valuation. The final section discusses a future direction for research in this area. The mean squared error (MSE) of the MLMC estimator is given by

$$\mathrm{MSE}_{\mathrm{MLMC}} = \mathrm{E}[(\widehat{Y} - \mathrm{E}[P])^2] = \mathrm{V}[\widehat{Y}] + (\mathrm{E}[\widehat{Y}] - \mathrm{E}[P])^2. \tag{2}$$

If the complexity theorem in Giles (2008) holds, both the variance and the square of bias error in (2) have the same upper bound, $\epsilon^2/2$. In comparison to the SMC method, the MLMC method reduces the variance, while leaving the bias unchanged. Variance reduction leads to reduction in computational complexity.

2 MODIFIED MULTILEVEL MONTE CARLO METHOD

2.1 Modified multilevel Monte Carlo estimators

We can rewrite the equation (1) as follows.

$$\mathrm{E}[\widehat{P}_L] = \mathrm{E}[\widehat{P}_{\ell_0}] + \sum_{j=1}^{F} \mathrm{E}[\widehat{P}_{\ell_j} - \widehat{P}_{\ell_{j-1}}], \tag{3}$$

where $\ell_F = L$ and $\ell_0 \ge 0$. Therefore, we can construct $2^{L-\ell_0-1}$ MMLMC estimators.

$$\ddot{Y} = \tilde{Y}_{\ell_0} + \sum_{j=1}^{F} \tilde{Y}_{\ell_j,\ell_{j-1}}, \quad \text{where} \quad \tilde{Y}_{\ell_0} = N_{\ell_0}^{-1}\sum_{i=1}^{N_{\ell_0}} \widehat{P}_{\ell_0}^i \quad \text{and} \quad \tilde{Y}_{\ell_j,\ell_{j-1}} = N_{\ell_j}^{-1}\sum_{i=1}^{N_{\ell_j}} \big(\widehat{P}_{\ell_j}^i - \widehat{P}_{\ell_{j-1}}^i\big)$$

for $0 < j \le F$. The variance of MMLMC estimators is $V[\ddot{Y}] = N_{\ell_0}^{-1}V_{\ell_0} + \sum_{j=1}^{F} N_{\ell_j}^{-1}V_{\ell_j,\ell_{j-1}}$, where V_{ℓ_0} or $V_{\ell_j,\ell_{j-1}}$ is the variance of a single sample at each level.

2.2 Discussion on the variance of MLMC and MMLMC

The mean squared error (MSE) of the MMLMC estimator is given by

$$\mathrm{MSE}_{\mathrm{MMLMC}} = \mathrm{E}[(\ddot{Y} - \mathrm{E}[P])^2] = \mathrm{V}[\ddot{Y}] + (\mathrm{E}[\ddot{Y}] - \mathrm{E}[P])^2, \tag{4}$$

where the first term on the right-hand side in (2), (4) is the variance of the estimator and the second term is the square of its bias due to discretization. Giles (2008) proves the MLMC complexity theorem. It claims that the computational complexity to attain MSE $< \epsilon^2$ is reduced from $O(\epsilon^{-3})$ to $O(\epsilon^{-2}(\ln \epsilon)^2)$ for a simple case. Similar to the MLMC complexity theorem in Giles (2008), if we use levels $\ell_0, \ell_1, \ldots, \ell_F$ and not levels $0, 1, \ldots, L$, we discuss that the MMLMC version of the complexity theorem holds in the next section.

$\mathrm{V}[\widehat{Y}]$ and $V[\ddot{Y}]$ are estimated using different numbers of simulation paths, so there will be numerical differences. Suppose that MMLMC can reduce variance by 10% compared to MLMC at the same computational cost. This suggests that MMLMC can achieve the same estimation accuracy as MLMC at nearly 10% less computational cost. For example, when performing Monte Carlo simulations in financial practice, it is desirable to minimize computational cost. Therefore, we are interested in how much variance can be reduced by MMLMC compared to MLMC.

3 Complexity Theorem of Modified multilevel Monte Carlo method

While MLMC uses all time steps h_{ℓ_j} from level 0 to the maximum level, MMLMC does not necessarily do so. Let us confirm that the following Lemma 1 holds. Since Lemma 1 holds, we can prove the MMLMC complexity theorem on the basis of the proof procedure of the MLMC complexity theorem in Giles (2008). This means that both the variance and the square bias error of each MMLMC estimator have the same upper bound, $\epsilon^2/2$. If we assume (ii) in the complexity theorem in Giles (2008), the MLMC estimator is an unbiased estimator. Then, due to (1) and (3), each MMLMC estimator is an unbiased and $\mathrm{E}[\widehat{P}_L] = E[\widehat{Y}] = E[\ddot{Y}]$. Therefore, the bias errors of the MMLMC and MLMC estimators are the same. However, $V[\ddot{Y}]$ and $V[\widehat{Y}]$ are not always the same except in a special case; for example, if $\ell_0 = 0$ and $\ell_j - \ell_{j-1} = 1, j = 1, 2, \ldots, F$, the MMLMC estimator coincides with the MLMC estimator. Here, our focus is the performance comparison of the three methods (MMLMC, MLMC, and SMC) in terms of variance estimation.

Lemma 1. *Let $h_{\ell_j} = M^{-\ell_j}T, j = 1, 2, \ldots, F$, where $\ell_j \geq 0$ for all j. Let $\ell_F = \lceil \ln \sqrt{2}c_1 T^\alpha \epsilon^{-1}/\alpha \ln M \rceil$. Then it holds that*

$$\sum_{j=0}^{F} h_{\ell_j}^{(\beta-1)/2} < T^{(\beta-1)/2}(1 - M^{-(\beta-1)/2})^{-1} \quad \textit{for } \beta > 1, \tag{5}$$

$$\sum_{j=0}^{F} h_{\ell_j}^{-(1-\beta)/2} \leq h_{\ell_j}^{(1-\beta)/2}(1 - M^{-(1-\beta)/2})^{-1} \quad \textit{for } \beta < 1. \tag{6}$$

Proof. For $\beta > 1$,

$$\sum_{j=0}^{F} h_{\ell_j}^{(\beta-1)/2} \leq \sum_{\ell_j=0}^{\ell_F} h_{\ell_j}^{(\beta-1)/2} = T^{(\beta-1)/2}\frac{1 - M^{-(\ell_F+1)(\beta-1)/2}}{1 - M^{-(\beta-1)/2}} < T^{(\beta-1)/2}(1 - M^{-(\beta-1)/2})^{-1}.$$

For $\beta < 1$,

$$\sum_{j=0}^{F} h_{\ell_j}^{-(1-\beta)/2} \leq \sum_{\ell_j=0}^{\ell_F} h_{\ell_j}^{-(\beta-1)/2} = h_{\ell_j}^{(1-\beta)/2} \sum_{\ell_j=0}^{\ell_F} (M^{-(1-\beta)/2})^{\ell_j} < h_{\ell_j}^{(1-\beta)/2}(1 - M^{-(1-\beta)/2})^{-1}.$$

□

Theorem 1 (MMLMC Complexity Theorem)**.** *Let P denote a function of the solution of SDE (1) for a given Brownian path $W(t)$, and, let $\widehat{P}_{\ell_1}$ denote the corresponding level ℓ_j of numerical approximation using a numerical discretization with time step $h_{\ell_j} = M^{-\ell_j}T$. If there exist independent estimators $\widehat{Y}_{\ell_j}$ based on N_{ℓ_j} Monte Carlo samples, and positive constants $\alpha \geq 1/2, \beta \geq 0, \gamma, c_1, c_2, c_3$ such that (i) $|E[\widehat{P}_{\ell_j} - P]| \leq c_1 h_{\ell_j}^\alpha$,* *(ii) $E[\widehat{Y}_{\ell_0}] = E[\widehat{P}_{\ell_0}]$ and $E[\widehat{Y}_{\ell_j}] = E[\widehat{P}_{\ell_j} - \widehat{P}_{\ell_{j-1}}]$ for $j = 1, 2, \ldots, F$, (iii) $V[\widehat{Y}_{\ell_j}] \le c_2 N_{\ell_j}{}^{-1} h_{\ell_j}^{\beta}$, (iv) C_{ℓ_j}, the computational complexity of $\widehat{Y}_{\ell_j}$, is bounded by $C_{\ell_j} \le c_3 N_{\ell_j} h_{\ell_j}^{-1}, \quad j = 1, 2, \ldots, F$, then, there exists a positive constant c_4 such that for any $\epsilon < e^{-1}$ there are values ℓ_F and $N_{\ell_j}, j = 0, 1, \ldots, F$, for which the modified multilevel estimator, $\widehat{Y} = \sum_{j=0}^{F} \widehat{Y}_{\ell_j}$, has a mean-square-error with bound MSE $\equiv E[(\widehat{Y} - E[P])^2] < \epsilon^2$ with a computational complexity $C = \sum_{j=0}^{F} C_{\ell_j}$ with bound*

$$C \le \begin{cases} c_4\epsilon^{-2}, & \beta > 1, \\ c_4\epsilon^{-2}(\ln\epsilon)^2, & \beta = 1, \\ c_4\epsilon^{-2-(\gamma-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

In a special case ($\ell_0 = 0$ and for all $1 \le j \le F,\ \ell_j - \ell_{j-1} = 1$), the MMLMC estimator is the original MLMC estimator in Giles (2008). Note that MMLMC uses levels $\ell_0, \ell_1, \ldots, \ell_j, \ldots, \ell_F$.

Proof. Let $\ell_F = \lceil \ln\sqrt{2}c_1 T^\alpha \epsilon^{-1}/\alpha \ln M \rceil$. Then it holds that

$$\frac{1}{\sqrt{2}} M^{-\alpha}\epsilon < c_1 h_{\ell_F}^{\alpha} < \frac{1}{\sqrt{2}}\epsilon. \tag{7}$$

Due to (i) and (ii), it holds that $(E[\widehat{Y}] - E[P])^2 \le \epsilon^2/2$. Set $N_{\ell_j}, j = 0, 1, \ldots, F$ as follows

$$N_{\ell_j} = \begin{cases} \lceil 2\epsilon^{-2}(\ell_F + 1)c_2 h_{\ell_j} \rceil, & (\beta = 1), \\ \lceil 2\epsilon^{-2} c_2 T^{(\beta-1)/2}(1 - M^{-(\beta-1)/2})^{-1} h_{\ell_j}^{(\beta+1)/2}, & (\beta > 1), \\ \lceil 2\epsilon^{-2} c_2 h_{\ell_F}^{-(1-\beta)/2}(1 - M^{-(1-\beta)/2})^{-1} h_{\ell_j}^{(\beta+1)/2}, & (\beta < 1). \end{cases} \tag{8}$$

Denote the variance of $\widehat{Y}$ as $V[\widehat{Y}]$ and denote the variance of $\widehat{Y}_{\ell_j}$ as V_{ℓ_j}. It can be shown that $V[\widehat{Y}] = \sum_{j=0}^{F} V_{\ell_j} < \epsilon^2/2$. When $\beta = 1$, from (iii), $V[\widehat{Y}] = \sum_{j=0}^{F} V_{\ell_j} \le \sum_{j=0}^{F} c_2 N_{\ell_j}^{-1} h_{\ell_j}$. Due to (8), $N_{\ell_j} > 2\epsilon^{-2}(\ell_F + 1)c_2 h_{\ell_j}$. Note that $F < \ell_F$, it holds that $V[\widehat{Y}] < ((F+1)/2(\ell_F+1))\,\epsilon^2 \le \epsilon^2/2$. When $\beta > 1$, from (iii) and (8), $V[\widehat{Y}] = \sum_{j=0}^{F} V_{\ell_j} \le \sum_{j=0}^{F} c_2 N_{\ell_j}^{-1} h_{\ell_j}^{\beta} \le (\epsilon^2/2) T^{-(\beta-1)/2}(1 - M^{-(\beta-1)/2}) \sum_{j=0}^{F} h_{\ell_j}^{(\beta-1)/2}$. Due to (5), it holds that $V[\widehat{Y}] < \epsilon^2/2$. When $\beta < 1$, from (iii) and (8), $V[\widehat{Y}] = \sum_{j=0}^{F} V_{\ell_j} \le \sum_{j=0}^{F} c_2 N_{\ell_j}^{-1} h_{\ell_j}^{\beta} \le (\epsilon^2/2) h_{\ell_F}^{-(1-\beta)/2}(1 - M^{-(\beta-1)/2}) \sum_{j=0}^{F} h_{\ell_j}^{-(1-\beta)/2}$. Due to (6), $V[\widehat{Y}] < \epsilon^2/2$. Here, it holds thtat MSE $\equiv E[(\widehat{Y} - E[P])^2] = V[\widehat{Y}] + (E[\widehat{Y}] - E[P])^2 < \epsilon^2$.

Incidentally, when using the standard Monte Carlo method, the computational cost is known to be $O(\epsilon^{-3})$. From here, we analyze that the computational cost of the MMLMC method required to achieve MSE $< \epsilon^2$ is smaller than that of the standard Monte Carlo method.

Due to (7), $h_{\ell_F}^{-1} < M\left(\epsilon/(\sqrt{2}c_1)\right)^{-1/\alpha}$. It holds that $\sum_{j=0}^{F} h_{\ell_j}^{-1} \le \sum_{\ell_j=0}^{\ell_F} h_{\ell_j}^{-1} = h_{\ell_F}^{-1} \sum_{\ell_j=0}^{\ell_F} M^{-\ell_j} < (M/(M-1)) h_{\ell_F}^{-1}$. For $\alpha \ge 1/2$ and $\epsilon < 1/e$, $\epsilon^{-1/\alpha} \le \epsilon^{-2}$. Thus it holds that $\sum_{j=0}^{F} h_{\ell_j}^{-1} \le (M^2/(M-1))(\sqrt{2}c_1)^{1/\alpha}\epsilon^{-2}$. Here set $N_j, j = 0, 1, \ldots, F$ as (8). When $\beta = 1$, from (8), $N_{\ell_j} \le 2\epsilon^{-2}(\ell_F+1)c_2 h_{\ell_j} + 1$. Also, for $\epsilon < 1/e$, $1 < \ln\epsilon^{-1}$ and $\ell_F \le (\ln\epsilon^{-1} + \ln(\sqrt{2}c_1 T^\alpha))/(\alpha \ln M) + 1$. So it holds that $\ell_F + 1 \le c_5 \ln\epsilon^{-1}$, where $c_5 = 1/(\alpha \ln M) + \max\left(\ln(\sqrt{2}c_1 T^\alpha)/\alpha \ln M, 0\right) + 2$. The computational complexity is given by $C = \sum_{j=0}^{F} C_{\ell_j} \le c_3 \sum_{j=0}^{F} N_{\ell_j} h_{\ell_j}^{-1} \le c_3(2\epsilon^{-2}(\ell_F+1)^2 c_2 + \sum_{j=0}^{F} h_{\ell_j}^{-1})$. Thus the computational complexity is represented by $C \le c_4\epsilon^{-2}(\ln\epsilon)^2$, where $c_4 = 2c_2c_3c_5^2 + c_3(M^2/(M-1))(\sqrt{2}c_1)^{1/\alpha}$. When $\beta > 1$, from (8), $N_{\ell_j} < 2\epsilon^{-2} c_2 T^{(\beta-1)/2}(1 - M^{-(\beta-1)/2})^{-1} h_{\ell_j}^{(\beta+1)/2} + 1$. Due to (iv), $C = \sum_{j=0}^{F} C_{\ell_j} \le c_3 \sum_{j=0}^{F} N_{\ell_j} h_{\ell_j}^{-1} \le c_3(2\epsilon^{-2} c_2 T^{(\beta-1)/2}(1 - M^{-(\beta-1)/2})^{-1} \sum_{j=0}^{F} h_{\ell_j}^{(\beta-1)/2} + \sum_{j=0}^{F} h_{\ell_j}^{-1})$. Thus the computational complexity is represented by $C \le c_4\epsilon^{-2}$, where $c_4 = 2c_2c_3 T^{\beta-1}(1 - M^{-(\beta-1)/2})^{-2} + c_3(M^2/(M-1))(\sqrt{2}c_1)^{1/\alpha}$. When $\beta < 1$, from (8), $N_{\ell_j} < 2\epsilon^{-2} c_2 h_{\ell_F}^{-(1-\beta)/2}(1 - M^{-(1-\beta)/2})^{-1} h_{\ell_j}^{(\beta+1)/2} + 1$. Due to (iv), $C = \sum_{j=0}^{F} C_{\ell_j} \le c_3 \sum_{j=0}^{F} N_{\ell_j} h_{\ell_j}^{-1} \le c_3(2\epsilon^{-2} c_2 h_{\ell_F}^{-(1-\beta)/2}(1 - M^{-(1-\beta)/2})^{-1} \sum_{j=0}^{F} h_{\ell_j}^{-(1-\beta)/2} + \sum_{j=0}^{F} h_{\ell_j}^{-1})$. Due to (6), $h_{\ell_F}^{-(1-\beta)/2}(1 - M^{-(1-\beta)/2})^{-1} \sum_{j=0}^{F} h_{\ell_j}^{-(1-\beta)/2} < h_{\ell_F}^{-(1-\beta)}(1 - M^{-(1-\beta)/2})^{-2}$. Also, from (7), $h_{\ell_F}^{-(1-\beta)} < (\sqrt{2}c_1)^{(1-\beta)/\alpha} M^{1-\beta} \epsilon^{-(1-\beta)/\alpha}$ and for $\epsilon < 1/e$, $\epsilon^{-2} < \epsilon^{-2-(1-\beta)/\alpha}$. Thus the computational complexity is represented by $C \le c_4\epsilon^{-2-(1-\beta)/\alpha}$, where $c_4 = 2c_2c_3(\sqrt{2}c_1)^{(1-\beta)/\alpha} M^{1-\beta}(1 - M^{-(1-\beta)/2})^{-2} + c_3(M^2/(M-1))(\sqrt{2}c_1)^{1/\alpha}$. □

4 NUMERICAL EXPERIMENTS

In Section 4.1, we discuss the optimal number of simulation paths for the MMLMC method. In Section 4.2, we test the MMLMC estimators for European vanilla, Lookback and Asian options valuation. We compare the MMLMC method with the MLMC method and compare the MMLMC method with the SMC method in terms of variance reduction.

4.1 Optimal number of simulation paths

Let a computational complexity $C = \sum_{j=0}^{F} N_{\ell_j} M^{\ell_j} = \sum_{j=0}^{F} N_{\ell_j} T/h_{\ell_j}$. We regard $V[\ddot{Y}] = N_{\ell_0}^{-1} V_{\ell_0} + \sum_{j=1}^{F} N_{\ell_j}^{-1} V_{\ell_j,\ell_{j-1}}$ as a partially differentiable function of $N_{\ell_j}, j = 1, 2, \ldots, F$. We denote $V_{\ell_j,\ell_{j-1}} = V_{\ell_j}$, since there is no possibility of misunderstanding. Then the optimal number of paths for each level, minimizing the variance of each MMLMC estimator, are given by $\ddot{N}^*_{\ell_j}$ as follows.

Theorem 2. *For $\epsilon > 0$, the optimal number of simulation paths that achieve $V[\ddot{Y}] < \epsilon^2/2$, are given by*

$$\ddot{N}_{\ell_j} = \left[2\epsilon^{-2} \sqrt{V_{\ell_j} h_{\ell_j}} \left(\sum_{i=0}^{F} \sqrt{V_{\ell_i}/h_{\ell_i}} \right) \right], \quad 0 \le j \le F, \tag{9}$$

where $[n]$ is the least integer greater than or equal to n. Furthermore, for fixed computational complexity $C^* > 0$, the optimal number of simulation paths that minimize the variance of the corresponding MMLMC estimator are given by $\ddot{N}^*_{\ell_j} = [C^* \sqrt{V_{\ell_j} h_{\ell_j}} / \sum_{i=0}^{F} T \sqrt{V_{\ell_i}/h_{\ell_i}}], \quad 0 \le j \le F$.

Proof. To minimize $V[\ddot{Y}] = \sum_{j=0}^{F} N_{\ell_j}^{-1} V_{\ell_j}$, we apply the Lagrange Multipliers method. We create the Lagrange equation as $L := \mathcal{L}(N_{\ell_0}, N_{\ell_1}, \ldots, N_{\ell_F}) = \sum_{j=0}^{F} N_{\ell_j}^{-1} V_{\ell_j} - \lambda(C - \sum_{j=0}^{F} N_{\ell_j} T/h_{\ell_j})$. Set the partial derivative $L_{N_{\ell_0}}, L_{N_{\ell_1}}, \ldots, L_F$ equal to zero, $L_{N_{\ell_j}} = -N_{\ell_j}^{-2} V_{\ell_j} + \lambda T/h_{\ell_j} = 0, \quad j = 0, 1, \ldots, F$. Thereby, $N_{\ell_j} = \sqrt{V_{\ell_j} h_{\ell_j}/\lambda T}, \quad j = 0, 1, \ldots, F$. If $V[\ddot{Y}] < \epsilon^2/2$, it holds that $V[\ddot{Y}] = \sum_{j=0}^{F} N_{\ell_j}^{-1} V_{\ell_j} = \sum_{j=0}^{F} \sqrt{\lambda T/V_{\ell_j} h_{\ell_j}} V_{\ell_j} < \epsilon^2/2$. We set $N_{\ell_j} = 2\epsilon^{-2} \sqrt{V_{\ell_j} h_j} (\sum_{i=0}^{F} \sqrt{V_{\ell_i}/h_{\ell_i}}), \quad 0 \le j \le F$. Therefore, we get $\ddot{N}_j = [N_{\ell_j}]$. Due to the definition of C, it holds that $2\epsilon^{-2} \sum_{i=0}^{F} \sqrt{V_{\ell_i}/h_{\ell_i}} = C/\sum_{j=0}^{F} T \sqrt{V_{\ell_j}/h_{\ell_j}}$. If we fix $C = C^*$, we get $\ddot{N}^*_{\ell_j} = [C^* \sqrt{V_{\ell_j} h_{\ell_j}} / \sum_{i=0}^{F} T \sqrt{V_{\ell_i}/h_{\ell_i}}], \quad 0 \le j \le F$. □

Note that if the MMLMC estimator coincides with the MLMC estimator, (9) is equal to the optimal number of simulation paths used in Section 5 of Giles (2008).

4.2 Numerical results

We set $M = 2$, $\ell_0 = 0$, $\ell_F = 8$, and $C^* = 1,000,000 \times 2^8$. Therefore, we can test 128 types of MMLMC estimators. We present each result for $dS_t = 0.03 S_t dt + 0.3 S_t dW_t, 0 \le t < T$, where $S_0 = 1$ and $T = 1$. We note that the results of the SMC estimator are based on $1,000,000$ simulation paths and 2^8 time-steps. European vanilla put, Lookback put and Asian put options payoffs at time T are $\max(K - S_T, 0)$, $\max_{0 \le t \le T} S_t - S_T$ and $E(S_T) - S_T$, respectively. We set the strike price $K = 1$. The numerical procedure is as follows. First, we set the initial $\ddot{N}_{\ell_j} = 10^5$. To calculate the optimal $\ddot{N}_{\ell_j}$, we run the MMLMC method using the initial paths. Second, we run the MMLMC method using the optimal paths and estimate the variance of the estimator. We repeat this procedure for all the MMLMC estimators. Figures 1, 2 and 3 show that more than 100 types of MMLMC estimators attain lower variance than that of the MLMC estimator in European Vanilla option and Lookback option valuations.

(European vanilla option) In comparison to the MLMC estimator, the finest MMLMC estimator, which has the lowest variance and is $\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{3,0} + \tilde{Y}_{5,3} + \tilde{Y}_{8,5}$, achieves variance reduction of about 20 % in Table 1. The MMLMC estimators reduce variance by 88-96 % of the SMC estimator.
(Lookback option) In comparison to the MLMC estimator, the finest MMLMC estimator, which is $\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{3,0} + \tilde{Y}_{5,3} + \tilde{Y}_{8,5}$, achieves variance reduction of little less than 28 % in Table 1. The MMLMC estimators reduce variance by 23-81 % of the SMC estimator.
(Asian option) In comparison to the MLMC estimator, the finest MMLMC estimator, which is $\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{2,0} + \tilde{Y}_{4,2} + \tilde{Y}_{6,4} + \tilde{Y}_{8,6}$, achieves variance reduction of little less than 9 % in Table 1. The MMLMC estimators reduce variance by 23-96 % of the SMC estimator.

Table 1. Estimated Variance of MMLMC (finest), MLMC and SMC

European Vanilla option price: 0.10			Lookback option price: 0.23		
MMLMC (finest)	MLMC	SMC	MMLMC (finest)	MLMC	SMC
6.84E-09	8.58E-09	1.87E-07	4.33E-09	6.02E-09	2.32E-08

Asian option price: -0.02		
MMLMC (finest)	MLMC	SMC
1.29E-09	1.41E-09	3.24E-08

Table 2. Optimal Number of Simulation Paths of MMLMC (finest), MMLMC (coarsest) and MLMC

European Vanilla option									
Level	0	1	2	3	4	5	6	7	8
MMLMC (finest)	9.8E+07	-	-	7.4E+07	-	1.2E+06	-	-	2.3E+05
MMLMC (coarsest)	5.7E+07	-	-	-	-	1.2E+06	-	-	7.8E+05
MLMC	8.8E+07	1.2E+07	5.7E+06	2.7E+06	1.3E+06	6.3E+05	3.1+E05	1.5E+05	7.7E+04

Lookback option									
Level	0	1	2	3	4	5	6	7	8
MMLMC (finest)	3.9E+07	-	-	9.5E+06	-	1.8E+06	-	-	3.3E+05
MMLMC (coarsest)	1.9E+07	-	-	-	-	-	-	-	9.2E+05
MLMC	3.3E+07	1.2E+07	6.6E+06	3.5E+06	1.8E+06	9.0E+05	4.6+E05	2.3E+05	1.1E+05

Asian option									
Level	0	1	2	3	4	5	6	7	8
MMLMC (finest)	6.5E+07	-	1.7E+07	-	3.3+E06	-	6.0E+05	-	1.3E+05
MMLMC (coarsest)	2.2E+07	-	-	-	-	1.2E+06	-	-	9.1E+05
MLMC	6.2E+07	1.7E+07	8.0E+06	3.5E+06	1.5E+06	6.6E+05	3.0+E05	1.4E+05	6.8E+04

5 Concluding Remarks

The numerical results show that all the MMLMC estimators perform better than the SMC estimator. The results also show that more than 100 types of MMLMC estimators perform better than the MLMC estimator and that the finest estimator reduces variance by more than 28 % of the MLMC estimator in two European style options valuation. Thinning some of the levels in the MLMC estimator improves the estimate of variance. We conjecture that the MMLMC method is beneficial for other options valuation. Future research may take three directions. First, it may be possible to apply the MMLMC method to other option pricing methods verified in Inui and Ano (2025). Second, it may be possible to apply this method to the valuation of American-style options. Finally, we plan to clarify why MMLMC is more effective than MLMC in reducing variance.

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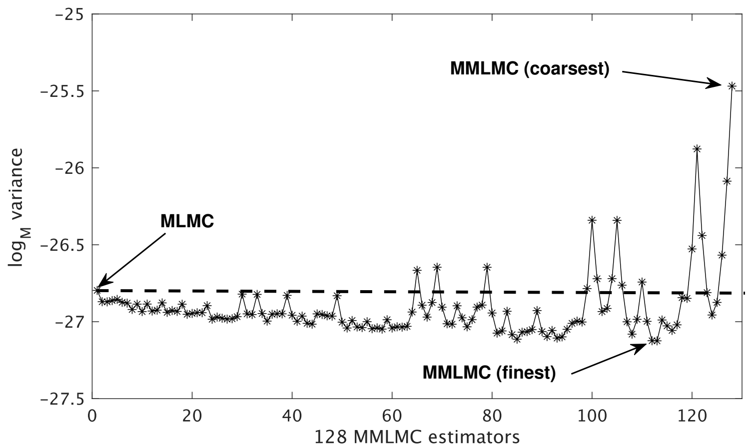

Figure 1. Variance of all MMLMC estimators in European Vanilla option valuation

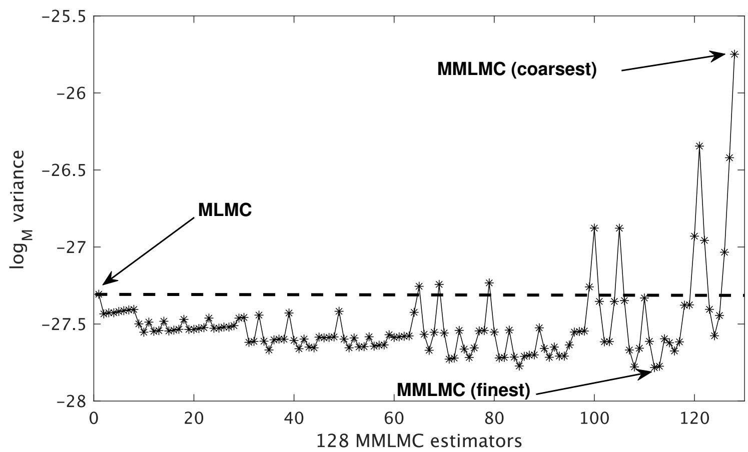

Figure 2. Variance of all MMLMC estimators in Lookback option valuation

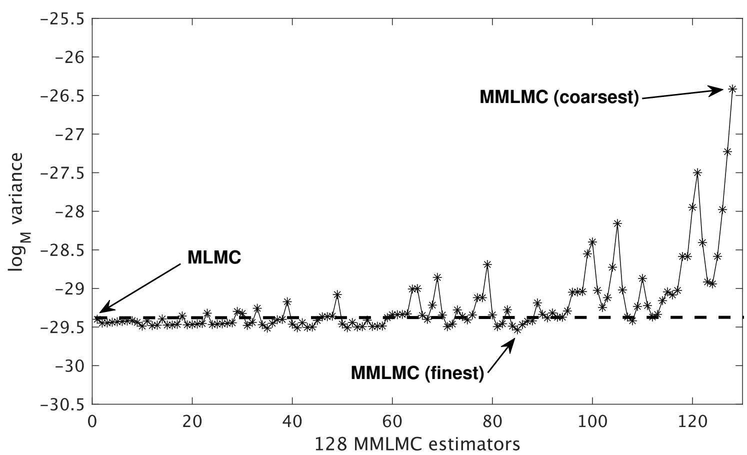

Figure 3. Variance of all MMLMC estimators in Asian option valuation

