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Full information problem with random horizon

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Abstract:

Consider observing a sequence of independent and identically distributed random variables drawn from some known distribution. A 'gambler' observes these variables sequentially, where no recall of previous observations is permitted, and has the objective to stop on several of these in order to maximise their expected reward. A 'prophet', by contrast, has clairvoyance of the entire sequence and can choose the optimal set of stops in hindsight – this achieving some 'edge' in pay-off over the gambler.

The performance gap (i.e. how well can a gambler do against a profit?) between the gambler and prophet is normally studied via a 'prophet-inequality' which is usually quite general. We contribute two explicit examples in the multiple stopping context with random horizon, where the observations are uniformly distributed with a discrete uniform horizon, and a geometric horizon (a bounded and unbounded horizon for contrast) – providing the structure of the rules, pay-offs and some numeric analysis.

Keywords: *Sequential decision analysis, optimal stopping, multiple optimal stopping, secretary problems, prophet inequalities*

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1 INTRODUCTION

Consider the problem where the gambler observes a sequence of random variables $X_1, X_2, \ldots, X_n$, where n is known, and the X_i are independent and identically distributed (iid) from some known distribution $F(x)$. The gambler, in the single stopping problem, may stop on any observation $X_m = x_m$ and receive x_m as their pay-off. In the multiple stopping context, the gambler may now stop on $k > 1$ variables and, after stopping on variables $X_{m_1}, X_{m_2}, \ldots, X_{m_k}$, receives the sum of these variables; see Entwistle et al. (2024) and Sofronov and Szajowski (2025) for a full overview of the formulation for the multiple stopping case where n is known.

This problem is a subset of a general class of other optimal stopping problems that all aim to find a sequential procedure to maximise the expected reward (see Section 13.4 of DeGroot (2005) for an overview), with the 'Secretary Problem' being the most well known (see Freeman (1983) and Ferguson (1989)).

As a means to benchmark the rewards achieved by 'optimal players' (which we refer to as 'gamblers'), the performance of the gambler is compared to that of a 'prophet' who gets to observe the entire sequence $X_1, X_2, \ldots, X_n$ and simply choose what they want (obviously, they will take the top k observations). A rich field of 'prophet inequalities' (PI) has been developed for certain situations. For example, in Krengel and Sucheston (1978), if the variables $X_1, \ldots, X_n$ are independent and non-negative, then the prophet's average reward M in the single stopping problem is no more than twice that of the gambler's reward V (i.e. $M \leq 2V$), and this bound is sharp. If the variables need not be independent, then in Hill and Kertz (1981), it was shown that $M \leq nV$. Also, if the non-negativity assumption is dropped, the ratio can be aribtrarily large. A survey of prophet inequalities may be found in Hill and Kertz (1992), also see Correa et al. (2019) for more recent developments.

Of interest to the authors is the case where n is *not* known – but, rather, is a random horizon (RH), H, with some known distribution which was first analysed in the context of the 'Secretary Problem' in Presman and Sonin (1973); see Entwistle and Sofronov (2025) for an overview of optimal stopping problems with random horizon. The gambler has the awkward prospect that they may be 'caught' out and not stop before the horizon and receive nothing. In contrast, the prophet has the luxury of observing both H as well as the sequence of variables. Some work has been done in this area with iid PI first introduced in Hajiaghayi et al. (2007) and developments with multiple stopping in Alijani et al. (2020) and more general classes of random horizon in Allaart (2004).

We consider the case where the observations are independent and identically distributed uniform $U(0,1)$ random variables (for their simple order statistics) and provide explicit formulations and results in the case of a discrete uniform horizon, which is bounded, and the unbounded geometric horizon, where the memoryless property can be utilised.

2 THE BOUNDED RANDOM HORIZON PROBLEM

In the case of where the horizon H is bounded, say, by the integer N (for example, H may follow a discrete uniform distribution on $[1, N]$), then the solution to the optimal stopping problem may still be found by backward induction with each variable discounted by a factor of its survival probability. Indeed, if we happen to arrive on observation X_N (the last), we *must* accept. Then on any observation X_i, $1 \leq i \leq N-1$, we balance the gain by stopping on X_i with the expected gain of continuing and being able to observe the next.

We define v_n as the value of the 'game' with n steps remaining, which is conditional on having reached this stage of the game. Then

$$v_n = \begin{cases} \mathbb{E}\max\{X_{N-n+1}, v_{n-1} \times P(H \geq N-n+2 | H \geq N-n+1)\} & n = 1, 2, \ldots, N, \\ 0 & n = 0, \end{cases}$$

where $P(H \geq N-n+2 | H \geq N-n+1)$ is the conditional probability that we will be able to progress to the next observation (if we decided to), given that we have already reached the $(N-n+1)$-th observation (as there are n potential observations left).

This is equivalent to observing the discounted sequence of random variables $Z_i = X_i S(i)$, where $S(i) = P(H \geq i | H \geq i-1)$.

2.1 Example: $X \sim U(0,1)$ with $H \sim U\{1, 2, \ldots, N\}$ (discrete uniform)

Suppose now that $H \sim U\{1, 2, \ldots, N\}$ is a discrete uniform with a fixed constant $N \in \mathbb{Z}^+$. Then the survival probability from observation $N-n+1$ to $N-n+2$ is given by $P(H \geq N-n+2 | H \geq N-n+1) = 1 - 1/n$.

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With the observations themselves coming from a $U(0,1)$ distribution, we have

$$\begin{aligned} v_n &= \mathbb{E}\max\left\{X_{N-n+1}, \left(1-\frac{1}{n}\right)v_{n-1}\right\} \\ &= \int_0^{(1-\frac{1}{n})v_{n-1}} \left(1-\frac{1}{n}\right)v_{n-1}\,dx + \int_{(1-\frac{1}{n})v_{n-1}}^1 x\,dx \\ &= \frac{1+\left[(1-\frac{1}{n})v_{n-1}\right]^2}{2} \qquad \text{with} \quad v_0 = 0, \end{aligned}$$

where the gambler receives, on average, V_N. Comparing with the standard case (with no random horizon), the $(1-\frac{1}{n})$ term operates as a discount factor.

Now consider the 'prophet' in this setting, who knows the result of the random variable $H = h$ and also observes the entire sequence $X_1, X_2, \ldots, X_h$. They would receive the maximum of these h variables. Let the prophet receive $M_1(N)$. Then we may show that

$$M_1(N) = 1 - \frac{1}{N}(H_{N+1}-1) \qquad \text{with} \quad H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For the multiple stopping case, we use the same notation as in Sofronov et al. (2006) and Nikolaev and Sofronov (2007), we define $v^{L,l}$ as the value of the sequence with l, $l \le k$, stoppings and L, $L \le N$ steps remaining. We have

$$\begin{aligned} v^{n,1} &= \mathbb{E}\left(\max\left\{X_{N-n+1}, \left(1-\frac{1}{n}\right)v^{n-1,1}\right\}\right), \quad 1 \le n \le N, \\ v^{n,k-i+1} &= \mathbb{E}\left(\max\left\{X_{N-n+1} + \left(1-\frac{1}{n}\right)v^{n-1,k-i}, \left(1-\frac{1}{n}\right)v^{n-1,k-i+1}\right\}\right), \end{aligned}$$

with $v^{0,1} = 0$ and $v^{k-i,k-i+1} = 0$ for $i = k-1, \ldots 1$. The optimal stopping rules can be defined with respect to thresholds in terms of these value functions (see Entwistle et al. (2024) for the same formulation without random horizon).

The prophet simply takes the top k observations (i.e. the top k order statistics) of the observed sequence (or just takes everything available in the case where $h \le k$). In other words, for a fixed $H = h$ the prophet receives, (on average)

$$m_k = \sum_{j=h-k+1}^{n} \frac{j}{n+1} = \frac{k(2h-k+1)}{2(h+1)} \qquad \text{for} \quad k \le h,$$

and $m_k = h/2$ if $k \ge h$. We remind the reader, if they are not familiar, that for the j-th order statistic of h standard uniform random variables, we have $\mathbb{E}(X_{(j)}) = j/(h+1)$, while $\mathbb{E}(X) = 1/2$. Summing over the horizon distribution, we have

$$M_k(N) = \sum_{h=1}^{k} \frac{1}{N}\cdot\frac{h}{2} + \sum_{h=k+1}^{N} \frac{1}{N}\cdot\frac{k(2h-k+1)}{2(h+1)} = \frac{k}{4N}(4N-3k+1) - \frac{k(k+1)}{2N}\left(H_{N+1} - H_{k+1}\right),$$

for $1 \le k \le N$, and $V_k(N) = M_k(N) = (N+1)/4$ for $k \ge N$. This agrees with the earlier expression for $k = 1$ and reduces to $(N+1)/4$ for $k = N$ which we note is simply $\mathbb{E}(H) \times \mathbb{E}(X)$.

Backward induction works well in this bounded horizon context but other techniques are required for unbounded horizons since the base case is 'at infinity' – although one simple method would be to truncate the horizon and consider the modified horizon H^* with $P(H^* = h) = P(H = h)$ for all $h \le n-1$ and $P(H^* = n) = P(H \ge n)$ which has negligible changes in performance for the gambler.

We now turn to the special case where the horizon is instead a geometric random variable, where the memoryless property holds. This results in easier equations to analyse since 'backward induction from infinity' is not needed.

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3 GEOMETRIC RANDOM HORIZON (UNBOUNDED)

We saw that in the bounded horizon case, the rule is multi-threshold. For an unbounded horizon posessing the memoryless property, the rule becomes a 'fixed-price' policy (i.e. single threshold) in the single stopping case. The multiple stopping case is also easy to analyse.

We let the value of the game with $k = 1$ stops for the gambler and prophet be $V_1(q)$ and $M_1(q)$, respectively. Suppose that $H \sim \mathrm{Geom}(1-q)$, a geometric distribution with with the probability of 'success' $1-q$. Then the 'survival' probability is $S(h) = q$ for all h and we have that

$$V_1(q) = \mathbb{E}\max\{X, qV_1(q)\},$$

where, due to the memoryless property, $V_1(q)$ appears again on the right-hand side since if we survive another observation then our expected gain has been 'rejuvenated'. Clearly, the rule is to accept X if and only if (iff) $X \geq qV_1(q)$.

It is also easy to extend to the multiple stopping formulation for k stops, by developing the sequences $V_1(q), V_2(q), \dots, V_n(q)$, where each $V_k(q)$ is the expected gain in the equivalent problem with k stops for any fixed q. Then $V_k(q)$ and $V_{k-1}(q)$ are related through the equation

$$V_k(q) = \mathbb{E}\left(\max\{X + qV_{k-1}(q), qV_k(q)\}\right) \qquad \text{and} \qquad V_0(q) = 0, \tag{1}$$

which reduces to the strategy where, in the situation with k stops left, we stop iff $X \geq c_k(q)$ for some threshhold $c_k(q)$ that will depend on k and q such that 'indifference' is obtained. In the other words, we choose $c_k(q)$ to satisfy $c_k + qV_{k-1}(q) = qV_k(q)$ which gives

$$\begin{aligned} V_k(q) &= (V_k(q) - V_{k-1}(q)) + (V_{k-1}(q) - V_{k-2}(q)) + \cdots + (V_2(q) - V_1(q)) + V_1(q) \\ &= \frac{1}{q}\sum_{m=1}^{k} c_m(q), \end{aligned} \tag{2}$$

and the dynamic programming equation can be solved to obtain a recursive expression for each $c_k(q)$.

3.1 Example: $X \sim U(0,1)$ with $H \sim \mathrm{Geom}(1-q)$

Gambler's Pay-off and Strategy
We use (1) to obtain

$$\begin{aligned} V_k &= \int_0^{c_k} qV_k \, dx + \int_{c_k}^{1} x + qV_{k-1} \, dx \\ &= c_k(c_k + qV_{k-1}) + \frac{1-c_k^2}{2} + qV_{k-1}(1-c_k), \end{aligned}$$

and then replace V_k with $\frac{1}{q}c_k + V_{k-1}$ (since c_k is defined through indifference) and then solve the resulting quadratic in c_k to obtain

$$c_k = \frac{1 - \sqrt{1 - q^2 + 2q^2(1-q)V_{k-1}}}{q}, \tag{3}$$

where from (2), we have that $V_k = \frac{1}{q}\sum_{m=1}^{k} c_m$. So we have a closed recurrence

$$c_k = \frac{1}{q}\left(1 - \sqrt{1 - q^2 + 2q(1-q)\sum_{m=1}^{k-1} c_m}\right), \quad \text{with base case } c_0 = 0. \tag{4}$$

Prophet's Pay-off and Strategy
Recall that for a fixed $H = h$, the prophet receives

$$m_k = \sum_{j=h-k+1}^{n} \frac{j}{n+1} = \frac{k(2h-k+1)}{2(h+1)} \quad \text{for } k \leq h,$$

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and $m_k = \frac{h}{2}$ if $k \geq h$ where we drop the notation $M_k(q)$ to m_k to indicate that this is for a fixed instance $H = h$ and, therefore, is not a function of q. So we have that, for a fixed horizon $H = h$,

$$m_{k+1} - m_k = \frac{h-k}{h+1} \qquad \text{for} \quad k < h,$$

and 0 otherwise. Then we sum over $P(H = h)$ to obtain

$$\begin{aligned} M_{k+1}(q) - M_k(q) &= (1-q) \sum_{h=k+1}^{\infty} q^{h-1} \cdot \frac{h-k}{h+1} \\ &= q^k + \frac{(k+1)(1-q)}{q^2} \ln(1-q) + \frac{(k+1)(1-q)}{q^2} \sum_{h=1}^{k+1} \frac{q^h}{h}. \end{aligned}$$

We may also observe that the expression $A_k(q) := M_{k+1}(q) - M_k(q)$ may be written in integral form

$$A_k(q) = q^k \int_0^1 \left(\frac{1-s}{1-qs} \right)^{k+1} ds,$$

which can help analyse asymptotic behaviour. These equations for $V_k(q)$ and $M_k(q)$ can be recursively applied to obtain explicit (although messy) equations for each but are suitable enough for numeric analysis.

3.2 Results

For $k = 1$, we have

$$V_1(q) = \frac{1 - \sqrt{1-q^2}}{q^2} \quad \text{and} \quad M_1(q) = \frac{q + (1-q)\log(1-q)}{q^2}$$

for $q \in (0, 1)$, giving a prophet-to-gambler ratio of

$$R(q) = \frac{q + (1-q)\log(1-q)}{1 - \sqrt{1-q^2}},$$

which is maximised at $q \approx 0.822059$. This is the value that the prophet would adversarially choose if they wished to be most competitive over the gambler, with $M/V \approx 1.19573$.

The difference, however, is given by

$$D(q) = \frac{q + (1-q)\log(1-q) - 1 + \sqrt{1-q^2}}{q^2},$$

which is maximised at $q \approx 0.901302$, with $M - V \approx 0.13041$.

If we extend this example to $k = 2$ items, we witness an interesting dynamic. For $k = 2$, we obtain

$$\begin{aligned} V_2(q) &= \frac{2 - \sqrt{1-q^2} - \sqrt{(1-q)(3+q-2\sqrt{1-q^2})}}{q^2} \qquad \text{and} \\ M_2(q) &= \frac{3-q}{q} + \frac{3(1-q)\ln(1-q)}{q^2}. \end{aligned}$$

The ratio $M_2(q)/V_2(q)$ is maximised at $q = 0.89275$ with a ratio ≈ 1.18301, and the difference $M_2(q) - V_2(q)$ is maximised at $q = 0.94705$ with a difference ≈ 0.24182. Both indicate a preference towards higher values of q as the prophet benefits from more choices over the gambler.

Interestingly, it seems that the optimal ratio is decreasing with more items, but the difference is increasing (at least to begin with). Clearly, for a fixed q, then in the limit $k \to \infty$,

$$\lim_{k\to\infty} V_k = \lim_{k\to\infty} M_k = \mathbb{E}(X) \cdot \mathbb{E}(H) = \frac{1}{2(1-q)},$$

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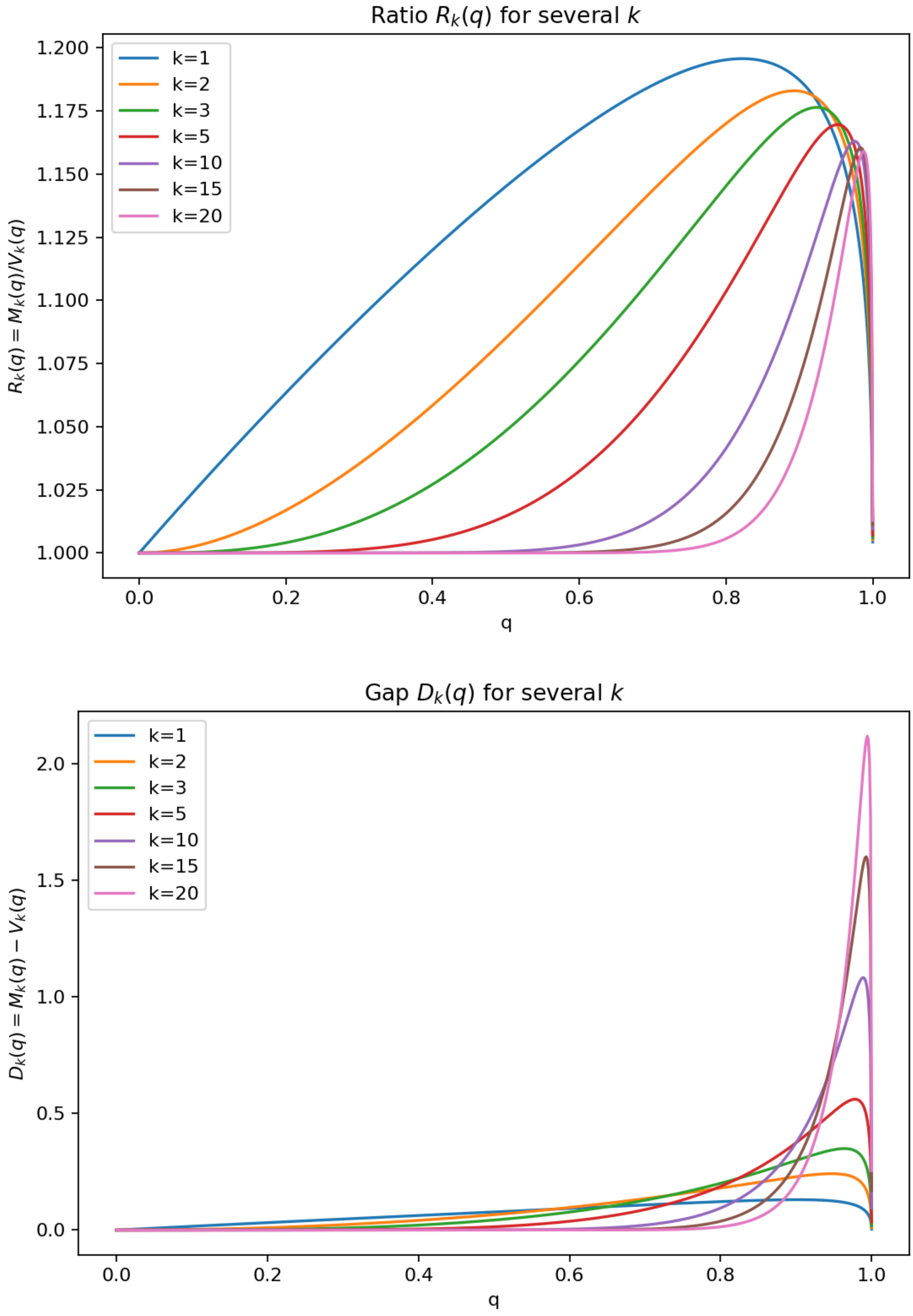

Figure 1. **The ratio $R_k(q)$ (top) and difference $D_k(q)$ (bottom) for values $k \in \{1, 2, 3, 5, 10, 15, 20\}$.**

since the strategy for both players reduces to simply taking every single item until the horizon occurs. This result is consistent if we analyse the equations found for a fixed k and take $k \to \infty$.

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Figure 1 shows the ratio $R_k(q)$ and difference $D_k(q)$ for various values of k and varying q. For $D_k(q)$, we note that for certain fixed values of q (typically smaller ones), the prophet will eventually prefer smaller values of k so as not to give the gambler enough 'edge' (which is evident in the intersection of various curves for $0.7 < q < 0.9$). For example, for $q = 0.9$ the difference is maximised at $k = 7$, which can be easily verified. Indeed, for a fixed q, we have already demonstrated that $D_k(q) \to 0$. However, if we are allowed to vary q for each k, the difference plot indicates that the prophet will always enjoy extra 'edge' for larger values of k by 'sliding' the value of q closer to 1. We let

$$F_k = \max_{q\in(0,1)} (M_k(q) - V_k(q)) = \max_{q\in(0,1)} D_k(q),$$

and consider the sequence $\{F_k\}_{k=1}^{\infty}$. We interpret F_k as the 'adversarial' best case scenario for the prophet – as if they are given the power to choose the value of q for each k. Then we can see that F_k is a strictly increasing sequence, whilst $D_k(q)$ is a unimodal sequence (that decreases to 0 in the limit as k gets large for a fixed q). The ratio plot shows a different story – with the prophet enjoying their edge when k is smaller, or if q is aggressively closer to 1, however the prophet prefers $k = 1$ and $q \approx 0.822059$ over all other situations.

4 CONCLUSION

In summary, we have analysed two explicit examples of full-information multiple-stopping problems with a random horizon in both the bounded and unbounded context. For a bounded horizon, we can view the problem as an equivalent stopping problem with a finite number of observations with discounting, which yields a multi-threshold strategy. For the unbounded geometric horizon, we took advantage of the memoryless property which yielded a fixed-price strategy to establish recursive equations for the gambler's payoff. Our computations reveal interesting patterns for the prophet-gambler gap in both ratio and difference forms.

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