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Re-Parameterization of Multi-regime STAR-GARCH Model

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Abstract: It is well known in the literature that the joint parameter estimation of the Smooth Autoregressive – Generalized Autoregressive Conditional Heteroskedasticity (STAR-GARCH) models poses many numerical challenges with unknown causes. This paper aims to uncover the root of the numerical difficulties in obtaining stable parameter estimates for a class of three-regime STAR-GARCH models using Quasi-Maximum Likelihood Estimator (QMLE). The paper also provides an easy and practical solution to alleviate the difficulties based on the findings.

The paper is divided into two parts. The first part investigates the numerical difficulties in maximizing the likelihood function by using computer simulations. Previous studies in the literature have identified that the threshold values and the transition rates are particular difficult to estimate. In light of this view, simulated data based on a pre-defined three-regime STAR-GARCH model will be generated and the values of the associated likelihood functions will be computed against different threshold values and transition rates.

The results show some interesting characteristics of the likelihood functions that have not been reported previously. Firstly, the log-likelihood functions of Exponential STAR-GARCH (ESTAR-GARCH) models tend to be flat around the global optimum near the true values of the transition rates. This explains the difficulties in estimating the transition rates by maximizing the log-likelihood functions using conventional gradient-based optimization algorithms. Secondly, the surfaces of the log-likelihood functions of the Logistic STAR-GARCH (LSTAR-GARCH) models tend to be lumpy in addition to being flat around the local optimums. This explains the sensitivity of QMLE relative to initial values. These findings have two implications: (i) the shapes of the log-likelihood functions are determined mostly by the choice of transition functions and (ii) it may be possible to transform the shapes of the log-likelihood functions by re-parameterising the model.

This paper proposes a simple re-parameterization of the three-regime STAR-GARCH models by transforming the transition rate parameter. The Monte Carlo simulation results show that the proposed method can alleviate the overall flatness and lumpy flatness of the log-likelihood functions for both LSTAR-GARCH and ESTAR-GARCH. These show promising signs in reducing estimation difficulties when jointly estimating the model parameters. Moreover, the results also open new channels for uncovering the statistical and structural properties of the three-regime STAR-GARCH model.

Keywords**:** *STAR, GARCH, Monte Carlo Simulation, reparameterization*

1. INTRODUCTION

The literature of nonlinear time series analysis has been growing rapidly in the last two decades. Interestingly, the literature focuses mainly on regime-switching models, such as the Smooth Transition Autoregressive (STAR) model of Teräsvirta (1994) and Markov-Switching (MS) model of Hamilton (1989). This is perhaps not surprising as many economic and financial variables exhibited regimes switching behaviours. For examples, Teräsvirta and Anderson (1992) applied STAR model to characterize dynamics of Gross National Product (GNP) during recession and expansion; Franses and van Dijk (2000) and Chan and McAleer (2003) followed the work of Lundbergh and Teräsvirta (1999) and applied the STAR models with Generalized Autoregressive Conditional Heteroskedastic errors (STAR-GARCH) to analyse the dynamics of stock returns.

Despite the popularity in applying regimes switching models in empirical studies, the statistical and structural properties for STAR and STAR-GARCH models are limited and the results are often restricted to the two-regime case. The lack of general structural and statistical properties makes valid inferences difficult to conduct for multi-regime switching models. Furthermore, the specification and the estimation of these models are not always straightforward, even in the two-regime case. As indicated in Haggen and Ozaki (1981) and Teräsvirta (1994), the transition rates in the STAR models are particularly difficult to estimate. This observation seems to be true also for STAR-GARCH models as demonstrated in Chan and McAleer (2002).

This paper aims to identify the causes underlying the difficulties in estimating the transition rates for three-regime STAR-GARCH models. Following the findings using computer simulations, the paper also proposes a practical and simple solution to alleviate the numerical difficulties in estimating the parameters of these models. The usefulness of the proposed method is investigated by Monte Carlo simulations. The results show promising signs for obtaining stable parameter estimates for a class of three-regime STAR-GARCH models.

The paper is organized as follows: Section 2 contains a concise review of the STAR-GARCH models. Section 3 investigates the log-likelihood functions of three-regime STAR-GARCH models using computer simulations. Following the results from Section 3, Section 4 proposes a re-parameterization of STAR-GARCH models in order to alleviate the numerical problem associated with maximizing the log-likelihood functions. Section 5 contains some concluding remarks.

2. LITERATURE REVIEW-MODEL SPECIFICATION, ESTIMATION, STRUCTURAL & STATISTICAL PROPERTIES,

In order to make STAR models more applicable for financial time series, Lundbergh and Teräsvirta (1999) proposed the STAR-GARCH model. It uses the STAR model of Teräsvirta (1994) for modelling the conditional mean and the GARCH model of Bollerslev (1986) for modelling the conditional variance. The specification of the multi-regime STAR-GARCH (MRSTAR – GARCH) model is:

$$\begin{aligned} y_t &= \sum_{i=1}^{m} \phi_i' x_t (G_{i-1}(s_{t-1};\gamma_{i-1},c_{i-1}) - G_i(s_{t-1};\gamma_i,c_i)) + \varepsilon_t \\ \varepsilon_t &= \eta_t \sqrt{h_t} \\ h_t &= \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \end{aligned} \tag{2.1}$$

where $k \geq 1$, $\phi_i = (\phi_{i1},\ldots,\phi_{ir})'$ and $G_i(s_t;\gamma_i,c_i)$ is the transition function with $G_0 = 1$. The transition function is a function of the threshold variable s_t with threshold value c_i and the smoothness parameter γ_i. The transition function must be continuous, at least twice differentiable and bounded between 0 and 1. Two of the most common transition functions adopted by the literature are the first-order logistic function and the first order exponential function. The transition function in the Logistic STAR-GARCH model (LSTAR-GARCH) is defined to be

$$G(s_t;\gamma;c)=\frac{1}{1+\exp(-\gamma(s_t-c))},\ \gamma>0, \tag{2.2}$$

and the transition function in the Exponential STAR-GARCH model (ESTAR-GARCH) is defined to be:

$$G(s_t;\gamma;c)=1-\exp\{-\gamma(s_t-c)\},\gamma>0. \tag{2.3}$$

van Dijk, Teräsvirta and Franses (2002) assert that the specification of the threshold variable s_t can be a combination of both endogenous and exogenous variables. A common specification of the threshold variable in financial applications is a linear combination of lagged dependent variable, that is $s_t=\sum_{i=1}^{n}\pi_i y_{t-i}$.

It is now widely accepted that the conditional variance of returns from high frequency financial time series is not constant over time. Therefore, the GARCH model of Bollerslev (1986) is included in equation (2.1) to capture the dynamics of the conditional variance in addition to the STAR model specified for the conditional mean. The structural and statistical properties of GARCH model are well established, see for example, Ling and McAleer (2003).

There are two approaches to estimate the parameters in STAR-GARCH models with m-regimes, namely, a two-stage procedure and a joint parameter estimation procedure. The two-stage procedure involves estimating the parameters in the conditional mean at the first stage using Non-linear Least Squares (NLS). The parameters in the conditional variance equation are estimated at the second stage by using the estimated residuals from the first stage. For a comprehensive survey on the two-stage procedure, see Teräsvirta (1994), van Dijk et al. (2002), and Lundbergh and Teräsvirta (1999).

One of the most common approaches for the joint parameter estimation procedure is to obtain the parameter estimates by maximising the log-likelihood function. Let $\theta=(\Gamma,\Omega)$ so that $\Gamma=(\phi_1',...,\phi_m',\gamma_1,...\gamma_m,c_1,...,c_m)$ is the vector of all the parameters in the conditional mean and $\Omega=(\omega,\alpha_1,...,\alpha_p,\beta_1,...,\beta_q)$ is the vector of parameter in the conditional variance. The Quasi Maximum Likelihood Estimator (QMLE) is defined to be

$$\begin{aligned}\hat{\theta}&=\arg\max_{\theta\in\Theta}\sum_{t=1}^{T}l_t(\theta)\\ l_t(\theta)&=-\frac{1}{2}\left(\log h_t+\frac{\varepsilon_t^2}{h_t}\right)\end{aligned} \tag{2.4}$$

where Θ is a compact subset of $\mathbb{R}^{(r+2)m+p+q+1}$. Haggen and Ozaki (1981) noticed that it is difficult to obtain QMLE as defined in (2.4) for STAR type models under different transition function. They explained that this could be the result of strong negative correlation between the transition rate, γ_i and the rest of the parameters. Teräsvirta (1994) confirmed that similar problem also exists for both LSTAR and ESTAR models.

In addition to the numerical difficulties in obtaining the QMLE for STAR-GARCH models, the knowledge concerning statistical and structural properties of STAR-GARCH models is still very much limited. Virtually all of the existing theoretical results applied only to the two regime case of STAR and STAR-GARCH models.

3. IDENTIFICATION OF ESTIMATION DIFFICULTIES – ANALYSIS OF THE LOG LIKELIHOOD FUNCTION

This section identifies the cause of the numerical difficulties in estimating three-regime STAR-GARCH models. This is done by investigating the surface of log-likelihood function using simulated data. The procedure of this investigation can be found as follows:

1. Simulate a set of data following a pre-defined three-regime STAR-GARCH(1,1) model
2. Given the data, compute the values of the log-likelihood function as defined in equation (2.4) with a range of transition rates and threshold values.
3. Investigate the surface of the log-likelihood functions using 3-dimensional plots and contour plots.
4. Use the simulated data above to individually investigate $\varepsilon-$neighbourhood plots of each parameter estimate around its optimum point.

Given that the parameter vector of the STAR GARCH model used in the log likelihood equation, θ is a 1x13 vector, the only way to assess log likelihood plots within a 3 dimensional plane is to couple 2 parameters at a time against the log likelihood value. There are 78 unique combinations of a set of two elements of θ. Hence there are 78 three dimensional plots and 78 Contour plots for ESTAR-GARCH and LSTAR-GARCH to assess. Of all the plots generated, the parameter of interest in this analysis is the transition rate, γ_i .

The simulation process generates one thousand observations following an MRSTAR GARCH process initially using the parameters in table 3.1. The first five hundred of these observations generated are trimmed to allow for the settling of data to the specified process. The remaining five hundred observations are then used to generate the required plots.

$\phi_{1,0}$	$\phi_{2,0}$	$\phi_{3,0}$	$\phi_{1,1}$	$\phi_{2,1}$	$\phi_{3,1}$	γ_1	γ_2	c_1	c_2	ω	α	β
0.1	0.3	.04	.0.35	.0245	.0478	0.21	0.7	-0.2	0.2	0.00001	0.2	0.78

Table 3.1 Input parameters for generating the log likelihood plots

Scilab, the software used in this exercise. It can only minimize an objective function, therefore deriving the parameter estimates using the generalized equation in (3.1), $l_t(\theta)$ changes such that max $l_t(\theta)$ becomes min $g_t(\theta)$ where $g_t(\theta)=- l_t(\theta)$. This implies the optimum is inverted in the plot results.

Selected results of the experiment are as follows:

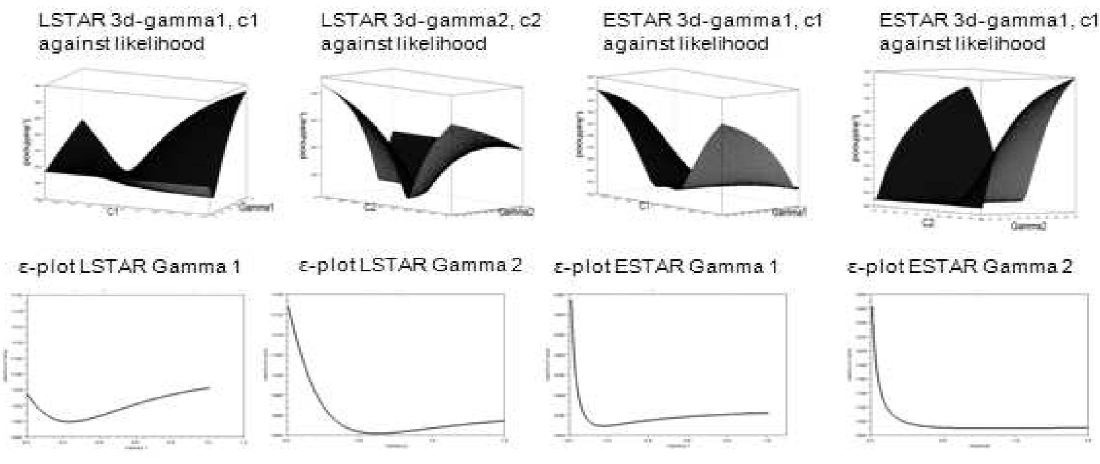


Figure 3.1: 3d and ε-neighbourhood plot results

As displayed by the 3d surface plot analysis, the log likelihood surfaces for both ESTAR and LSTAR plots display a hinge or "V" shape around the optimum region. This creates a ridge that is flat. When compared to other parameters of the MRSTAR GARCH model, this flatness is found to be consistent with all parameters around their optimum, but is far more pronounced with transition rate plots for both ESTAR and LSTAR.

Moreover, the flatness affects ESTAR more severely than LSTAR. This is made evident by the hinge associated with ESTAR transition rate plots. Around the optimum they appear more "U" shaped as opposed to "V" shaped. It was also found that for both ESTAR and LSTAR transition rate plots, a much more complex shape exists when compared to all other parameters of the MRSTAR-GARCH model. This complexity is more sever with LSTAR than it is with ESTAR.

Closer inspection revealed that the hinge associated with LSTAR transition rates is saw tooth shaped for the majority of LSTAR plots. This saw toothed characteristic is not clearly evident to the naked eye and was discovered when magnifying the plots around the optimum region. This warranted further inspection.

A contour plot analysis provided further insight into the saw toothed characteristic identified for LSTAR transition plots. The contour analysis confirms that this characteristic is associated with LSTAR only. The ramification of this is that numerical difficulties in optimizing the log likelihood function is subject to the specification of the transition functions.

The effect of the saw toothed shape on LSTAR transition rate plots is that it creates lumpiness around the optimum region. This creates many local optimums. Moreover, each local optimum also has an alarmingly large and flat neighbourhood. Overall, this suggests that the log-likelihood function for LSTAR-GARCH has a lumpy surface with flat peaks.

ε-neighbourhood plots can be used to further investigate the degree of flatness around the optimum. All the parameters in the MRSTAR-GARCH have an unmistakable optimum point with the exception of the transition rates[1]. In particular the second transition rate behaves like an asymptote making exception for the graphical scale.

In the strictest sense, unequivocal cause cannot be established. However, it is clear that the flatness of the log-likelihood function associated with the transition rates creates some (if not all) of the difficulties in estimating MRSTAR-GARCH models. Section 4 will provide a plausible solution to this issue.

4. RE-PARAMETERIZATION OF THE TRANSITION FUNCTION

This section provides a simple transform of the transition rates γ_i to alleviate the numerical difficulties in optimizing the log likelihood function. Let

$$\gamma_i = \frac{1}{\lambda_i^2}. \tag{4.1}$$

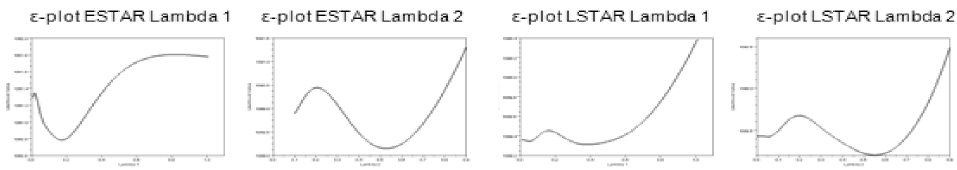
The transform has the following effect on the transition functions: for the logistic function:

$$G(s_t;\lambda;c) = \frac{1}{1+\exp(-\frac{1}{\lambda^2}(s_t - c))}. \tag{4.2}$$

Likewise for the exponential transition function:

$$G(s_t;\lambda;c) = 1 - \exp\{-\frac{1}{\lambda^2}(s_t - c)^2\} \tag{4.3}$$

This transforms can alleviate the flatness around the optimum in the log-likelihood function as shown in Figure 3.2:



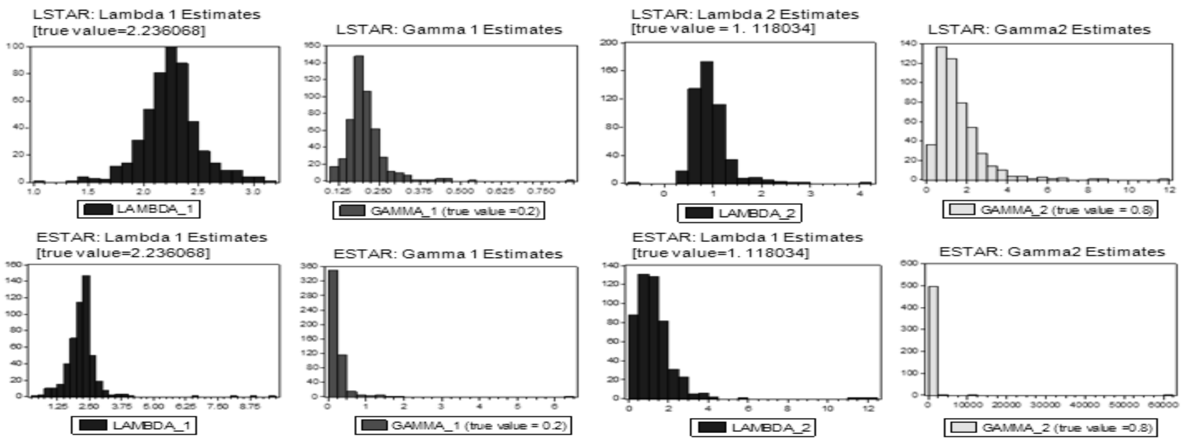
[1] All the plots have been omitted for brevity, but they are available upon request.

Figure 3.2: ε-neighbourhood plot results

The practical usefulness of this transform is investigated by Monte Carlo experiments. In addition to the original parameter vector for the GDP, a new parameter vector is also used in the Monte Carlo experiment to ensure the proposed method performs well for a range of parameters. The second parameter vector is defined as follows:

$\phi_{1,0}$	$\phi_{2,0}$	$\phi_{3,0}$	$\phi_{1,1}$	$\phi_{2,1}$	$\phi_{3,1}$	λ_1	λ_2	c_1	c_2	ω	α	β
0.01	0.03	0.04	0.0035	0.0245	0.0478	2.236	1.118	-1.5	1.5	0.00001	0.16	0.67

which implies the two transition rates are $\gamma_1 = 0.2$ and $\gamma_2 = 0.8$. The data is re-simulated generating 1000 observations, the first 500 are trimmed and the model is then estimated giving parameter estimates. This procedure is repeated 500 times creating 500 estimates of each parameter which is then presented in a histogram. The results from the Monte Carol experiments under both parameter vectors are very similar. Therefore, only the results under the second parameter vector are include for brevity. The results under the first parameter vector can be available upon request. The results are as follows:



Comparing the transition rates estimates, without the transform for LSTAR both are marred with outliers and the estimator for these parameters is strongly skewed. The implementation of the transform has a significant effect on mitigating outliers and consequently insinuates an asymptotically normally distributes estimator for this parameter.

The reduction in outliers is not as prominent for the transition rates in the ESTAR case but is still evident. This is far outweighed by the notion that without the transform, the optimization algorithm used the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm; commonly default for many applications, kept crashing and returning its initial values. The transform alleviates this serious issue and shows strong evidence towards the potential of an asymptotically normal estimator for these parameters as well. ESTAR however requires further refinement to reduce the skewness present in the lambda estimates.

5. CONCLUSION

Through a three tier analysis of the log likelihood function of an MRSATR-GARCH model with three regimes it was found that estimation difficulty is most likely caused by relative flatness for ESTAR and a lumpy likelihood surface with flat minima regions for LSTAR. The core culprit parameter was found to be the transition rates.

With an elementary transform on the transition rate and accentuates the minima region when assessed with an ε-neighbourhood plot, vast improvements in the estimation difficulties are achieved not only with transition rates, but with all the other parameters of the MRSTAR-GARCH model. Prior to the transform most algorithms crashed, with the transform this is averted. Although preliminary the transform with further refinement shows great promise. A beneficial by-product of the transform on the previously unknown statistical properties of transition rates is that they appear to have signs of being asymptotically normal, which is an important subject for future research.

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