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Scarf Instability and Production: A Simulation

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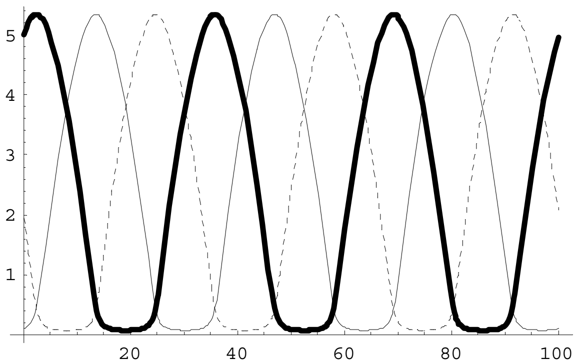
Abstract: The Walrasian stability argument in the 1950s appears to have been terminated by Scarf's instability paper [1960]. His analysis led to the creation of Applied General Equilibrium (AGE), in which Scarf Algorithm guarantees the computation of general equilibrium prices. Nowadays, AGE and Computable General Equilibrium are fundamental tools in evaluating economic policies or systems. In AGE, however, production and utility functions are specified by Cobb-Douglass type or CES type functions. If it is the case, the classical Newton method is known to suffice (in a lot of AGE models) in computing the general equilibrium prices. Thus, it may be worthwhile to examine why the instability emerges in the Scarf paper. The Scarf paper [1960] is formulated as a pure-trade model, in which there are three households and three goods. Each household has Harrod-type utility function, whose indifference curves are L-shaped. With the assumption of skewed preference on these households the model gave rise to the instability: cyclic trajectory paths as shown in the following figure.



This paper examines how the exclusion of production contributes to the instability through providing concrete feature for each good. In this paper, first, the original abstract Scarf model is reproduced. Next, it is assumed that the first household is the workers' class, and the second household is the entrepreneurs' class, while the third household is the capitalists' class. In order to provide the concrete feature for each good, it is assumed that the first good is leisure hours (or labor); the second good is consumption good, while the third good is capital goods, such as machines or buildings. In this paper, the production of consumption good is conducted by utilizing labor and capital. The first household, the workers' class, possesses solely the initial endowment of leisure hours, and supplies a part of leisure hours for production, while it demands commodity and leisure hours. The second household, the entrepreneurs' class, possesses solely the initial endowment of consumption goods, and supplies a part of consumption good, while it demands consumption goods and capital goods. Here, capital goods, such as computers, are also utilized as an instrument for hobby of households. The third household, the capitalists' class, possesses solely the initial endowment of capital goods, and supplies a part of capital goods, while it demands labor for domestic service and capital goods. The entrepreneurs' class conducts production, employing labor and capital. The total supply of consumption good is the sum of entrepreneurs' new product and a part of his initial endowment. In checking whether the instability prevails by assuming four different production functions, we examine the effect of production in stability analysis. They are Cobb-Douglas type; CES type with positive substitution parameter, CES type with negative substitution parameter, and Harrod type, whose isoquants are L-shaped. Utilizing simulation approach, we actually compute general equilibrium (GE) prices and the eigenvalues of Jacobian matrix at GE. For all these types, the real parts of them are negative; Walrasian tatonnement process is locally stable for all these cases. In other words, the exclusion of production in Scarf [1960] is shown to be indispensable for the instability.

Keywords: *General Equilibrium, Walrasisn adjustment, Scarf instability*

1. INTRODUCTION

The general equilibrium (GE) analysis was initiated in the 1870s by Leon Walras. Combined with the marginal revolution in those days, GE became one of the fundamental tools in economics. Famous IS-LM analysis in macroeconomics is a special case of GE. Simply counting the number of equations and variables, Walras himself, however, was satisfied with the existence of the GE price system. It was in 1954, that Arrow and Debreu proved rigorously the existence, utilizing Topology. There remained the problem of the computation of GE price system. In this problem, Walras himself was also satisfied with the stability "argument", saying that when market excess demand (supply) for each commodity raises (reduces) its price this price mechanism can compute the GE prices. In 1959, Arrow, Block and Hurwicz proved that when the market excess demand functions satisfy the gross substitutability the Walrasian tatonnement is globally stable: starting from arbitrary initial point the price trajectory path on the Walrasian tatonnement differential equation converges to the GE prices.

The stability analysis in the 1950s appears to be terminated by Scarf's instability paper [1960]. The model of Scarf [1960] is formulated by a pure-trade model, in which there are three households and three goods. Each household has Harrod-type utility function, whose indifference curves are L-shaped. With the assumption of skewed preference on these households the model gave rise to the instability. The present paper examines how the exclusion of production contributes to the Scarf instability through providing concrete feature for each good. In this paper, first, the original abstract Scarf model is reproduced. Next, it is assumed that the first household is the workers' class, and the second household is the capitalists' class, while the third household is the entrepreneurs' class. The entrepreneur's class conducts production, employing labor and capital. Checking whether the instability prevails by assuming different production functions, we examine the effect of production in stability analysis. We start with the reproduction of original Scarf model.

2. ORIGINAL SCARF MODEL

In a famous paper [1960], Scarf constructed a few instability models. In this section one of them is reproduced. In this model, there are three households and three goods. Each household has Harrod-type utility function, whose indifference curves are L-shaped. Suppose that the ith household has one unit of the ith good, whose utility function, $u_i[x_1, x_2, x_3]$, is assumed as in what follows where x_i is the quantity of the ith good (i=1,2,3).

$$\begin{aligned} u_1[x_1, x_2, x_3] &= \min\{x_1, x_2\} \\ u_2[x_1, x_2, x_3] &= \min\{x_2, x_3\} \\ u_3[x_1, x_2, x_3] &= \min\{x_3, x_1\} \end{aligned} \qquad (1\text{-}1)$$

The ith household maximizes utility subject to income constraint, with income, m_i:

$$\max u_i[x_1, x_2, x_3] \text{ s.t. } p_1x_1+ p_2x_2+ p_3x_3= m_i \qquad (2)$$

where p_i is the price of the ith good (i=1,2,3). The demand function of ith household for the jth good, $x_{ij}[p_1, p_2, p_3, m_i]$, is given by the following:

$$\begin{aligned} &x_{11}[p_1, p_2, p_3, m_1]= m_1/(p_1+p_2), x_{12}[p_1, p_2, p_3, m_1]= m_1/(p_1+p_2), x_{13}[p_1, p_2, p_3, m_1]=0, \\ &x_{21}[p_1, p_2, p_3, m_2]=0, x_{22}[p_1, p_2, p_3, m_2]= m_2/(p_2+p_3), x_{23}[p_1, p_2, p_3, m_2]= m_2/(p_2+p_3), \qquad (3) \\ &x_{31}[p_1, p_2, p_3, m_3]= m_3/(p_3+p_1), x_{32}[p_1, p_2, p_3, m_3]=0, x_{33}[p_1, p_2, p_3, m_3]= m_3/(p_3+p_1) \end{aligned}$$

In (3), for individual demand function each good is *gross complement* with each other: $\partial x_{ij}/\partial p_k \leq 0$ ($j \neq k$). It is assumed that the ith household has one unit of the ith good, so that

$$m_i = p_i \qquad (i=1,2,3). \qquad (4)$$

The general equilibrium is the situation in which market excess demand equals to zero for each market for good, where the market excess demand function for the jth good is assumed by

$$z_j[p_1, p_2, p_3]=\Sigma x_{ij}[p_1, p_2, p_3, p_i] -1. \quad (j=1,2,3) \qquad (5)$$

where the summation is over i. Note that the market excess demand functions are not necessarily *gross complement.* Indeed, $\partial z_1/\partial p_3$=2/441 (*gross substitute*), while $\partial z_1/\partial p_2$=–2/25 (*gross complement*)when p_1=1/2, p_2=2, p_3=10. The general equilibrium prices are computed as

$$p_1 = p_2 = p_3 = \alpha \qquad (6\text{-}1)$$

where α is an arbitrary positive number. At general equilibrium, the eigenvalues of Jacobian matrix, $\{\partial z_i/\partial p_j\}_{i,j=1,2,3}$, are $\{3^{1/2}/4\boldsymbol{i}, -3^{1/2}/4\boldsymbol{i}, 0\}$, which implies the local instability, since all the real parts of them are zero.

The *non-normalized* Walrasian tatonnement process is formulated by the following differential equations.

$$\mathrm{d}\, p_i[\tau]/\mathrm{d}\tau = z_j[p_1[\tau], p_2[\tau], p_3[\tau]] \quad (i=1,2,3) \qquad (7\text{-}1)$$

The trajectory path of $p_1[\tau]$ on (7-1), starting from $p_1[0]$=1/10, $p_2[0]$=2, $p_3[0]$=5, is drawn by the solid curve in Figure 1. The one of $p_2[\tau]$ is drawn by the dashed curve, while the one of $p_3[\tau]$ is drawn by the thick curve in Figure 1.

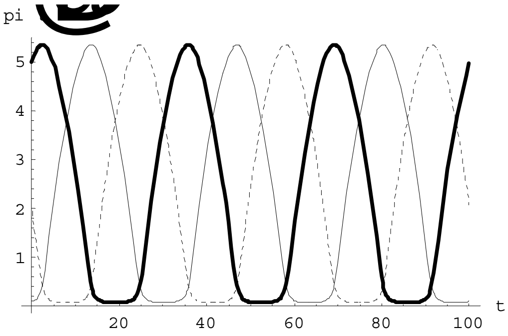


Figure1. Non-normalized tatonnement.

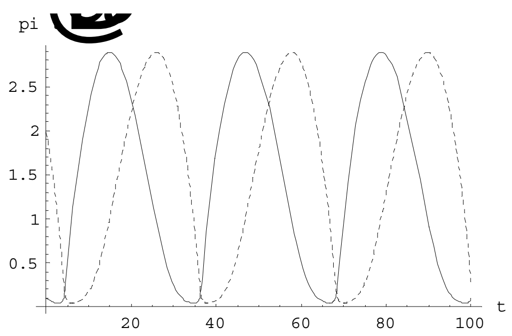


Figure2. Normalized tatonnement.

When the third good is selected as the *numeraire*, the *normalized* Walrasian tatonnement process is formulated by the following differential equations.

$$\mathrm{d}\, p_i[\tau]/\mathrm{d}\tau = z_j[p_1[\tau], p_2[\tau], 1] \quad (i=1,2) \qquad (7\text{-}2)$$

The trajectory path of $p_1[\tau]$ on (7-2), starting from $p_1[0]$=1/10, $p_2[0]$=2, p_3=1, is drawn by the solid curve, while the one of $p_2[\tau]$ is drawn by the dashed curve, in Figure 2.

3. INCORPORATION OF PRODUCTION: COBB-DOUGLAS TYPE CASE

In this section, production is incorporated into the original Scarf model. It is assumed that the first household is the workers' class, and the second household is the entrepreneurs' class, while the third household is the capitalists' class. In order to provide the concrete feature for each good, it is assumed that the first good is leisure hours (or labor), the second good is a consumption good, while the third good is a capital good, such as computers or machines in general or buildings. In this paper, the production of consumption good is conducted by utilizing labor and capital. Production function, x_2=f[x_1, x_3], is assumed to be of Cobb-Douglas type.

$$x_2 = \mathrm{f}[x_1, x_3] = x_1^{1/3} x_3^{1/3} \qquad (8\text{-}1)$$

This function is under decreasing returns to scale, so that the profit accrues to the entrepreneurs. By the profit maximization of the entrepreneurs, the demand for labor, x_1^D, the demand for capital, x_3^D, the supply of consumption good, x_2^S, and profit, π, are derived.
It is assumed that the first household is the workers' class with 1 unit of initial endowment of leisure hours. The first household maximizes utility subject to income constraint, with income, m_1:

$$\max u_1[x_1, x_2, x_3] \;\; \text{s.t.} \;\; p_1x_1 + p_2x_2 + p_3x_3 = m_1 = p_1 \qquad (2\text{-}1)$$

where p_1 is wage rate, and 1 is the initial endowment of leisure. From this maximization, demand function for leisure, x_{11}^D, demand function for consumption good, x_{12}^D, and supply function of labor, $x_{11}^S = 1 - x_{11}^D$, are computed. It is assumed that the second household with initial endowment of 1 unit of consumption good is the entrepreneurs' class, who owns the firm. The second household maximizes utility subject to income constraint, with income, m_2:

$$\max u_2[x_1, x_2, x_3] \text{ s.t. } p_1x_1+ p_2x_2+ p_3x_3= m_2= p_2+\pi \qquad (2\text{-}2)$$

where π is the profit distribution from the firm, and the initial endowment of consumption good is 1. From this maximization, demand function for capital good, x_{23}^D, demand function for consumption good, x_{22}^D, and supply function of consumption good, $x_{22}^S=1-x_{22}^D$, are computed. It is assumed that the third household is the capitalists' class with 1 unit of initial endowment of capital goods. The third household maximizes utility subject to income constraint, with income, m_3:

$$\max u_3[x_1, x_2, x_3] \text{ s.t. } p_1x_1+ p_2x_2+ p_3x_3= m_3= p_3 \qquad (2\text{-}3)$$

where p_3 is rental price of capital goods, and the initial endowment of capital goods is 1. From this maximization, demand function for domestic labor, x_{31}^D, demand function for capital good, x_{33}^D, and supply function of capital good, $x_{33}^S=1-x_{33}^D$, are computed. It is assumed that the capitalist can live without consumption good.

In order to compute the general equilibrium prices, excess demand functions are defined. The first good's excess demand function, $z_1[p_1, p_2, p_3]$, is defined as $z_1=x_{31}^D+x_1^D-x_{11}^S$, the second good's excess demand function, $z_2[p_1, p_2, p_3]$, is defined as $z_2= x_{12}^D-x_{22}^S-x_2^S$, while the third good's excess demand function, $z_3[p_1, p_2, p_3]$, is defined as $z_3= x_3^D+x_{23}^D-x_{33}^S$. The general equilibrium prices are computed as

$$p_1=1.38896, p_2=1.1144, p_3=1 \qquad (6\text{-}2)$$

The eigenvalues of the Jacobian matrix, $\{\partial z_i/ \partial p_j\}_{i,j=1,2}$, at (6.2) are $\{-0.0865861+0.181521\, \boldsymbol{i}, -0.0865861-0.181521\, \boldsymbol{i}\}$, so that the *normalized* Walrasian *tatonnement* process, formulated in (7-2), is locally stable.

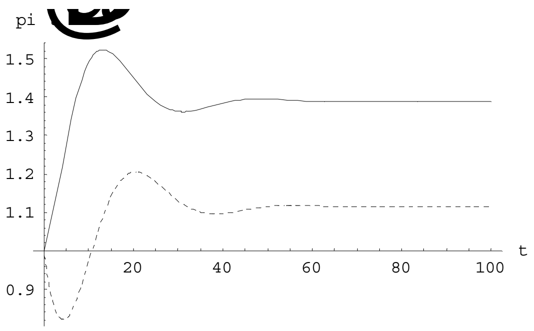


Figure3. The trajectory path on (7.2) when $p_1[0]=p_2[0]=1$.

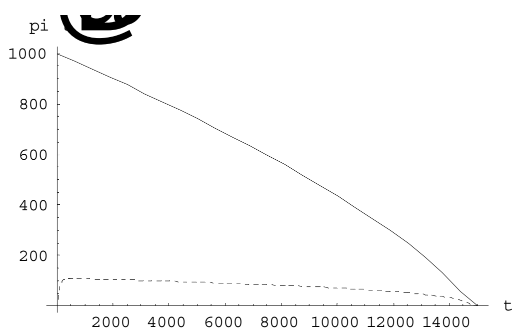


Figure4. The trajectory path on (7.2) when $p_1[0]=1000$, $p_2[0]=1$

As shown in Figure 3 the trajectory path shows the convergence to (6-2) when the initial point is near (6-2): when $p_1[0]=p_2[0]=1$.Even when the initial point is far away from (6.2), the trajectory path may be convergent as shown in Figure 4: when $p_1[0]=1000, p_2[0]=1/1000$.

When the first good is selected as the *numeraire*, the *normalized* Walrasian *tatonnement* process is formulated by the following differential equations.

$$d\, p_i [\tau]/d\tau= z_j[1, p_2[\tau], p_3[\tau]] \quad (i=2,3) \qquad (7\text{-}3)$$

The general equilibrium prices are computed as

$$p_1=1, p_2=0.719961, p_3=0.802322 \qquad (6\text{-}3)$$

The eigenvalues of the Jacobian matrix, $\{\partial z_i/ \partial p_j\}_{i,j=2,3}$, at (6.3) are $\{-0.168712+0.349395\, \boldsymbol{i}, -0.168712-0.349395\boldsymbol{i}\}$, so that the *normalized* Walrasian *tatonnement* process, formulated in (7-3), is locally stable. On (7.3), however, when the initial point is far away from (6.3), for example $p_2[0]=100$, $p_3[0]=1$, the trajectory path is *not* convergent: *i.e.* $p_2[53]=0$.

4. INCORPORATION OF PRODUCTION: CES TYPE CASE I

In this section, production is incorporated into the original Scarf model. Production function, x_2=f[x_1, x_3], is assumed to be of Constant Elasticity Substitution (CES) type. CES production function is defined by the following.

$$x_2=f[x_1, x_3]=((c_1\, x_1)^{-t}+(c_3\, x_3)^{-t})^{-n/t} \qquad (8\text{-}2)$$

Furthermore, it is assumed that

$$c_1=1,\ c_3=1,\ n=1/2,\ t=-1/2 \qquad (9\text{-}1)$$

The assumption of n<1 implies that the production function is under decreasing returns to scale, so that the profit accrues to the entrepreneurs. As in the previous section, the demand for labor, x_1^D, the demand for capital, x_3^D, the supply of consumption good, x_2^S, and profit, π, are computed. In exactly the same way as in the previous section, the general equilibrium prices are computed as

$$p_1= 10.7557,\ p_2= 1.0295,\ p_3=1 \qquad (6\text{-}3)$$

The eigenvalues of the Jacobian matrix, $\{\partial z_i/ \partial p_j\}_{i,j=1,2}$, at (6.3) are {– 0.169613, – 0.00487542}, so that the *normalized* Walrasian *tatonnement* process, formulated in (7-2), is locally stable.

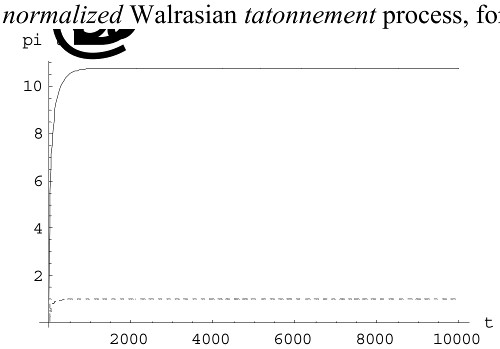


Figure 5. The trajectory path on (7.2) when $p_1[0]=p_2[0]=1$.

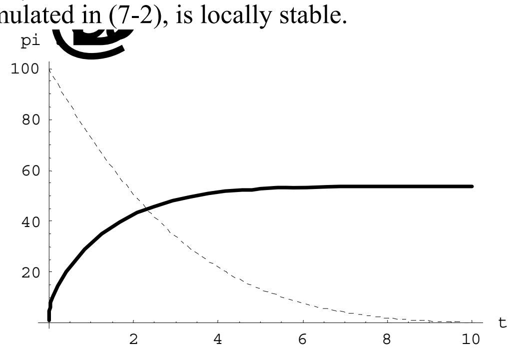


Figure 6. The trajectory path on (7.3) when $p_2[0]=100$, $p_3[0]=1$.

As shown in Figure 5 the trajectory path shows the convergence to (6-3) when the initial point is near (6-3): when $p_1[0]=p_2[0]=1$.Even when the initial point is far away from (6.3), the trajectory path may be convergent: when $p_1[0]=100$, $p_2[0]=1/100$. When the first good is selected as the *numeraire*, the *normalized* Walrasian *tatonnement* process is formulated by (7-3). The general equilibrium prices are computed as

$$p_1=1,\ p_2=0.0957171,\ p_3=0.0929741 \qquad (6\text{-}4)$$

The eigenvalues of the Jacobian matrix, $\{\partial z_i/ \partial p_j\}_{i,j=2,3}$, at (6.4) are {–10.5352, –1.05046}, so that the *normalized* Walrasian *tatonnement* process, formulated in (7-3), is locally stable. On (7.3), however, when the initial point is far away from (6.4), for example $p_2[0]=100$, $p_3[0]=1$, the trajectory path is *not* convergent as shown in Figure 6: *i.e.* $p_2[17]=0$. In figure 6, the trajectory path of p_2 is depicted by the dashed curve, while the one of p_3 is depicted by the thick curve.

5. INCORPORATION OF PRODUCTION: CES TYPE CASE II

In this section, production is incorporated into the original Scarf model. Production function, x_2=f[x_1, x_3], is assumed to be of CES type, defined by (8-2). It is assumed that

$$c_1=1,\ c_3=1,\ n=1/2,\ t=1/2 \qquad (9\text{-}2)$$

The assumption of n<1 implies that the production function is under decreasing returns to scale, so that the profit accrues to the entrepreneurs. In exactly the same way as in the previous section, the general equilibrium prices are computed as

p_1= 1.1965, p_2= 1.05671, p_3=1 (6-5)

The eigenvalues of the Jacobian matrix, $\{\partial z_i/\partial p_j\}_{i,j=1,2}$, at (6.5) are {– 0.024592+0.218165***i***, – 0.024592– 0.218165 ***i***}, so that the *normalized* Walrasian *tatonnement* process, formulated in (7-2), is locally stable.

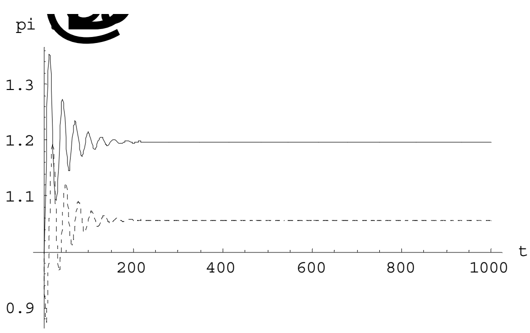


Figure 7. The trajectory path on (7.2) when $p_1[0]=p_2[0]=1$.

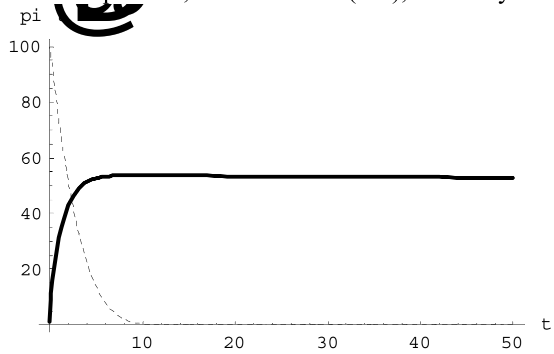


Figure 8. The trajectory path on (7.3) when $p_2[0]=100$, $p_3[0]=1$.

As shown in Figure 7 the trajectory path shows the convergence to (6-5) when the initial point is near (6-5): when $p_1[0]=p_2[0]=1$.Even when the initial point is far away from (6-5), the trajectory path may be convergent: when $p_1[0]=100$, $p_2[0]=1/100$. When the first good is selected as the *numeraire*, the *normalized* Walrasian *tatonnement* process is formulated by (7-3). The general equilibrium prices are computed as

p_1=1, p_2= 0.883169, p_3= 0.835769 (6-6)

The eigenvalues of the Jacobian matrix, $\{\partial z_i/\partial p_j\}_{i,j=2,3}$, at (6.4) are {–0.0428926+0.311367***i***, – 0.0428926– 0.311367***i***}, so that the *normalized* Walrasian *tatonnement* process, formulated in (7-3), is locally stable. On (7.3), however, when the initial point is far away from (6.6), for example $p_2[0]=100$, $p_3[0]=1$, the trajectory path is *not* convergent as shown in Figure 8: *i.e.* $p_2[17]=0$. In figure 8, the trajectory path of p_2 is depicted by the dashed curve, while the one of p_3 is depicted by the thick curve.

6. INCORPORATION OF PRODUCTION: HARROD TYPE CASE

In this section, production is incorporated into the original Scarf model where production function, $x_2=f[x_1, x_3]$, is assumed to be of Harrod type. Harrod type production function is defined by the following.

$$x_2=f[x_1, x_3]=\min\{x_1^{\beta}, x_3^{\beta}\} \quad (8\text{-}3)$$

Harrod type utility function assumed in (1-1) in Section I is a special case of (8-3). In this section, in order to guarantee positive profit, we assume that

$$\beta=1/2. \quad (9\text{-}3)$$

Isoquants for (8-3) when (9-3) is assumed are the same L-shaped as in the indifference curves for (1-1). In exactly the same way as in the previous section, the general equilibrium prices are computed as

p_1= 1.86229, p_2= 1.21166, p_3=1 (6-7)

The eigenvalues of the Jacobian matrix, $\{\partial z_i/\partial p_j\}_{i,j=1,2}$, at (6.7) are {– 0.0614896+0.145188***i***, – 0.0614896– 0.145188 ***i***}, so that the *normalized* Walrasian tatonnement process, formulated in (7-2), is locally stable. As shown in Figure 9 the trajectory path shows the convergence to (6-7) when the initial point is near (6-7): when $p_1[0]=p_2[0]=1$.Even when the initial point is far away from (6-7), the trajectory path may be convergent: when $p_1[0]=100$, $p_2[0]=1/100$. When the first good is selected as the *numeraire*, the *normalized* Walrasian *tatonnement* process is formulated by (7-3). The general equilibrium prices are computed as

p_1=1, p_2= 0.650629, p_3= 0.536974 (6-8)

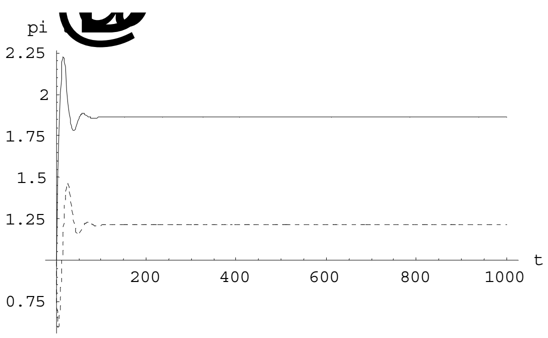


Figure 9. The trajectory path on (7.2) when $p_1[0]=p_2[0]=1$.

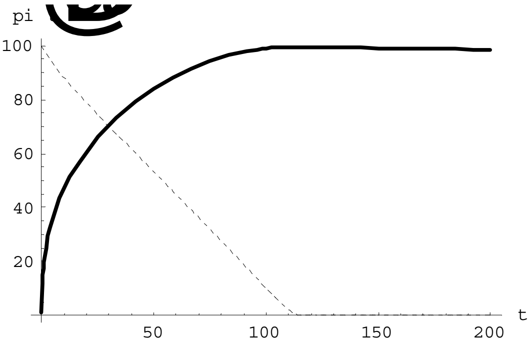


Figure 10. The trajectory path on (7.3) when $p_2[0]=100$, $p_3[0]=1$.

The eigenvalues of the Jacobian matrix, $\{\partial z_i / \partial p_j\}_{i,j=2,3}$, at (6.8) are {–0.182522+0.515465***i***, – 0.182522–0.515465 ***i***}, so that the *normalized* Walrasian *tatonnement* process, formulated in (7-3), is locally stable. On (7.3), however, when the initial point is far away from (6.8), for example $p_2[0]=100$, $p_3[0]=1$, the trajectory path is *not* convergent as shown in Figure 10: *i.e.* $p_2[125]=0$..

6. CONCLUSIONS

The aim of this paper is to examine how crucial the assumption of excluded production in Scarf [1960] is. Scarf [1960] constructed examples of unstable Walrasian *tatonnement* process. One of them assumed the exclusion of production. It is formulated by a pure-trade model, in which there are three households and three goods. Each household has Harrod-type utility function, whose indifference curves are L-shaped. With the assumption of skewed preference on these households the model gave rise to the instability. The present paper incorporated production into the Scarf pure-trade model. The entrepreneurs'class is assumed to produce consumption good utilizing labor and capital. The entrepreneurs, workers, and capitalists are the three households corresponding to Scarf model. They have the same Harrod type utility functions as in Scarf model. This paper examined the stability of Walrasian *tatonnement* process by simulation approach. Specifying parameters numerically for four production functions, we computed the eigenvalues of the Jacobian matrix of excess demand functions at general equilibrium, after actually computing general equilibrium prices. The four production functions consist of Cobb-Douglas type, CES type with negative substitution parameter, CES type with positive substitution parameter, and Harrod type. For each production function all the real parts of eigenvalues of the Jacobian matrix of excess demand functions at general equilibrium are negative. Thus, for all the production functions the Walrasian *tatonnement* process is locally stable. It was shown that the assumption of excluded production in Scarf [1960] is indispensable for the instability.

REFERENCES

Arrow, K.J., Block, H.D., and Hurwicz, L. [1959], "On the Stability of the Competitive Equilibrium II", *Econometrica* 26.pp.89-109.

Arrow, K.J., and Debreu, G. [1954], "Existence of an Equilibrium for a Competitive Economy", *Econometrica* 22. pp.265-290.

Fukiharu, T. and K. Imamura, [2008], "Scarf Instability and Production: A Simulation", http://home.hiroshima-u.ac.jp/fukito/index.htm.

Scarf, H. [1960], "Some Examples of Global Instability of the Competitive Equilibrium," *International Economic Review* 1, pp.157-172