18th World IMACS / MODSIM Congress, Cairns, Australia 13-17 July 2009
http://mssanz.org.au/modsim09

The Robustness of a Nash Equilibrium Simulation Model

Eitaro Aiyoshi[1], Atsushi Maki[2] and Takashi Okamoto[3]

[1]*Faculty of Science and Technology, Keio University, Kanagawa, Japan*
[2]*Faculty of Business and Commerce, Keio University, Tokyo, Japan*
[3]*Graduate School of Engineering, Chiba University, Chiba, Japan*
Email: maki@fbc.keio.ac.jp

Abstract: Aiyoshi and Maki (2009) proposed a Nash equilibrium model applying continuous time replicator dynamics to the analysis of oligopoly markets. This paper considered a game problem under the simultaneous constraints of the allocation of product and market shares. The model assumes that a Nash equilibrium solution can be applied and derived the gradient system dynamics that can attain the Nash equilibrium solution without violating the simplex constraints. Models assumed that a minimum of three firms exist within a market, and that these firms behave to maximize their profits, defined as the difference between sales and cost functions with conjectural variations.

Before conducting an empirical analysis based on observations of oligopoly markets in the real world, we have to assess the robustness of the Nash equilibrium model by changing profit and cost function parameters, as well as the initial production values and market shares of the firms. This is necessary in order to assess how well observations in the real world match those forecasts by the model. When the model is fragile, no policy implications could be extracted from the model.

The paper considers differences of the converged values in the number of firms included in the model, in the numbers of the commodities included in the model, in the specification of firms' profit and cost functions, and in the initial values for the level of production and market share. The approach facilitates understanding of the robustness of attaining equilibrium in an oligopoly market.

Keywords: *Nash equilibrium, replicator dynamics, oligopoly*

1. INTRODUCTION

Oligopoly markets prevail in developed countries in both the industrial and service sectors. Dixit (1988) analyze the U.S. automobile industry. Klepper (1990) analyze the airline industry. In this study, we propose a simple Nash equilibrium model and use a simulation method to derive an optimal solution for production decisions by rival firms in oligopoly markets. Aiyoshi and Maki (2009) proposed a Nash equilibrium model that applies continuous time replicator dynamics to the analysis of oligopoly markets. In this paper, we considered a game problem under the simultaneous constraints of allocation of product and market share.

Before conducting empirical analysis using observation on oligopoly markets in the real world, we have to check the robustness of the Nash equilibrium model by changing the parameters of firms' profit and cost functions as well as the initial values of the amount of production and firms' market share. This process is necessary to conduct forecasting and simulation using real-world observations after estimating the model. We sometimes obtain unrealistic solutions as a result of the fragility of the model. In these cases, no policy implications can be drawn.

The paper is organized as follows. Section 2 provides a general explanation of the double constraint interference model, which concentrates on the double allocation problem of production capacity and market share. Section 3 introduces a normalized Nash equilibrium solution for the profit maximization of players' functions modeled in the double constraint interference model. Section 4 describes the application of numerical methods to the Nash equilibrium solution. Section 5 proposes a simulation model and reports the results. Section 6 presents the conclusions.

2. NONCOOPERATIVE NASH EQUILIBRIUM MODEL AND RESOURCE ALLOCATION

Consider a continuous game problem with P players and N strategy variables, governed by duplicate simplex constraints. The pth player's strategy variables are $\boldsymbol{x}^p = (x_1^p, \cdots, x_N^p) \in R^N$ and $i = 1, \cdots, N$, $p = 1, \cdots, P$. The variable matrix X that contains all variables is

$$X = (\boldsymbol{x}^1, \cdots, \boldsymbol{x}^P) = \begin{pmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} x_1^1 & \cdots & x_1^P \\ \vdots & \ddots & \vdots \\ x_N^1 & \cdots & x_N^P \end{pmatrix} \tag{1}$$

(where $\boldsymbol{x}^p$, $p = 1, \cdots, P$ are column vectors; and $\boldsymbol{x}_i$, $i = 1, \cdots, N$ are row vectors). Let the pth player's profit function be $E^p(X)$. An unconstrained game problem is formulated as

$$\max_{\boldsymbol{x}^p} \; E^p(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^p, \cdots \boldsymbol{x}^P), \quad p = 1, \cdots, P, \tag{2}$$

where $\boldsymbol{x}^p$ is the pth player's only known variable and the other players' variables ($\boldsymbol{x}^1, \cdots, \boldsymbol{x}^{p-1}, \boldsymbol{x}^{p+1} \cdots \boldsymbol{x}^P$)are unknown parameters. Consider the game problem constrained by the simultaneous allocation of products and market share as

$$\max_{\boldsymbol{x}^p} \; E^p(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^p, \cdots \boldsymbol{x}^P) \tag{3a}$$

$$\text{subj. to} \sum_{i=1}^{N} x_i^p = a^p, \quad p = 1, \cdots, P, \quad \sum_{p=1}^{P} x_i^p = b_i, \quad i = 1, \cdots, N, \tag{3b}$$

$$x_i^p \geq 0, \; i = 1, \cdots, N, \; p = 1, \cdots, P \tag{3c}$$

where, by definition,

$$\sum_{p=1}^{P} a^p = \sum_{i=1}^{N} b_i. \tag{4}$$

As an example, let P firms produce N types of products in a market; p represents the number of firms, and i represents a product type. The problem, represented by Eq. (3), is called a game problem with double allocation constraints, in which the allocation of production ability and market share is considered simultaneously. The properties of Nash equilibrium solutions, which are assumed to be rational solutions for non-cooperative game problems, differ depending on their equality constraints.

3. THE NORMALIZED NASH EQUILIBRIUM SOLUTION FOR THE CONSTRAINT INTERFERENCE PROBLEM

The Nash equilibrium solution $\bar{X}$ for the interference-type problem is denoted by Eq. (3), with double simplex constraints. In this case, stationary conditions for each player do not exist, unlike in the constraint independent–type problem. A maximization problem for the *p*th player, under the condition that other players' strategies $\bar{\boldsymbol{x}}^1,\cdots,\bar{\boldsymbol{x}}^{p-1},\bar{\boldsymbol{x}}^{p+1},\cdots,\bar{\boldsymbol{x}}^P$ are given, is expressed as

$$\max_{x^p} E^p(\bar{\boldsymbol{x}}^1,\cdots,\bar{\boldsymbol{x}}^{p-1},\boldsymbol{x}^p,\bar{\boldsymbol{x}}^{p+1},\cdots\bar{\boldsymbol{x}}^P) \tag{5a}$$

$$\text{subj. to} \sum_{i=1}^{N} x_i^p = a^p,\ \ p=1,\cdots,P,\ \ x_i^p = b_i - \sum_{\substack{q=1\\ q\neq p}}^{P} \bar{x}_i^q,\ \ i=1,\cdots,N, \tag{5b}$$

$$x_i^p \geq 0,\ i=1,\cdots,N,\ p=1,\cdots,P, \tag{5c}$$

$$\sum_{p=1}^{P} a^p = \sum_{i=1}^{N} b_i. \tag{6}$$

In Eq. (5), the *p*th player's strategy $\boldsymbol{x}^p$ that satisfies Eq. (5b) is determined uniquely, because the other players' strategy variables are already given. There is no freedom to maximize the function E^p. Therefore, to define the nontrivial Nash equilibrium solutions for constraint interference-type problems, we introduce the normalized Nash equilibrium solution proposed by Rosen (1965), which has the flexibility of maximization and for which stationary conditions can be derived. The normalized Nash equilibrium solution $\bar{X}$ for the constraint interference–type problem is defined by relaxing interference among players in the constraint Eq. (5b) and considering the problem of maximizing the sum of all players' profit functions:

$$\max_{X} \sum_{p=1}^{P} E^p(\bar{\boldsymbol{x}}^1,\cdots,\bar{\boldsymbol{x}}^{p-1},\boldsymbol{x}^p,\bar{\boldsymbol{x}}^{p+1},\cdots\bar{\boldsymbol{x}}^P) \tag{7a}$$

$$\text{subj. to} \ \sum_{i=1}^{N} x_i^p = a^p,\ \ p=1,\cdots,P, \qquad \sum_{p=1}^{P} x_i^p = b_i,\ \ i=1,\cdots,N, \tag{7b}$$

$$x_i^p \geq 0,\ i=1,\cdots,N,\ p=1,\cdots,P \tag{7c}$$

$$\sum_{p=1}^{P} a^p = \sum_{i=1}^{N} b_i. \tag{8}$$

Notice that Eq. (7a) is dependent on unknown parameters $\bar{X}=(\bar{\boldsymbol{x}}^1,\cdots,\bar{\boldsymbol{x}}^P)$, and the variable $X=(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^P)$ is maximized simultaneously. Let the function $F: R^{N\times P}\times R^{N\times P}\to R^1$ be defined by

$$F(X;\bar{X}) = \sum_{p=1}^{P} E^p(\bar{\boldsymbol{x}}^1,\cdots,\bar{\boldsymbol{x}}^{p-1},\boldsymbol{x}^p,\bar{\boldsymbol{x}}^{p+1},\cdots\bar{\boldsymbol{x}}^P). \tag{9}$$

We define $\bar{X}$ as the local normalized Nash equilibrium solution for the constraint interference–type problem in Eq. (3), when the neighborhood $B(\bar{X})\subseteq R^{N\times P}$ of $\bar{X}$ exists such that the following inequality holds:

$$F(\bar{X};\bar{X}) \geq F(X;\bar{X}) \ \ \forall X\in B(\bar{X})\cap S, \tag{10}$$

where $S=\{X \mid X \text{ satisfies Eq.(7b)(7c)}\}$. Note that the normalized Nash equilibrium solution is not a solution for the simple maximization of the sum of all players' profit functions, F, but defines the maximum point, F, with respect to variable X in F, given the value of $\bar{X}$ in F as a parameter (that is, it is defined as a fixed point of the maximization operation).

4. SEARCH DYNAMICS OF THE NASH EQUILIBRIUM SOLUTION FOR THE CONSTRAINT INTERFERENCE –TYPE PROBLEM

We investigate the dynamic used to search for the normalized Nash equilibrium solution for the constraint interference–type problem expressed in Eq. (3), in which the double constraints of allocating production ability and market share are imposed simultaneously. In order to apply the results of the above constraint interference–type problem to the double constraint case directly, we transform the $N\times P$ matrix variable X into the $N\times P$ dimensional column vector variable as $\boldsymbol{X}=(\boldsymbol{x}^{1T},\cdots,\boldsymbol{x}^{PT})^T$, and reformulate the double constraints of Eq. (3b) as the linear equality constraint of the vector-matrix form $\boldsymbol{AX}=\boldsymbol{c}$ with a $(P+N)\times(N\times P)$ coefficient matrix $\boldsymbol{A}$, expressed as

$$\boldsymbol{X}=(\boldsymbol{x}^{1T},\cdots,\boldsymbol{x}^{PT})^T,\ A=\begin{bmatrix} 1 \cdots 1 & 0 \cdots 0 & & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 & & 0 \cdots 0 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \cdots & \vdots \ddots \vdots \\ 0 \cdots 0 & 0 \cdots 0 & & 1 \cdots 1 \\ 1 \cdots 0 & 1 \cdots 0 & & 1 \cdots 0 \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \cdots & \vdots \ddots \vdots \\ 0 \cdots 1 & 0 \cdots 1 & & 0 \cdots 1 \end{bmatrix},\ \boldsymbol{c}=(a^1,\cdots,a^P,b_1,\cdots,b_N)^T. \tag{11}$$

Here an arbitrary element of equality $\boldsymbol{AX}=\boldsymbol{c}$ must be satisfied, under the balancing conditions of Eq. (8), and then $\boldsymbol{A}=N+P-1$ ranked. Let $\bar{\boldsymbol{A}}$ be the $(P+N-1)\times(N\times P)$ matrix in which an arbitrary row of matrix $\boldsymbol{A}$ is deleted. We can propose a dynamic to search for the normalized Nash equilibrium solution of a game problem with a double interference constraint allocation–type problem as follows:

$$\frac{d\boldsymbol{X}(t)}{dt}=Q_{\bar{\boldsymbol{A}}}^{M}(\boldsymbol{X}(t))M^{-1}(\boldsymbol{X}(t))\nabla F(\boldsymbol{X}(t);\boldsymbol{X}(t)), \tag{12}$$

where

$$\frac{d\boldsymbol{X}(t)}{dt}=\begin{pmatrix} d\boldsymbol{x}^1(t)/dt \\ \vdots \\ d\boldsymbol{x}^P(t)/dt \end{pmatrix}, \tag{13a}$$

$$Q_{\bar{\boldsymbol{A}}}^{M}(\boldsymbol{X})=I-M^{-1}(\boldsymbol{X})\bar{\boldsymbol{A}}^T(\bar{\boldsymbol{A}}M^{-1}(\boldsymbol{X})\bar{\boldsymbol{A}}^T)^{-1}\bar{\boldsymbol{A}}, \tag{13b}$$

$$M^{-1}(\boldsymbol{X})=\mathrm{diag}(1/x_i^p)\quad (N\times P)\times(N\times P)\mathrm{matrix} \tag{13c}$$

$$\nabla F(\boldsymbol{X};\boldsymbol{X})=\begin{pmatrix} \nabla_{\boldsymbol{x}^1}E^1(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^P) \\ \vdots \\ \nabla_{\boldsymbol{x}^P}E^P(\boldsymbol{x}^1,\cdots,\boldsymbol{x}^P) \end{pmatrix}. \tag{13d}$$

Here the $(N\times P)\times(N\times P)$ variable metric projection matrix $Q_{\bar{\boldsymbol{A}}}^{M}(\boldsymbol{X})$ cannot be expressed by a simple formula, because the inverse $(\bar{\boldsymbol{A}}M^{-1}(\boldsymbol{X})\bar{\boldsymbol{A}}^T)^{-1}$ cannot be formulated explicitly.

5. SIMULATIONS OF THE NORMALIZED NASH EQUILIBRIUM SOLUTION FOR THE CONSTRAINTS OF DOUBLE RESOURCE ALLOCATION

5.1 The Three-Person, Three-Strategy Game (Benchmark)

Consider a three-person (P = 3) game with three products (N = 3). Even in the simplest model, there is no loss of generality from the model described in Section 2. As a concrete example, consider three automobile companies, each of which produces three types of automobiles: budget, midlevel, and luxury. The decision variables are $\boldsymbol{x}^1=(x_1^1,x_2^1,x_3^1)^T$, $\boldsymbol{x}^2=(x_1^2,x_2^2,x_3^2)^T$ and $\boldsymbol{x}^3=(x_1^3,x_2^3,x_3^3)^T$, where the subscripts indicate the product and the superscripts indicate the firm. The profit functions of each firm are

$$E^p(X)=-\sum_{i=1}^{N}f_i^p(x_i^p)-\sum_{i=1}^{N}\sum_{\substack{q=1\\q\neq p}}^{P}\theta_{pqi}x_i^p x_i^q, \tag{14}$$

where θ_{pqi} is the loss parameter suffered by the ith product when player p produces x_i^p and player q produces x_i^q. In the economics of firms, gain is the corporate profit and loss represents the various kinds of conjectural costs. The constraints are production capacity and market share as expressed by Eq. (3b), and $a^1=a^2=a^3=1$ and $b_1=b_2=b_3=1$ for simplicity. The gain for firm 1 from products 1, 2, and 3 is indicated, respectively, as

$$f_1^1(x_1^1)=-2(x_1^1-1.4)^2+3.2\,,\ f_2^1(x_2^1)=-1.9(x_2^1-1.3)^2+2.8\,,$$

$$f_3^1(x_3^1)=-1.8(x_3^1-1.2)^2+2.4 \tag{15}$$

The difference in the function, f_i^p is due to the production technology differences among products and firms. The profit function $E^1(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ for firm 1 is specified as

$$E^1(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3) = f_1^1(x_1^1) + f_2^1(x_2^1) + f_3^1(x_3^1)$$
$$-(\theta_{121}x_1^1x_1^2 + \theta_{122}x_2^1x_2^2 + \theta_{123}x_3^1x_3^2 + \theta_{131}x_1^1x_1^3 + \theta_{132}x_2^1x_2^3 + \theta_{133}x_3^1x_3^3),$$

where $\theta_{111}, \theta_{112}$, and θ_{113} are assumed to be zero. For firm 2, the gain functions for products 1, 2, and 3, respectively, are

$$f_1^2(x_1^2) = -2.1(x_1^2 - 1.5)^2 + 3.8 \quad f_2^2(x_2^2) = -1.9(x_2^2 - 1.4)^2 + 3.2$$
$$f_3^2(x_3^2) = -1.7(x_3^2 - 1.3)^2 + 2.6\,. \tag{16}$$

The profit function, $E^2(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$, for firm 2 is specified as

$$E^2(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3) = f_1^2(x_1^2) + f_2^2(x_2^2) + f_3^2(x_3^2)$$
$$-(\theta_{211}x_1^2x_1^1 + \theta_{212}x_2^2x_2^1 + \theta_{213}x_3^2x_3^1 + \theta_{231}x_1^2x_1^3 + \theta_{232}x_2^2x_2^3 + \theta_{233}x_3^2x_3^3),$$

where $\theta_{221}, \theta_{222}$, and θ_{223} are assumed to be zero. For firm 3, the gain functions for products 1, 2, and 3, respectively, are

$$f_1^2(x_1^2) = -2.2(x_1^2 - 1.6)^2 + 4.4 \quad f_2^2(x_2^2) = -1.9(x_2^2 - 1.5)^2 + 3.6$$
$$f_3^2(x_3^2) = -1.6(x_3^2 - 1.4)^2 + 2.8 \tag{17}$$

The profit function, $E^3(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$, for firm 3 is specified as

$$E^3(\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3) = f_1^3(x_1^3) + f_2^3(x_2^3) + f_3^3(x_3^3)$$
$$-(\theta_{311}x_1^3x_1^1 + \theta_{312}x_2^3x_2^1 + \theta_{313}x_3^3x_3^1 + \theta_{321}x_1^3x_1^2 + \theta_{322}x_2^3x_2^2 + \theta_{323}x_3^3x_3^2),$$

where $\theta_{331}, \theta_{332}$, and θ_{333} are assumed to be zero. To choose the values for the parameters, θ_{ijk} (i, j, k = 1, 2, 3) except $\theta_{111}, \theta_{112}, \theta_{113}, \theta_{221}, \theta_{222}, \theta_{223}$, $\theta_{331}, \theta_{332}$, and θ_{333}, we conducted many experiments before selecting two sets of parameters. The set indicated in the first simulation shows the internal solutions for firms and commodities. The set of parameters indicated in the second simulation shows that each firm specializes in the production of at least one commodity. This is a case for product differentiation within an oligopoly market. In the first simulation, we assigned the 18 values of θ_{pqi} as 2.0, resulting in the normalized Nash equilibrium solution for the decision variable $X = (\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$. Table 1 indicates the changes in the normalized Nash equilibrium value for $X = (\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ from the initial values to the converged values.

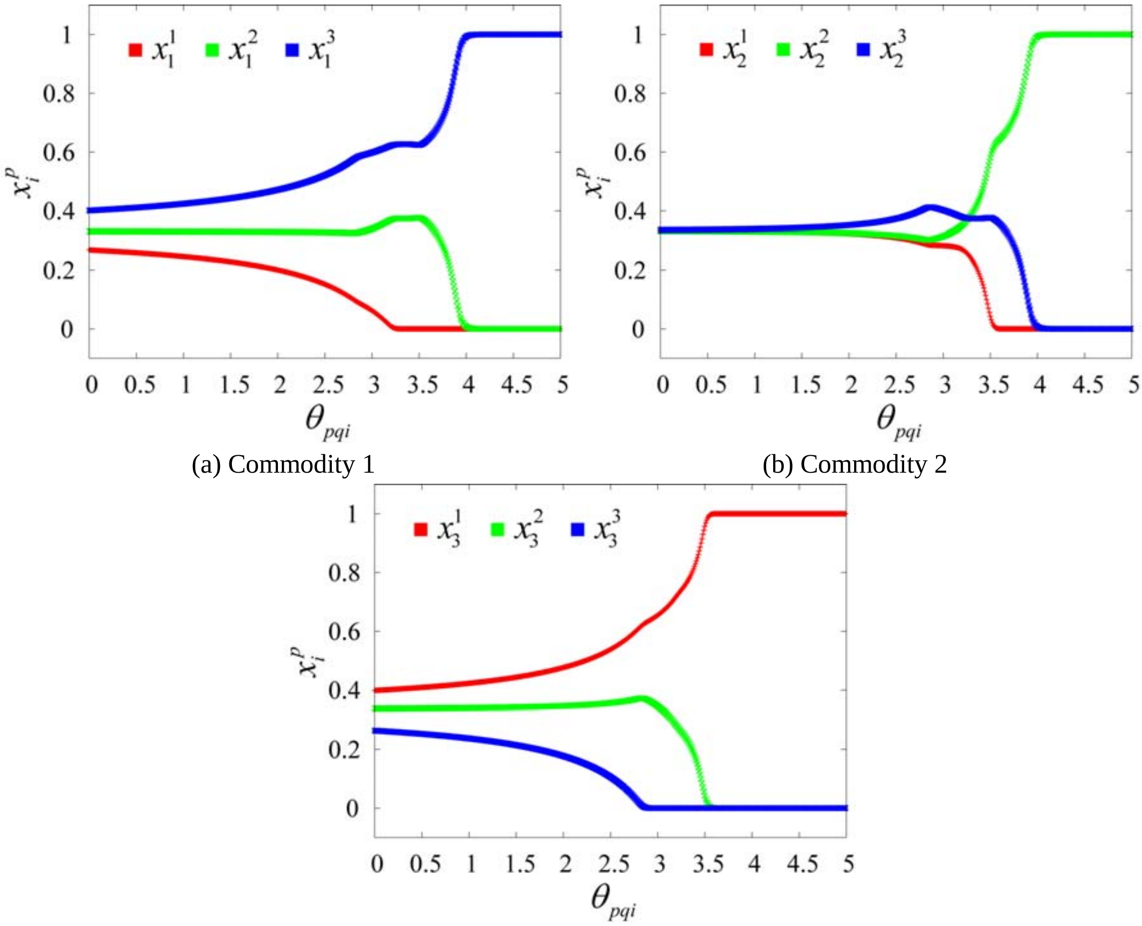
Table 1. Benchmark Values of *X*

Initial values of $X(0)$	Converged values
$X(0) = \begin{pmatrix} 0.33 & 0.33 & 0.34 \\ 0.33 & 0.34 & 0.33 \\ 0.34 & 0.33 & 0.33 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0.200 & 0.323 & 0.476 \\ 0.329 & 0.325 & 0.346 \\ 0.471 & 0.352 & 0.177 \end{pmatrix}$

5.2 Changes in Converged Values Caused by Changes in θ_{pqi}

In the second simulation, we change the conjectural variation of firm *p* against firm *q* from 0.0 to 5.0. Figure 1 indicate the normalized Nash equilibrium solution for $X = (\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$ by changing the parameters of θ_{pqi}. The horizontal axis indicates the values of conjecture from 0 to 5; the vertical axis indicates the market share for commodities 1–3. Figure 1(a) indicates changes in the market share of commodity 1 by the three firms. When the value of θ_{pqi} is relatively small, all firms produce commodity 1. When the conjecture exceeds the point of 4 on the horizontal axis, firm 3 (blue plots) produces only commodity 1, and firms 1 (red plots) and 2 (green plots) do not produce commodity 1 (Fig. 1(a)). The tendency is the same for commodity 2 and 3 (Figs. 1(b) and 1(c)). In the oligopoly market, conjectural variation plays an important role in determining the share of the products within and among firms. When the values of conjectural variation are small, each firm produces a full set of commodities. In contrast, when the values of the conjectural variation are large, product specialization takes place within an oligopoly market.

Figure 1. Changes in the Market Share as a Result of Changes in the Parameters of θ_{pqi}



(a) Commodity 1 (b) Commodity 2

(c) Commodity 3

5.3 Initial Values

In the third simulation, we change the initial values of matrix X. Table 2 indicates the converged values of X in a different set of parameters of θ_{pqi}, namely, $\theta_i^{pq} = 3.6$ and $\theta_i^{pq} = 4.2$. In both cases although the initial values of $X(0)$ are different, the converged values are the same for $\theta_i^{pq} = 3.6$ and $\theta_i^{pq} = 4.2$. This experiment shows that the convergence method is robust with respect to changes in the initial values. However, when the initial values of $X(0)$ are in the neighborhood of the boundary, the converged values are sometimes different from one another.

Table 2. Initial Values of $X(0)$ and Converged Values of X

Initial values $X(0)$	Converged values $\left(\theta_i^{pq} = 3.6\right)$	Converged values $\left(\theta_i^{pq} = 4.2\right)$
$X(0) = \begin{pmatrix} 0.33 & 0.33 & 0.34 \\ 0.33 & 0.34 & 0.33 \\ 0.34 & 0.33 & 0.33 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0.000 & 0.000 & 1.000 \\ 0.356 & 0.644 & 0.000 \\ 0.644 & 0.356 & 0.000 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$X(0) = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.6 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0.000 & 0.000 & 1.000 \\ 0.356 & 0.644 & 0.000 \\ 0.644 & 0.356 & 0.000 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

5.4 Changes in Equality Constraints

We have two kinds of data for quantity produced: share data and quantity data. This experiment examines the applicability of the calculation method for both shares and quantity. The sum of the share becomes unity, while the sum of the quantities need not become unity. We set ***a* = (1.2, 1.0, 0.8)** and ***b* = (0.9, 1.0, 1.1)—**that is, the total of vector ***a*** or vector ***b*** is not unity but three. This experiment shows that the method is applicable not only the case of shares but also to the case of quantity produced.

5.5 Three-Person, Four-Strategy Game

This experiment extends the number of firms from three to four, which yields the converged values indicated in Table 3.

Table 3. Values of *X* in a Three-Person, Four-Strategy Game

Initial values $X(0)$	Converged values $\left(\theta_i^{pq} = 2.0\right)$	Converged values $\left(\theta_i^{pq} = 4.2\right)$
$X(0) = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0.046 & 0.165 & 0.312 & 0.477 \\ 0.245 & 0.235 & 0.247 & 0.273 \\ 0.459 & 0.349 & 0.191 & 0.000 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0.0 & 0.0 & 0.25 & 0.75 \\ 0.0 & 0.5 & 0.5 & 0.0 \\ 0.75 & 0.25 & 0.0 & 0.0 \end{pmatrix}$

5.6 Four-Person, Three-Strategy Game

This experiment extends the number of commodities from three to four, which yields the converged values indicated in Table 4.

Table 4. Values of *X* in a Four-Person, Three-Strategy Game

Initial values $X(0)$	Converged values $\left(\theta_i^{pq} = 2.0\right)$	Converged values $\left(\theta_i^{pq} = 4.2\right)$
$X(0) = \begin{pmatrix} 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0.033 & 0.235 & 0.482 \\ 0.172 & 0.232 & 0.346 \\ 0.325 & 0.254 & 0.171 \\ 0.471 & 0.279 & 0.000 \end{pmatrix}$	$\hat{X} = \begin{pmatrix} 0.0 & 0.0 & 0.75 \\ 0.0 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.0 \\ 0.75 & 0.0 & 0.0 \end{pmatrix}$

6. CONCLUSIONS

Because oligopoly firms usually produce a variety of products, it is important to simultaneously understand both the determination of market shares among firms and the product mix within a firm. As the total amount of production for both firms and commodities is a priori given in the model as the constraint, managers are able to consider production strategy relying on the profit functions in the model. After the functional form and parameters are fixed, the convergent process is managed by the replicator dynamics algorithm. Using the Nash equilibrium simulation model, we can generate various optimal paths for production by changing the conjectures of firms. We confirmed that the present algorithm is good in assessing policy simulation.

REFERENCES

Aiyoshi. E., and A. Maki. 2009. **“**A Nash Equilibrium Solution in an Oligopoly Market: The Search for Nash Equilibrium Solutions with Replicator Equations Derived from the Gradient Dynamics of a Simplex Algorithm.” *Mathematics and Computers in Simulation*, forthcoming.

Dixit, A. K. 1988. “Optimal Trade and Industrial Policies for the U.S. Automobile Industry.” In R. Feenstra ed. *Empirical Methods for International Trade*, 141–65.

Klepper, G. 1990. “Entry into the Market for Large Transport Aircraft.” *European Economic Review* 34 (4): 775–803.

Rosen, J. B. 1965. Existence and Uniqueness of Equilibrium Points for Concave *N*-Person Games.” *Econometrica* 33 (3): 520–34.