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L-curve for hedging instrument selection in CVaR-minimizing portfolio hedging

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Abstract: Derivative contracts are popular instruments of risk management in financial and commodity markets. An optimal selection of a small subset from the large set of available hedging derivative instruments is an important practical problem. In this paper, the L-curve phase-plot strategy of regularization is generalized to this situation to provide a quick exploratory tool for optimal selection of hedging instruments.

In this paper, the optimal hedging problem is formulated as a one-period static portfolio selection model. The conditional value-at-risk (CVaR) of the portfolio loss distribution is adopted as a measure of hedging risk. CVaR is rapidly gaining popularity as a risk measure in portfolio analysis due to its coherence property. It is is closely related to VaR (Value-at-Risk), which is a current international industry benchmark in risk analysis. CVaR, however, is free of some limitations of VaR, which lacks sub-additivity and convexity and therefore contradicts the diversification principle and poses difficulties for optimization.

We consider a problem of reducing the risk of a target portfolio, consisting of derivative instruments on the underlying asset, by purchasing or selling the underlying asset and the derivative instruments on the same underlying asset. Such problem is typically ill-posed. The objective function considered in this paper is the CVaR of portfolio loss distribution with a generalized L1-norm penalty on portfolio decision vector in the form of proportional transaction costs. The advantage of such formulation is twofold: (1) it provides a required regularization of the problem; (2) the L1-norm penalty is known for causing some of the vector components to become exactly zero when the regularization parameter increases, thus performing automatic subset selection.

In this paper, we introduce a phase plot (or a parametric function) that relates the generalized L1-norm of the portfolio decision vector and the CVaR of the portfolio loss distribution, with the regularization parameter as an independent parameter of the function. We call such phase plot an “L-Curve” by analogy with Tikhonov regularization, due to its distinctive L-shape. The L-curve is calculated in this paper by formulating the problem as a Linear Programming problem and using CPLEX as the LP-solver. Numerical simulations reveal that the phase-plot of generalized L1-norm versus CVaR indeed possesses a pronounced L-shape and exhibits two distinctive regions: (1) a region of a rapid decrease in generalized L1-norm of the portfolio decision vector with increase in CVaR, and (2) a region of a slow decrease in the generalized L1-norm with increase in CVaR. The regularization parameter that corresponds to the transition between the two regions (or the “corner” of the L-curve) is a candidate for an optimal trade-off between minimization of the CVaR and the transaction costs (generalized L1-norm of portfolio decision vector). It is shown in this paper that in case of hedging a portfolio of derivatives with CVaR as a hedging risk measure, the L-curve can be interpreted as the trade-off curve between the deterministic and stochastic components of portfolio risk.

The properties of the regularized solution have been established via numerical simulations. Amongst these properties is the existence of a threshold value of the regularization parameter below which the solution is unbounded.

Keywords: *Conditional value-at-risk (CVaR), optimal hedging, proportional transaction costs, one-period portfolio selection, L-curve, generalized L1-norm penalty*

1. INTRODUCTION

Derivative contracts are popular instruments of risk management in financial and commodity markets. Hedge performance is strongly dependent on the hedging instruments used. While reduction in risk can be achieved by using a large number of derivative instruments, the transaction costs of such portfolio may be very large. Thus, an optimal selection of a small subset from the large set of available derivative instruments is an important practical problem. The purpose of this paper is to generalize the L-curve phase-plot strategy of regularization to the problem of optimal selection of hedging instruments.

The conditional value-at-risk (CVaR) is adopted in this paper as a hedging risk measure. CVaR is rapidly gaining popularity in portfolio analysis, as it is closely related to value-at-risk (VaR), which is a current international benchmark in risk management. However, VaR is not typically used in the portfolio selection problems, as it is not a coherent measure of risk (in the sense of Artzner et al, 1999). It lacks sub-additivity and convexity and therefore contradicts the diversification principle and poses difficulties for optimization. CVaR is a coherent risk measure (see Rockafellar and Uryasev, 2000, 2002) and is free of limitations of VaR.

We consider a problem of reducing the risk of a target portfolio, consisting of the derivative instruments on the underlying asset, by purchasing or selling the underlying asset and the derivative instruments on the same underlying asset. A similar problem has been studied by Alexander et al, 2006, and it has been shown that such a problem is typically ill-posed. Proportional transaction costs have been introduced as an additional objective (Alexander et al, 2006) and it was illustrated via numerical simulations that such a formulation can produce portfolios with smaller transaction costs and fewer instruments. However, the issue of the optimal choice of the regularization parameter was not addressed by Alexander et al, 2006. This is the subject of this paper.

In this paper, we consider the objective function in the form of the CVaR of the portfolio loss distribution with a generalized L1-norm penalty in the form of the proportional transaction costs. The use of the generalized L1-norm penalty has several advantages in this problem: (1) it provides the required regularization; (2) it induces some of the components of the portfolio decision vector to become exactly zero when the regularization parameter increases, thus performing automatic subset selection (Tibshirani, 1996); and (3) it adds the transaction costs as an additional performance criterion. The focus of this paper is on the selection of optimal regularization parameter. We introduce a phase-plot of the generalized L1-norm of the portfolio decision vector versus the CVaR of the portfolio loss distribution, with the regularization parameter as an independent parameter of the function. Such function has a distinctive L-shape and is called "L-Curve" in case of Tikhonov regularization (Hansen, 1992; Hansen and O'Leary, 1993). For Tikhonov regularization, the corner of the L-curve (or an L-corner) is known to produce the solution for optimal regularization parameter for a wide range of problems, including the problems where the structure of errors is not known or when the errors are highly correlated and other methods for optimal regularization parameter selection may not be successful (which may be the case in the problem of interest). Although the problem studied in this paper is significantly different from Tikhonov regularization problem, it appears that the phase-plot in this case possesses a distinctive L-shape and therefore can provide a quick exploratory tool for identification of the approximate value of the optimal regularization parameter. The interpretation of the L-curve for this problem is suggested.

As an example, we consider a problem of hedging a short call option with the underlying asset and European options, using synthetic data. The penalized portfolio hedging problem is formulated as a linear programming problem and solved using CPLEX. The properties of the regularized solution as well as the implications for optimal selection of the regularization parameter are discussed.

2. MATHEMATICAL MODEL

2.1 Profit/Loss Function for Portfolio Hedging

Consider a set of hedging instruments consisting of the underlying asset S and $n-1$ derivative instruments $V_i,\ i=1,...,n-1$ (note that the formulation can be readily extended for arbitrary number of underlying assets).

The random vector $\mathbf{V} \in \Re^n$ denotes the prices of hedging instruments. Assume that the price of any derivative instrument is a function of time and the underlying price (which is the case for non-path-dependent derivative instruments), then $\mathbf{V}(t,\mathrm{S}) \equiv \{S(t), V_1(t,\mathrm{S(t)}),...,V_{n-1}(t,\mathrm{S(t)})\}$. Denote by $\mathbf{X} \equiv (x_0, x_1,.., x_{n-1})$ $\in \Re^n$ a portfolio decision vector with components $x_i,\ i=1,...,n-1$, representing the positions for the i-th instruments V_i and x_0 representing the position for the underlying asset S (positive values correspond to "buying" and negative to "selling"). The target portfolio value at time t is a random variable $\Pi_0(t,S)$. For the sake of simplicity, we assume the underlying process dynamics in the form

$$S(t) = S_0 \exp(\mu t + \sigma\sqrt{t}\Omega) . \tag{1}$$

Consider a single period $[0,\tau]$ portfolio hedging model. We assume, for simplicity of presentation, that the portfolio is acquired at $t=0$ and all hedging instruments are disposed of at the end of period (it is straightforward to extend the model for more realistic situations). The portfolio profit at the end of period is given by

$$\Pi(S,\mathbf{X}) = \Pi_0(\tau,S) + B(\tau,S,\mathbf{X}), \tag{2}$$

where B is the cash account. If we ignore the interest earned on cash account for simplicity, then

$$B(\tau,S,\mathbf{X}) = B(0,S(0),\mathbf{X}) + \mathbf{X}^T \bullet \mathbf{V}(\tau,\mathrm{S}(\tau)), \tag{3}$$

where the superscript "T" denotes the transpose operator. The cash account at the beginning of time horizon is given by

$$B(0,S,\mathbf{X}) = -\Pi_0(0,S(0)) - \mathbf{X}^T \bullet \mathbf{V}(0,\mathrm{S}(0)), \tag{4}$$

Substituting (3) and (4) into (2) gives

$$\Pi(S,\mathbf{X}) = \Delta\Pi_0 + \mathbf{X}^T \bullet \Delta\mathbf{V} . \tag{5}$$

The portfolio loss is given by

$$f_{loss}(S,\mathbf{X}) = -\Pi(S,\mathbf{X}) = -\Delta\Pi_0 - \mathbf{X}^T \bullet \Delta\mathbf{V} . \tag{6}$$

2.2 Conditional Value-at-Risk (CVaR) as a Risk Measure

The conditional value-at-risk at probability level α (denoted by α-CVaR or $\phi_\alpha(\mathbf{X})$) of the random variable f_{loss} associated with the decision vector $\mathbf{X}$ and the risk factor S at time horizon τ is defined as a mean of the α-tail distribution of this random variable (Rockafellar and Uryasev, 2002), where α-tail cumulative distribution function is defined as

$$\Psi_\alpha(\mathbf{X},\xi) = \begin{cases} 0, & \xi < \xi_\alpha(\mathbf{X}), \\ [\Psi(\mathbf{X},\xi) - \alpha]/(1-\alpha), & \xi \ge \xi_\alpha(\mathbf{X}), \end{cases} \tag{7}$$

where $\Psi(\mathbf{X},\xi) = P[f_{loss}(S,\mathbf{X}) \le \xi]$ is a standard cumulative distribution function of portfolio random loss variable $f_{loss}(S,\mathbf{X})$, $\xi_\alpha(\mathbf{X})$ is the value-at-risk (VaR) at a given confidence level α of distribution for $f_{loss}(S,\mathbf{X})$,

$$\alpha\text{-VaR} \equiv \xi_\alpha(\mathbf{X}) = \min_{\xi \in R}\{\Psi(\mathbf{X},\xi) \ge \alpha\} .$$

When the random variable S is represented by scenarios $S_1,...,S_M$ with probabilities $p_1,...,p_M$, the α-VaR is given by $\xi_\alpha(\mathbf{X}) = f_{loss}^{sort}(S_{j_\alpha},\mathbf{X})$, and α-CVaR is (Rockafellar and Uryasev, 2002)

$$\phi_\alpha(\mathbf{X}) = \frac{1}{1-\alpha}\left[\left(\sum_{j=1}^{j_\alpha} p_j^{sort} - \alpha\right)\xi_\alpha(\mathbf{X}) + \sum_{j=j_\alpha+1}^{M} p_j^{sort} f_{loss}^{sort}(S_j,\mathbf{X})\right], \tag{8}$$

where j_α is defined so that $\sum_{j=1}^{j_\alpha-1} p_j^{sort} < \alpha \le \sum_{j=1}^{j_\alpha} p_j^{sort}$, $f_{loss}^{sort}(S,\mathbf{X})$ and p^{sort} are sorted (in the ascending order) samples.

The main advantage of CVaR over VaR as a risk measure is that CVaR is a coherent risk measure (as defined by Artzner et al, 1999). This means that CVaR is linearly homogeneous, convex (and therefore sub-additive), monotonic and translation invariant.

Optimization of CVaR

Rockafellar & Uryasev (2000, 2002) have shown that minimizing α-$CVaR$ with respect to $\mathbf{X}$ is equivalent to minimizing a function $F_\alpha(\mathbf{X},\xi)$ with respect to $(\mathbf{X},\xi)$:

$$\min_{\mathbf{X}} \phi_\alpha(\mathbf{X}) = \min_{\mathbf{X},\xi} F_\alpha(\mathbf{X},\xi) \, , \tag{9}$$

where
$$F_\alpha(\mathbf{X},\xi) = \xi + \frac{1}{(1-\alpha)} E\{[f_{loss}(S,\mathbf{X}) - \xi]^+\} \, , \tag{10}$$

$$[a]^+ \equiv \max\{0, a\} \, . \tag{11}$$

If the loss function $f_{loss}(S,\mathbf{X})$ is convex with respect to $\mathbf{X}$, then $\phi_\alpha(\mathbf{X})$ is also convex with respect to $\mathbf{X}$, and $F_\alpha(\mathbf{X},\xi)$ is jointly convex in $(\mathbf{X},\xi)$ (Rockafellar and Uryasev, 2002). The integral $F_\alpha(\mathbf{X},\xi)$ (10) can be estimated via the first moment approximation by generating sample points S_i, $i = 1,...,M$, with probabilities p_i,

$$F_\alpha(\mathbf{X},\xi) \approx \xi + \frac{1}{(1-\alpha)} \sum_{j=1}^{M} p_j [f_{loss}(S_j,\mathbf{X}) - \xi]^+ \, . \tag{12}$$

2.3 Optimal Hedging with Generalized L1-Norm Penalty

It can be shown that the portfolio loss function is not unique when the number of hedging derivative instruments exceeds the threshold, which is a function of the number of underlying assets. It has been shown (Alexander et al, 2006) that the loss function of the portfolio described in Section 2.1 is not unique if the number of hedging instruments is $n > 3$. The portfolio selection problem in this case is ill-posed. Consequently, for its solution, some form of regularization is required. As in Alexander et al, 2006, we consider the objective function in the form of CVaR of portfolio loss distribution with the penalty in the form of the proportional transaction costs

$$\Xi(\mathbf{X},\lambda) = \phi_\alpha(f_{loss}(S_j,\mathbf{X})) + \lambda\beta\sum_{i=1}^{n} V_i^0 |x_i| \, , \tag{13}$$

where $V_i^0 \equiv V_i(0,S(0))$ represent the initial prices of hedging instruments, β is the fraction of the initial instrument price taken as the transaction cost and λ is the regularization parameter. Such penalty has the form of a generalized L1-norm of the portfolio decision vector. The L1-norm penalty is known for reducing the vector components towards zero. It also induces some of them to become exactly zero when the regularization parameter increases, thus performing automatic subset selection (Tibshirani, 1996). Thus, such regularization has several benefits for portfolio hedging problem, as it simultaneously provides the required regularization, performs the automatic selection of hedging instruments and adds the transaction costs as an additional objective criterion.

Using (9)-(13), the optimal hedging problem can be formulated in the form

$$\mathbf{X}_\lambda = \arg\min_{\mathbf{X}} \{ \, \Xi(X,\lambda) \, \} = \arg\min_{\mathbf{X},\xi} \left\{ \xi + \frac{1}{(1-\alpha)M} \sum_{j=1}^{M} [f_{loss}(S_j,\mathbf{X}) - \xi]^+ + \lambda\beta\sum_{i=1}^{n} V_i^0 |x_i| \right\}, \quad \lambda \ge 0. \tag{14}$$

Note that the bounds on the portfolio weights are not included into the formulation.

We now show that, if CVaR is used as a portfolio risk measure, the transaction costs represent a deterministic component of the portfolio risk. Indeed, the transaction costs can be included into the portfolio selection model (Section 2.1) by adding them as an expense to the cash account B. Thus,

$$B(0,S,\mathbf{X}) = -\Pi_0(0,S(0)) - \mathbf{X}^T \bullet \mathbf{V}(0,S) - \beta\sum_{i=1}^{n} V_i^0 |x_i|, \tag{15}$$

For a single period $[0,\tau]$ portfolio model (and ignoring the interest earned on cash account) we have $B(\tau,S,\mathbf{X}) = B(0,S,\mathbf{X}) + \mathbf{X}^T \bullet \mathbf{V}(\tau,S)$ and the portfolio profit at the end of time horizon is given by

$$\Pi(S,\mathbf{X}) = \Pi_0(\tau,S) + B(\tau,S,\mathbf{X}) = \Pi_0(\tau,S) - \Pi_0(0,S) + \mathbf{X}^T \bullet \mathbf{V}(\tau,S) \\ - \mathbf{X}^T \bullet \mathbf{V}(0,S) - \beta\sum_{i=1}^{n} V_i^0 |x_i| \quad = \Delta\Pi_0 + \mathbf{X}^T \bullet \Delta\mathbf{V} - \beta\sum_{i=1}^{n} V_i^0 |x_i|. \tag{16}$$

The loss of the portfolio with transaction costs $f_{loss}^{tc}(S,\mathbf{X})$ is given by

$$f_{loss}^{tc}(S,\mathbf{X}) = -\Delta\Pi_0 - \mathbf{X}^T \bullet \Delta\mathbf{V} + \beta\sum_{i=1}^{n} V_i^0 |x_i| = f_{loss}(S,\mathbf{X}) + \beta\sum_{i=1}^{n} V_i^0 |x_i|. \tag{17}$$

Using the translation invariance property of CVaR yields

$$\phi_\alpha\{f_{loss}^{tc}(S,\mathbf{X})\} = \phi_\alpha\{f_{loss}(S,\mathbf{X})\} + \beta\sum_{i=1}^{n} V_i^0 |x_i|. \tag{18}$$

Equation (18) shows that the transaction costs add a deterministic component to the risk of the portfolio without the transaction costs. It also implies that minimization of CVaR of the portfolio with transaction costs is equivalent to solving the penalized problem (14) with λ=1. However, such a choice of the regularization parameter suggests that equal weights are allocated to deterministic and stochastic components of portfolio risk. This is unlikely to reflect the investor/broker preference. In practice, the optimal choice of the regularization parameter should reflect the optimal trade-off between the deterministic and stochastic components of portfolio risk, which will be discussed in the following section.

2.4 L-Curve for Regularization Parameter Selection

In this section, we introduce the L-curve phase-plot as a tool for optimal selection of hedging instruments. In the context of Tikhonov regularization, the L-curve was introduced by Hansen, 1992, as a phase-plot of the L2-norm of the regularized solution versus the risk measure in the form of L2-norm of the corresponding residual vector. There is a considerable amount of research on L-curve for Tikhonov regularization, in which case the generalized singular value decomposition gives a way for analytical treatment of the properties of the L-curve. It was shown that the corner of the L-curve (defined as a point with the largest curvature) always exists for Tikhonov regularization and provides good approximation to the optimal regularization parameter for a wide range of problems, even when the correlated errors in the signal are present and other methods may fail to do so. While the problem studied here is considerably different from Tikhonov regularization problem, the L-curve plot provides a convenient way to display the information about the regularized solution of an ill-posed problem. This is known to be the case for regularization of ill-posed problems in general and provides a motivation for our study.

We introduce the L-curve as a parametric function that relates the generalized L1-norm of the portfolio decision vector and the CVaR of the portfolio loss distribution, with the regularization parameter as an independent parameter of the function

$$\{u(\lambda), \upsilon(\lambda)\}_{\lambda\in(0,\infty)}, \quad u(\lambda) = \phi_\alpha(\mathbf{X}(\lambda)); \; \upsilon(\lambda) = \beta\sum_{i=1}^{n} V_i^0 |x_i(\lambda)|. \tag{19}$$

As was shown in Section 2.3, the generalized L1-norm of the portfolio decision vector in the form of the proportional transaction costs represents an additional deterministic component of the portfolio risk. The optimal choice of the regularization parameter in this case reflects an optimal balance between the deterministic and stochastic components of the portfolio risk.

3. NUMERICAL STUDY

In this section, we consider an example of hedging a short call option C with the underlying asset S and $n-1$ call options on the same underlying asset $C^{(i)}, i = 1,...,n-1$. Thus, $\mathbf{V}(t,S) = \{S, C^{(1)},...,C^{(n-1)}\}$. The target portfolio is $\Pi_0 = -C(t)$. We generate *M* scenarios for the price of the underlying asset at the end of period using Monte Carlo simulations. The prices of the target and hedging call options are calculated using Black-Scholes formula. The entire solution path (19) is calculated by reducing the problem to linear programming problem and using CPLEX as the LP solver.

In all cases below, the target portfolio consists of at the money short call option with maturity $T = 30$ days. The parameters in the model for the underlying asset price (1) are taken as: $S_0 = 100$, $\mu = 0.1, \sigma = 0.2$. The 95%-CVaR of the target option over the 5 days horizon is 4.63. Figures 1 and 2 show typical results of calculations. In Figure 2, the set of hedging options consists of all possible combinations of call options with maturities $T_i = \{5, 15\}$ days and strikes $K_i = \{95, 100, 105\}$ (thus, producing the set of 6 hedging options). In Figure 1, there are 21 hedging call options formed by all possible combinations of options with maturities $T_i = \{10, 15, 20, 25, 35, 40, 45\}$ days and strikes $K_i = \{95, 100, 105\}$.

One can see that in both cases (as also in other cases considered by the authors), the generalized L1-norm (transaction costs) as a function of CVaR (see Figures 1b, 2b) has a pronounced L-shape. The regularized solutions for CVaR and transaction costs as functions of the regularization parameter are monotonic functions, while the number of hedging instruments with non-zero weights as a function of the regularization parameter can be non-monotonic for smaller values of the regularization parameter (see Figures 1a, 2a). It is not clear at this stage whether this behavior is due to an insufficient number of Monte Carlo simulations for the examples studied.

The number of hedging instruments with non-zero weights as a function of the regularization parameter resembles a step-function (See Figures 1a, 1b). The CVaR and the generalized L1-norm as functions of the regularization parameter (Figures 1a, 2a) are also discontinuous functions. As a result, the L-curve possesses the regions of unattainable values that correspond to the discontinuities in the CVaR and the generalized L1-norm.

The numerical simulations reveal the existence of the threshold value of the regularization parameter, below which the solution is unbounded. In Figure 2, such value is approximately 0.05, with all available hedging instruments present at the threshold value. In Figure 1, the threshold value is about 0.055, with only 7 out of 22 hedging instruments present.

4. DISCUSSION AND CONCLUSIONS

This paper introduces the phase-plot (the parametric function) that relates the generalized L1-norm of the portfolio decision vector, in the form of the proportional transaction costs, and the CVaR of the loss distribution, with the regularization parameter as an independent parameter of the function. Numerical simulations show that such a parametric curve possesses a pronounced L-shape. It has two distinctive regions of different behavior: 1) a region of a rapid decrease in L1-norm of the solution with increase in CVaR, and 2) a region of a slow change in generalized L1-norm with increase in CVaR. A regularization parameter that corresponds to the transition point between the two regions presents a candidate for an optimal regularization parameter. Indeed, moving away from this point would lead to decrease in one of the objective functions (CVaR or generalized L1-norm) at the expense of the much larger relative increase in another.

We have shown that, for portfolio hedging with CVaR as a risk measure, the transaction costs represent the deterministic component of the portfolio risk and therefore the L-curve can be viewed as a trade-off curve between the deterministic and stochastic components of the portfolio risk.

The properties of the penalized solution revealed via the numerical simulations are:

- The transaction costs (the generalized L1-norm of the portfolio decision vector) and the CVaR of the portfolio loss function are monotonic functions of the regularization parameter; they are essentially discontinuous;
- The number of hedging instruments with non-zero weight as a function of the regularization parameter is a step function, which can be non-monotonic for smaller values of the regularization parameter;
- The L-curve possesses the regions of unattainable values that correspond to the discontinuities in the CVaR and the generalized L1-norm functions;
- There exists a threshold value of the regularization parameter below which the solution is unbounded. Therefore the parameter values below the threshold should be excluded from consideration. The number of hedging instruments with non-zero weights at the threshold value of the regularization parameter can be smaller that the total number of hedging instruments available.

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a) b)

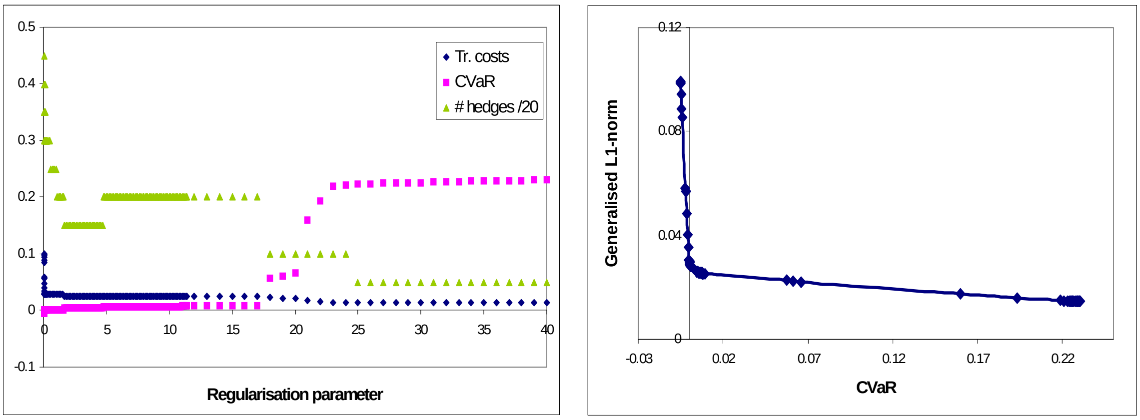

Figure 1: Hedging a short call with expiry of 30 days with the underlying asset and 21 hedging options. Hedge horizon is 5 days; 30000 Monte Carlo simulations; a) the regularised solutions for CVaR, transaction costs and the number of hedging instruments with non-zero weights as functions of the regularisation parameter; b) the L-curve.

a) b)

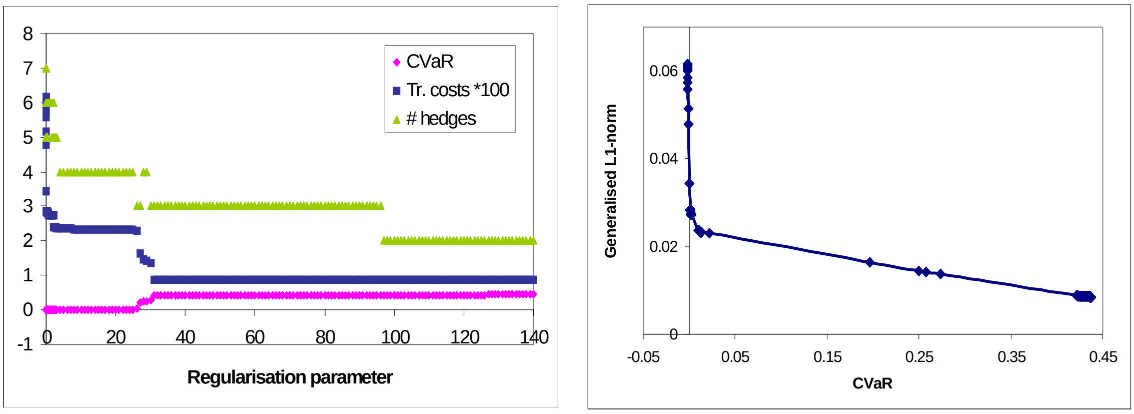

Figure 2: Hedging a short call with expiry of 30 days with the underlying asset and 6 hedging options. Hedge horizon is 5 days; 20000 Monte Carlo simulations; a) the regularised solutions for CVaR, transaction costs and the number of hedging instruments with non-zero weights as functions of the regularisation parameter; b) the L-curve.

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