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Images and Barriers on the road to Real Options Valuation

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Abstract: Traditional methods for evaluating investment decisions, such as Net Present Value, don't properly account for the flexibility inherent in many investment projects. This has been recognized and the attempt to value the such flexibilities is known as Real Options Analysis. This type of investment analysis involves applications of exotic option pricing theory to the evaluation of investment decisions by firms. Many investment projects involve particular types of flexibility and these situations have been identified and recognized as various types of options. This relates to investment decisions about non tradeable assets such as real estate development and mining projects. The decision to delay commencement of a mining project contingent on commodity prices rising enough to make a mining operation viable can be thought of as a type of call option. The decision to temporarily shut down a mining operation due to low commodity prices can be thought of and valued as a type of put option. Some of these options may have barrier features, where the investment project gets cancelled due to commodity prices falling below a critical level. This may the thought of as a "down and out barrier option". Many multi-stage investment decisions can be thought of as compound options, which are options over options. In this paper we consider how to combine the theory of exotic multi-period, multi-asset options and the theory of barrier option pricing to the evaluation of investment decisions. In particular we consider the valuation of compound options with various levels of complexity, and the valuation of barrier versions of these options. We show how to mathematically model such situations, and we derive closed form valuation formulae for evaluating some of them and discuss how to apply numerical methods in other cases. We illustrate the ideas and methods in the context of a hypothetical gold mining project. We derive an analytic formula for the option to delay a mining project which, once commenced, includes the right to further expand production and the right to close down production. The right to delay the commencement of this project is a compound call option over the underlying asset plus a call option plus a put option. The valuation formula for this investment opportunity involves the bivariate normal distribution.

Keywords:, *Black Scholes Option Pricing, Method of Images, Barrier Options, Dual Expiry Options, Compound Options, Monte Carlo Simulation, Binomial Option Pricing Method*

1. INTRODUCTION

The Net Present Value method is the most widely used method for Investment appraisal, having replaced traditional approaches such as Accounting Rate of Return and Payback Period methods. However, it is a still flawed methodology. The NPV method involves forecasting the expected future cashflows, discounting these at a risk adjusted discount rate and then subtracting the initial outlay required to get the project underway. The future expected cashflows are treated as being known with certainty, with a risk adjusted discount rate used to account for the uncertainty. This approach ignores the flexibility that an investor may have to vary the project, either before it commences or once it is underway. The NPV method doesn't properly value the flexibility inherent in many investment projects.

Modern approaches to investment appraisal attempt to remedy this defect by considering the flexibility to vary the project as a kind of option. This is called the real options approach to investment appraisal. The options are "real" in the sense they are options over non tradeable "real" projects rather than exchange-traded assets. Situations involving particular types of flexibility have been identified as existing in many types of investment projects including mining, real estate and pharmaceutical projects. Many researchers and practitioners have identified these situations as particular types of (exotic) options. This facilitates attempts to apply option pricing theory to estimate the economic value of the project and of the flexibility inherent in it.

Some examples of these flexibilities include: The option to delay the commencement of a project and the payment of the initial outlay can be viewed as a call option over the project; the option to abandon the project once it is operational, and recover the salvage value of the project can be thought of and valued as a put option: the option to expand the project once it is operational, at some cost, can be valued as a call option. The project may have multiple options to expand / contract at various future times and this can be thought of as a type of compound option.

We shall illustrate these types of real options in the context of a gold mining project. The options involved are often complicated exotic options, which are difficult to value analytically, possibly requiring numerical approaches such as Monte Carlo Simulation and lattice (binomial tree) methods for evaluation.

In this paper we consider the *barrier* versions of these types of options, and show how such options may be easily evaluated using recently developed option pricing technology.

Barrier features for options over projects

Corporations are subject to resource constraints (limited capital, limited ability to borrow) and there are alternative investments that compete for funding. In addition, governments and regulators may intervene to cancel or take over a project in some circumstances. During the waiting period to when the decision to proceed or not must be made, it is possible that the banks who lend money to the project sponsor will cancel the funding if the gold price drops below level B. This can be thought of as a down-and-out type of barrier. It is possible that if the gold price rises above some level B then the sponsor might have some other more profitable project to invest in, or the government may nationalise the project. This could be thought of as an up-and-out type of barrier. Accordingly it is plausible that a real option to delay investment in a gold mine could be an up and out or a down and out type of barrier call option. Barrier options contain provisions which allow them to be effectively cancelled if the price of some underlying asset (e.g. a commodity price) drops below a threshold barrier level, which may represent some threshold for profitability of an enterprise. Alternatively, the option may come into existence when the commodity price rises above this level.

We briefly outline relevant results from the theory of barrier option pricing and illustrate its use in conjunction with both analytic formulae for some of the simpler real options, and with numerical methods for real options that can only be valued numerically.

REVIEW OF ASSET PRICING THEORY

2.1 The Black Scholes Option Pricing Model

This was developed in the early 1970's (Black and Scholes 1973). Using the idea that markets are efficient, an option over a stock has an economic value that can be considered to be a function of 2 variables x (the market price of the stock) and t, (the time elapsed since the option was written). The model assumes the stock price process is geometric Brownian motion, and makes various other "idealised" assumptions about market frictions. Let $V(x,t)$ be the value at time t of some option contract defined over the stock with current value x. Then this function $V(x,t)$ satisfies the Black Scholes Partial Differential Equation, on the domain $D=\{(x,t): x>0, 0<t<T\}$, subject to the boundary (terminal) condition $V(x,T)=f(x)$

$$\frac{\partial V}{\partial t}+(r-y)x\frac{\partial V}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}-rV=0 \qquad (2.1)$$

The parameters r, y, σ, T are respectively the risk free interest rate, the dividend yield, the volatility of the asset and the time when the option contract matures. For instance, a call option has $V(x,T)=f(x)=\max(x-K,0)$ which is the payoff at maturity, and $V(0,t)=0$ meaning that the option value is zero if the stock price falls to zero. The solution of the pde, subject to the relevant boundary conditions on the relevant domain for a european call is:

$$C = xe^{-y\tau}N(d_1)-Ke^{-r\tau}N(d_2) \qquad \text{where } d_1,d_2=\frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right)+\left(r-y\pm\frac{1}{2}\sigma^2\right)\tau\right] \qquad (2.2)$$

2.2 The discounted risk neutral expectations approach to option valuation

Initially, PDE methods were used to derive option pricing formulae. An alternative approach (Harrison and Pliska, 1981) to obtaining the option price is to compute the expectation of the option payoff under the equivalent martingale measure (also known as the risk neutral distribution) The expected payoff is then discounted at the risk free interest rate. That this is mathematically equivalent to solving the PDE subject to the boundary conditions, is due to a celebrated theorem of Feynman & Kac.

2.3 Numerical methods

There are numerical methods for applying this discounted expectations approach. These include Monte Carlo simulation (Boyle and Schwartz 1977) and the binomial method (Cox, Ross and Rubinstein 1979). The binomial method is a discrete time, discrete state space approximation which models the asset price distribution as "log-binomial" rather than log-normal. These are typically applied in cases where it is not possible to derive analytically tractable formulae using either a discounted expectations or a pde approach.

2.4 Down and Out Barrier Options

The down and out option $V_{DO}(x,t)$ satisfies the PDE and the boundary conditions $V_{DO}(x,T)=f(x)$ and $V_{DO}(b,t)=0, t<T$ on the domain $D=\{(x,t): x>b, 0<t<T\}$

$$\frac{\partial V_{DO}}{\partial t}+(r-y)x\frac{\partial V_{DO}}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V_{DO}}{\partial x^2}-rV_{DO}=0 \qquad (2.3)$$

This option is cancelled if the stock price crosses the level *B* (the barrier level) from above before the option maturity date. Otherwise, it provides the same payoff as the standard option (e.g. standard call or put) if the barrier level is not crossed by the stock price before maturity. These types of options were first considered by Merton (1973). There are 4 basic types of barrier option. These are the down and out, the down and in, the up and out and the up and in. Space prevents us from going into the details of these other options. However all 4 types can be expressed in terms of a linear combination of the non barrier version of the option and the down and out version of the option. Details of how to do this are covered in the paper by Buchen (2001)

The method of images for the down and out option

The solution to the above problem is $V_{DO}(x,t)=V_B(x,t)-V_B^*(x,t)$ where $V_B(x,t)$ is the solution of another pde problem with domain $D=\{(x,t): x>0, 0<t<T\}$ and boundary condition $V_B(x,T)=f(x)I_{(x>B)}$

$$\frac{\partial V_B}{\partial t}+(r-q)x\frac{\partial V_B}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V_B}{\partial x^2}-rV_B=0 \qquad (2.4)$$

$V_B^*(x,t)$ is called the "image" of $V_B(x,t)$ with respect to the barrier $x=B$

$$V_B{}^*(x,t) = \left(\frac{B}{x}\right)^{\alpha} V_B\left(\frac{B^2}{x},t\right) \text{ where } \alpha = 2\frac{(r-q)}{\sigma^2} - 1 \qquad (2.5)$$

For a call option with $K > B$ we have $f(x)I_{(x>B)} = f(x) = \max(x-K,0)$ so that $V_B(x,t) = V(x,t)$

This method of images converts a difficult problem into another problem with a simpler boundary condition, which is easier to solve. Usually it is relatively easy to compute $V_B(x,t)$, either analytically or numerically. From this we can compute the down and out version of the option. The method of images is discussed in the paper by Buchen (2001) and a recent mathematical proof is provided in Konstandatos (2008).

The NPV method of valuation and risk neutral valuation

In asset pricing of projects and equity, the NPV method is often applied. This involves estimating the expected cashflows provided by a project and discounting these at a risk adjusted discount rate. The risk adjustment is often done via the capital asset pricing model and adds a risk premium to the risk free rate. The risk premium is based on the correlation between the returns on the asset and the returns on the stock market index. If this correlation is zero then the discount rate is the risk free rate. Risk neutral valuation does the risk adjustment to the expected cashflow instead of to the discount rate. If the valuation is done consistently we should arrive at the same valuation with either method. Both approaches in effect relate the value of the project to the value of other assets in the economy and are forms of relative valuation.

2. GOLD MINE PROJECT EXAMPLE

We consider a simple mining project that does not include any options to expand or contract, once the project begins. The project sponsor has the option to delay the start of the project till time T_0. We have the option to invest in a gold mine project at time T_0 in the future. If we decide to invest we have to outlay an initial amount of capital K_0 at time T_0. In return we receive profits at times $T_1, T_2, ..., T_n$ of amount $(X_{T_i} - C)$ at time T_i, where X_{T_i} is the market price of gold at time T_i and C is the cost of extracting and processing the gold each period. We assume that this cost of extraction is constant. We assume that once we commit to the project we are "locked in to the project" and to this cashflow.

The present value of the project

At time T_0 we have $PV = \sum_{i=1}^{n} E\{(X_{T_i} - C)\} e^{-r(T_i - T_0)}$ and we shall use a risk neutral approach to computing the PV. The risk neutral expectation of the future gold price at time T_i is the forward gold price (as seen at time T_0) for delivery at time T_i: $E\{X_{T_i} \mid X_{T_0}\} = X_{T_0} e^{(r-q)(T_i - T_0)}$. The parameter q is the income yield on gold.

It follows that $\sum_{i=1}^{n} E\{G_{T_i}\} e^{-r(T_i - T_0)} = G_{T_0}\left[\sum_{i=1}^{n} e^{-q(T_i - T_0)}\right] = X_{T_0} A(q,1,n)$

and that $\sum_{i=1}^{n} E\{X_{T_i}\} e^{-r(T_i - T_0)} = X\left[\sum_{i=1}^{n} e^{-r(T_i - T_0)}\right] = XA(r,1,n)$ where $A(r,1,n) = \sum_{i=1}^{n} e^{-r(T_i - T_0)}$

The value of the project at time T_0 is therefore

$$PV = \alpha_0 \left(X_{T_0} - \beta_0\right) \text{ where } \alpha_0 = A(q,1,n) \text{ and } \beta_0 = CA(r,1,n)/A(q,1,n) \qquad (2.6)$$

The net present value of the project at time T_0 is $NPV = \alpha_0 \left(X_{T_0} - \beta_0\right) - K_0$

The NPV at the date of the decision to proceed with the project and the value of the delay option

Using the NPV methodology, we check whether the NPV is positive and if it is then we would proceed with the investment. If not then we don't proceed. We have the right but not the obligation to proceed with the project at time T_0. If we consider the value of this opportunity at some time $t < T_0$ we see that this is the value of an option over the project's value at time T_0. The payoff on the option to delay the commencement of the project is $\max(PV - K_0, 0)$. This payoff is equivalent to

$$\alpha_0 \left[X_{T_0} - (\beta_0 + K_0/\alpha_0) \right]^+ \qquad (2.7)$$

This can be thought of and valued as α_0 units of a call option over gold with a maturity date of T_0 with an exercise price of $\tilde{K}_0 = (\beta_0 + K_0/\alpha_0)$. This can be valued using the standard black scholes formula for a call.

The option to abandon

Suppose that after the project commences we have the option to abandon the project at time T_m. If we do abandon the project, we receive a cashflow of S_m at time T_m (the salvage value), but we forgo the future project cashflows of $\left(X_{T_i} - C\right)$ at time T_i for $i = m+1, m+2, ..., n$.

The value at time T_m of the future cashflows we forgo is

$$V_{T_m} = \alpha_m \left[X_{T_m} - \beta_m \right] \text{ where } \alpha_m = A(q, m+1, n) \text{ and } \beta_m = CA(r, m+1, n)/A(q, m+1, n)$$

The payoff from the option to abandon is

$$\left(S_m - V_{T_m}\right)^+ = \alpha_m \left([S_m/\alpha_m + \beta_m] - X_{T_m}\right)^+ \qquad (2.8)$$

Which is the payoff from α_m units of a put option over the asset X, with exercise price $\tilde{K}_m = [S_m/\alpha_m + \beta_m]$ and with maturity T_m. This can be valued using the standard black scholes formula for a put.

The option to expand

We may have the option to expand the project at time T_m and this could involve opening a new mine, with differences in remaining life, cost of extraction and initial cost. For instance it may be that there is another deposit of ore that can be mined but the cost of extraction is higher at $\tilde{C}$ per period, the new project produces cashflows at times T_i for $i = m+1, m+2, ..., N$ where $N \neq n$ and the initial outlay required at time T_m is of amount K_m. The value of this new project at time T_m is

$$V_{T_m} = \tilde{\alpha}_m \left[X_{T_m} - \tilde{\beta}_m \right] \text{ where } \tilde{\alpha}_m = A(q, m+1, N) \text{ and } \tilde{\beta}_m = \tilde{C} A(r, m+1, N)/A(q, m+1, N)$$

At time T_m we can choose to expand by starting up this new project at a cost of K_m. The payoff from our option to expand is thus

$$\tilde{\alpha}_m \left(X_{T_m} - \tilde{K}_m\right)^+ \qquad (2.9)$$

which is the payoff from $\tilde{\alpha}_m$ units of a call option over the asset X, with exercise price $\tilde{K}_m = \left(K_m/\tilde{\alpha}_m + \tilde{\beta}_m\right)$ and with maturity T_m

The value of the flexible version of the project

When we commence the project, we may have the right to vary the project in the future by either expanding it or abandoning it at a fixed future time. We can do a valuation of this flexible version of the project at time T_0. Its value will be the value of the original fixed project plus the value of the call option to expand the project at time T_m plus the value of the put option to abandon the project at time T_m. The time T_0 value of each of these components is a function of the gold price X_{T_0}. It can be shown that this value is a monotonic increasing function of the gold price. The function involves the univariate normal distribution.

The value of the project at time T_0 is

$$V_{T_0}\left(X_{T_0}\right)=\alpha_0\left(X_{T_0}-\tilde{K}_0\right)+\alpha_m P\left(X_{T_0},T_m-T_0,\tilde{K}_m\right)+\tilde{\alpha}_m C\left(X_{T_0},T_m-T_0,\tilde{K}_m\right) \qquad (2.10)$$

The value of the option to delay this flexible version of the project

Define the constant a by the equation $V_{T_0}\left(a\right)=K_0$. This is the value of the gold price that makes the option to delay worth exercising at the time when the decision to proceed or not must be made. The payoff from exercising the option to commence the project at time T_0 is $\left(V_{T_0}\left(x\right)-K_0\right)^+=\left(V_{T_0}\left(x\right)-K_0\right)I_{(x>a)}$. The value of this payoff at an earlier time *t* is $V\left(x,t\right)=e^{-r(T_0-t)}E\left[\left(V_{T_0}\left(X_{T_0}\right)-K_0\right)I\left(X_{T_0}>a\right)\right]$. The right to delay the start of the project till time T_0 is a compound call option over the project plus the options to expand and the option to abandon. The valuation formula (2.11) for this option to delay flexible version of the project is obtained using methods for pricing dual expiry exotic options as in Buchen (2004). The formula for the value at time *t* when the gold price is *x* involves the bivariate normal distribution and the formula is:

$$V\left(x,t\right)=\alpha_0 C_a\left(x,T_0-t,\tilde{K}_0\right)+\alpha_m P_a^+\left(x,T_0-t,T_m-t,\tilde{K}_m\right)+\tilde{\alpha}_m C_a^+\left(x,T_0-t,T_m-t,\tilde{K}_m\right) \qquad (2.11)$$

where

$$C_a\left(x,T_0-t,K\right)=xe^{-q(T_0-t)}N\left(d\left(x,T_0-t,\max\left(a,K\right)\right)\right)-Kxe^{-q(T_0-t)}N\left(d'\left(x,T_0-t,\max\left(a,K\right)\right)\right)$$

$$\begin{aligned}P_a^+\left(x,T_0-t,T_m-t,K\right)= \; & Ke^{-r(T_m-T_0)}N_2\left(d'\left(x,T_0-t,a\right),-d'\left(x,T_m-t_0,K\right),-\rho\right)\\ & -xe^{-q(T_m-T_0)}N_2\left(d\left(x,T_0-t,a\right),-d\left(x,T_m-t,K\right),-\rho\right)\end{aligned} \qquad (2.12)$$

$$\begin{aligned}C_a^+\left(x,T_0-t,T_m-t,K\right)= \; & xe^{-q(T_m-T_0)}N_2\left(d\left(x,T_0-t,a\right),d\left(x,T_m-t,K\right),\rho\right)\\ & -Ke^{-r(T_m-T_0)}N_2\left(d'\left(x,T_0-t,a\right),d'\left(x,T_m-t,K\right),\rho\right)\end{aligned} \qquad (2.13)$$

$$\rho=\sqrt{\frac{T_0-t}{T_m-t}},$$

$$d\left(x,\tau,k\right)=\frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{k}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)\tau\right),$$

$$d'\left(x,\tau,k\right)=d\left(x,\tau,k\right)-\sigma\sqrt{\tau}$$

$N\left(x\right)$ is the cdf of the univariate normal distribution

$N_2\left(x,y,\rho\right)$ is the cdf of the bivariate normal distribution where both arguments are standard normals and they have correlation ρ

Valuation formula for down and out barrier version of compound call option on the project

Using the method of images, the value of the down and out barrier option to delay the commencement of the flexible project can be expressed via the following formula (2.14). For ease of exposition we shall assume the barrier level *B* is lower than the critical gold price level *a* that makes the option to delay worth exercising. This means that the payoff from exercising the right to commence the project at time T_0 is

$\left(V_{T_0}\left(x\right)-K_0\right)^+ I_{(x>B)}=\left(V_{T_0}\left(x\right)-K_0\right)^+$ and that $V_B\left(x,t\right)=V\left(x,t\right)$

In turn this means that

$$V_{DO}(x,t) = V_B(x,t) - V_B^*(x,t) \qquad (2.14)$$

Which is a formula for the value of a down and out barrier compound call option over the value of the project for the case $B<a$. When $B>a$ we can write down a formula for $V_B(x,t) = e^{-r(T_0-t)}E\left[V(x,T_0)I(x>B))\right]$ and it will be similar to the formula for $V(x,t)$ above but the definitions of the d , d' terms will be different.

Adding additional complexity:

The above formulae involve an option to delay a project which itself includes options to expand or abandon the project at future times. The valuation formula is quite complicated and it involves the cumulative bivariate normal distribution. It is conceivable that if the project is expanded, the expanded part of the project may include further options to expand again, or to cancel the expansion. This would make the option to delay a compound option on a compound option. It is possible to derive a valuation formula for such an option but it is yet more complicated and the formula would involve many terms and the cumulative trivariate normal distribution. Methods for doing this have been developed by Skipper and Buchen (2009). We can then compute the value of a barrier version of this compound compound option using the method of images.

Numerical modeling:

The cumulative multivariate normal distribution can be computed only with numerical methods, in particular monte carlo simulation. As the level of complexity / flexibility allowed for in the project increases, analytic valuation formulae become intractable. Accordingly the valuation of the complex compound options and the barrier versions of them may should be done via a numerical method such as monte carlo simulation or the binomial method. Monte Carlo and binomial modelling of European Options requires modeling the asset price at a particular date or finite set of dates. However barrier options are path dependent options so monte carlo modeling of them is more difficult, requiring the modeling of the asset price trajectory over time instead of at a single time. The same applies to the binomial method. The method of images for pricing barrier options converts the complicated path dependent option into a combination of non path dependent European options and so simplifies the modeling substantially.

Discussion And Conclusions

We have demonstrated how we may value a gold mining project which includes the flexibility to expand or abandon at some future time, along with the value of the option to delay such a project and the value of a barrier version of such an option. The option to delay this flexible project can the thought of as a type of compound call option. The valuation of these projects can be done using analytic or numerical methods. As the amount of flexibility allowed for increases, so does the complexity of the valuation process and the need for numerical modeling increases. The method of images substantially simplifies the modeling and the valuation of “barrier” versions of real options when such complexity is present.

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