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Optimisation of proximity measures for aircraft with uniform 3D straight motion

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Abstract. The engineering specifications that define aircraft proximity management functions are currently under an intense review. Simultaneously, in an operational context, the frequency of aircraft proximity incidents is anticipated to increase due to the growth in traditional air travel, and the introduction of aerial vehicle operations (without a human pilot), and of personalised jets.

The design of the airspace rules and procedures needs to be dependable thus assuring that minimum miss-distances and response times will not be violated during all the close proximity situations that arise in actual operations. These important assurances need to be captured as design requirements and objectives (Fulton, 1999, 2002). The development of generalised optimisation strategies and mathematical solutions are used to baseline system performance.

An important operational issue in all air transport systems (it is equally applicable to sea transport) and a specific design requirement is to be able to determine the closest point of approach (CPA) (Tarnopolskaya, Fulton, 2008, 2009; Krozel, Peters, 1997) between aircraft trajectories. The CPA identifies a local minimum in relative range (miss-distance). Several systems have been proposed to manage the varying degrees of proximity arising. These are classified by the time required to reach CPA. The analysis in this paper is generally applicable for all time intervals and aircraft speeds that permit earth curvature effects to be ignored. It is most appropriate for systems that come under the generic classifications of Airborne Collision Avoidance System (ACAS) and Airborne Separation Assurance System (ASAS).

Two aircraft with uniform velocities are considered. The first aircraft, titled own-aircraft is the point of reference. The second aircraft is considered a threat or intruder for own-aircraft. The traditional formulation is presented in Section 2. It is posed in relative space using the state-vector $\mathbf{r} = (r, \phi, \theta)$ that specifies, in polar coordinates, the instantaneous relative range, bearing, and heading between the aircraft. Some recently published results (Gates et al, 2008) are refined and interpreted further. Then, in Section 3, the same situation is re-presented in a Cartesian Co-ordinate system such as an earth reference frame. This latter formulation is suitable for new systems due to the advent of the widespread use of satellite navigation systems which provide aircraft 3D position as (latitude, longitude, altitude) and 3D velocity vectors.

This paper is foundational to, and contributes towards, a more generalised concept of CPA where aircraft may have either linear or turning motion. In Section 3, a new approach to specifying proximity is used to provide deterministic methods to find the location of the CPA. A geometrical characterization of Fermat's method for stationary points in vector form leads to the identification of a fixed reference point that lies on the line that is the common perpendicular to the two flight-trajectories. The fixed reference point is then used to determine the location of the CPA between the aircraft trajectories. The method can be used to more accurately specify aircraft proximity management functions when developing algorithms either for aircraft avionics or for air traffic management systems.

Keywords: *Collision avoidance, closest point of approach, proximate.*

1. INTRODUCTION

Own-aircraft, F, is initially at a position, $\underline{P}_F$. An intruder aircraft, T, is initially at a position, $\underline{P}_T$. The situation of aircraft moving with 3D straight motion can be posed in a Cartesian frame with the aircraft positions and trajectories as shown in Figure 1. Assume that two 3D lines (trajectories) Γ and Δ are given and they are not parallel. Their vector equations are given by:

$$\begin{aligned} \Gamma: &\quad \underline{r} = \underline{P}_F + \underline{V}_F t \qquad \forall t \in \mathbb{R} \\ \Delta: &\quad \underline{r} = \underline{P}_T + \underline{V}_T t \qquad \forall t \in \mathbb{R} \end{aligned} \tag{1}$$

where $\underline{\boldsymbol{V}}_F$ and $\underline{\boldsymbol{V}}_T$ are the velocities of own-aircraft and the intruder respectively. The frame origin is set at point O_F, the intersection of the line, Γ, and its common perpendicular with the line, Δ. The X axis $\left(\overrightarrow{O_F X}\right)$ is aligned parallel to, and coincident with, the common perpendicular. The Y axis $\left(\overrightarrow{O_F Y}\right)$ is coincident with the line, Γ. Consequently, the XY-plane is the plane determined by the X axis and Y axis. And, the Z axis $\left(\overrightarrow{O_F Z}\right)$ is directed upwards, and orthogonal to the XY-plane. The line, Δ, makes an angle, γ, with the XY-plane. The point O_T, is the intersection point of the line, Δ, and the X axis. The angle, γ, is measured with respect to the normal mathematical convention with angles increasing in a positive (anti-clockwise) direction.

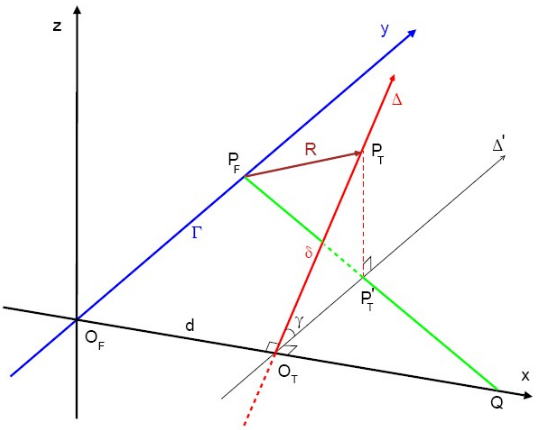

Figure 1: Construction for defining the miss distance between Γ and Δ.

2. AIRCRAFT WITH UNIFORM STRAIGHT MOTION IN 3D RELATIVE SPACE

Aircraft T appears to move with a relative velocity, $\underline{\boldsymbol{V}}_R$, with respect to own-aircraft given by:

$$\underline{V}_R = \underline{V}_T - \underline{V}_F = \left\|\underline{V}_R\right\| \underline{U}_R \tag{2}$$

where $\underline{U}_R$ is the unit vector for $\underline{V}_R$. The Line of Sight (LOS) vector is, the instantaneous relative displacement (range) between the aircraft. It is a function of time given by the fundamental vector equation:

$$\underline{R} = \underline{R}_0 + t\underline{V}_R = \left\|\underline{R}_0\right\| \underline{U}_{LOS} + tV_R \underline{U}_R \qquad where \qquad \underline{R}_0 = \underline{\mathrm{P}}_T - \underline{\mathrm{P}}_F \tag{3}$$

The initial unit vector for the LOS between the two aircraft is $\underline{U}_{LOS}$ as depicted in Figure 2 (A). In general the LOS vector will rotate as the intruder's position follows a path in 3D relative space. The LOS vector (of the threat aircraft, T, with respect to own-aircraft, F) rotates with an angular velocity, $\underline{\Omega}^{TF}$, given by:

$$\underline{\Omega}^{TF} = \frac{\underline{R} \times \underline{V}_R}{\left\|\underline{R}\right\|^2} \tag{4}$$

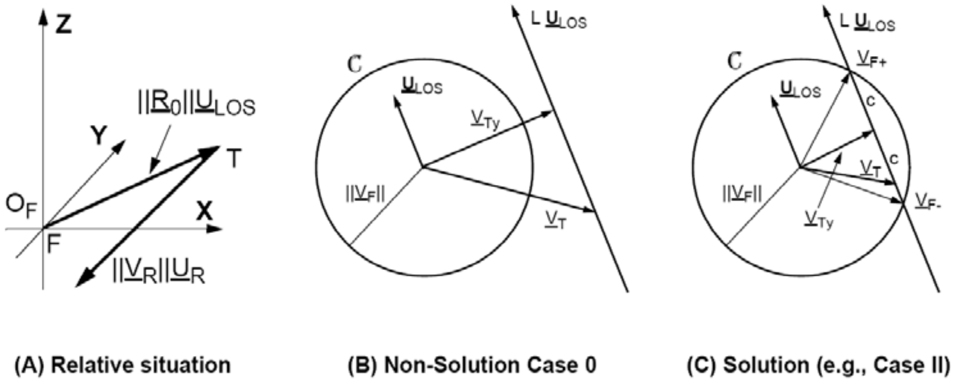

Figure 2: Geometry of relative situation

A more complete treatment of the rotation of the LOS vector formulated in terms of the angular momentum of a general rotating body for aircraft T is given in Cartesian tensor notation in Zipfel (2000).

The miss distance between the aircraft at the CPA is given in terms of the LOS vector by:

$$R_{MD} = \|\underline{R}_0\| \cdot \|\underline{U}_{LOS} \times \underline{U}_R\| \quad \Rightarrow \quad \underline{R}_{MD} = \|\underline{R}_0\| (\underline{U}_R \times (\underline{U}_{LOS} \times \underline{U}_R)) \tag{5}$$

A collision will occur for a non zero $\|\underline{R}_0\|$ when $\|\underline{R}_{MD}\| = 0$

$$\Rightarrow \|\underline{U}_{LOS} \times \underline{U}_R\| = 0 \quad \Rightarrow \quad \underline{U}_{LOS} = \pm \ \underline{U}_R \qquad \text{since } \|\underline{\boldsymbol{U}}_{\text{LOS}}\| = 1,\ \|\underline{\boldsymbol{U}}_{\text{R}}\| = 1, \text{ and } \underline{\boldsymbol{U}}_{\text{LOS}} \parallel \underline{\boldsymbol{U}}_{\text{R}}.$$

$$\Rightarrow \|\underline{V}_R\| \underline{U}_{LOS} = \pm \|\underline{V}_R\| \underline{U}_R = \pm \underline{V}_R \quad \text{and} \quad \underline{V}_F = \underline{V}_T \pm \|\underline{V}_R\| \underline{U}_{LOS} \tag{6}$$

It is similar to that which Gates et al. (2008) have shown. Alternatively, the solution vectors for a collision may be geometrically constructed as the intercept(s) of a circle, $\mathbb{C}$, of radius $\|\underline{\boldsymbol{V}}_{\text{F}}\|$ with a line drawn parallel to $\underline{\boldsymbol{U}}_{\text{LOS}}$ passing through the vector $\underline{\boldsymbol{V}}_{\text{TY}}$ whose origin coincides with the centre of $\mathbb{C}$ as shown in Figure 2. The infeasibility of solution can be readily established as shown in Figure 2 (B). Feasible solutions are shown in Figure 2(C) where $V_F^2 - V_{TY}^2 \geq 0$,

$$\underline{V}_F = \underline{V}_{TY} \pm c\underline{U}_{LOS} \quad \text{and} \quad \underline{V}_{TY} = \underline{U}_{LOS} \times \underline{V}_T \times \underline{U}_{LOS} \quad \text{and} \quad c = \sqrt{V_F^2 - V_{TY}^2} \tag{7}$$

$$\underline{V}_{F+} = \underline{V}_{TY} + c\underline{U}_{LOS} \quad \text{and} \quad \underline{V}_{F-} = \underline{V}_{TY} - c\underline{U}_{LOS} \tag{8}$$

The square of the norm of the relative range, $\|\underline{\text{R}}\|^2$ is given by:

$$\text{R}^2 = \|\underline{R} \bullet \underline{R}\| = \underline{V}_R \bullet \underline{V}_R t^2 + \underline{V}_R \bullet \underline{R}_0 t + \underline{R}_0 \bullet \underline{R}_0 \tag{9}$$

The time at which the relative range is stationary may be found by differentiation of Equation 9. This yields the first instance in this paper of Fermat's equation for stationary points (Sanford, 1930; Ball, 1960) – in calculus see Maak (1963, p82), and online Paolini (2003):

$$\frac{\text{d}}{\text{dt}}\|\underline{\text{R}}\|^2 = 2\underline{R}_0 \bullet \underline{V}_R + 2t\underline{V}_R \bullet \underline{V}_R = 0 \qquad \text{by } \underline{R}_0 \text{ and } \underline{V}_R \text{ being time independent.} \tag{10}$$

$$\Rightarrow \quad t = -\frac{\underline{R}_0 \bullet \underline{V}_R}{\|\underline{V}_R\|^2} \tag{11}$$

Equation 9 is a quadratic polynomial, with respect to t, that has a positive quadratic coefficient and therefore the stationary point identified by Equation 11 is a true minimum.

An indicator for closure is given by: $\kappa = \underline{U}_{LOS} \bullet (\underline{V}_T - \underline{V}_F) = \underline{U}_{LOS} \bullet \underline{V}_R$. The CPA will occur at a future time if κ is negative, that is, with a reducing relative range. If κ is positive, the CPA would have occurred in the motion at an earlier time. Otherwise κ is zero and the relative range does not change. The feasibility of solution is shown graphically in Figure 2 (B and C) and in tabular form in Table 1.

Table 1: Feasibility of solution with a collision

		Own-aircraft	
Intruder	Speed regime	V_{F-}	V_{F+}
CASE 0	$\|\|V_F\|\| < \|\|V_{TY}\|\|$	No Solution	No Solution
CASE I			
V_{T+}	$\|\|V_T\|\| < \|\|V_F\|\|$	Divergence	Tail-chase for F $\kappa < 0$
	$\|\|V_{TY}\|\| < \|\|V_F\|\| < \|\|V_T\|\|$	Divergence	Divergence
CASE II			
V_{T-}	$\|\|V_T\|\| < \|\|V_F\|\|$	Divergence	$\kappa < 0$
	$\|\|V_{TY}\|\| < \|\|V_F\|\| < \|\|V_T\|\|$	Tail-chase for T $\kappa < 0$	$\kappa < 0$

A pilot, when needing to avoid proximity with another aircraft, would like to know all possible velocity vectors which will bring own-aircraft within a given miss-distance of an intruder aircraft. Holding $\underline{\boldsymbol{V}}_T$ constant and using the relationship in Equation 8, a cone comprised of all possible $\underline{\boldsymbol{U}}_R$ vectors can be found by rotating $\underline{\boldsymbol{U}}_R$ about $\underline{\boldsymbol{U}}_{LOS}$. This is illustrated in Figure 3. The general equation of the surface of such a cone with half angle θ, axial vector, $\underline{\boldsymbol{a}}$, and vertex, $\underline{\boldsymbol{V}}_T$, is: $\underline{a} \bullet \left(\dfrac{\underline{r} - \underline{V}_T}{\left\| \underline{r} - \underline{V}_T \right\|} \right) = \cos(\theta)$

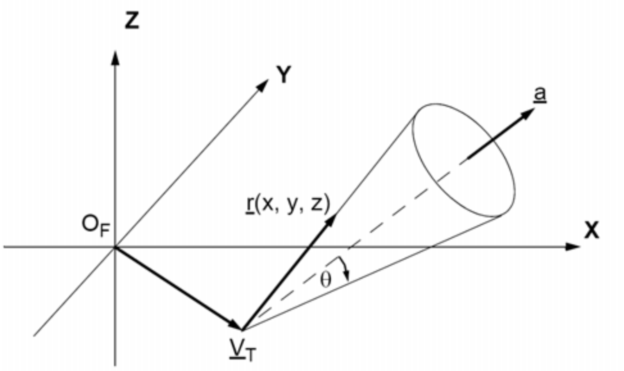

Figure 3: The $\underline{V}_R$ cone of velocities

The equation of a point $\mathbf{\underline{r}}$ (x, y, z) on the surface of this cone with x-axis aligned to $\underline{\mathbf{a}} = \underline{\boldsymbol{U}}_{LOS}$ and with vertex, $\underline{\boldsymbol{V}}_T$, is given by:

$$\hat{\underset{\sim}{i}} \bullet \left(\frac{(x - V_{Tx})\hat{\underset{\sim}{i}} + (y - V_{Ty})\hat{\underset{\sim}{j}} + z\hat{\underset{\sim}{k}}}{\sqrt{(x - V_{Tx})^2 + (y - V_{Ty})^2 + z^2}} \right) = \cos(\theta) \quad \text{by } \underline{V}_T = (V_{Tx}, V_{Ty}, 0) \tag{12}$$

$$\Rightarrow \quad \tan^2(\theta)\left(x - V_{Tx}\right)^2 = \left[(y - V_{Ty})^2 + z^2\right] \tag{13}$$

Now let $\beta = \tan(\theta)$ and $h = \left(x - V_{Tx}\right) \quad \Rightarrow \quad \beta^2 h^2 = (y - V_{Ty})^2 + z^2$. This is the equation of a circle with radius βh, centred at (h + V_{Tx}, V_{Ty}, 0). The circle lies in the $(\vec{O}_y, \vec{O}_z)$ plane located at (h + V_{Tx}, 0, 0). Therefore, $y = \left(y - V_{Ty}\right) = \beta h.\cos(\phi)$ and $z = \beta h.\sin(\phi) \quad \forall \phi \in \left[0, 2\pi\right]$

$$\Rightarrow \qquad (x, y, z) = \left(x + V_{TX},\ \beta h \cos(\phi) + V_{TY},\ \beta h \sin(\phi)\right) \tag{14}$$

where ϕ is the azimuth angle defining rotation about the axis of the cone. Since the equation for constant $\|\underline{V}_F\|$ is given by, $V_F^{\ 2} = x^2 + y^2 + z^2$ (a sphere), the contour of intersection between the cone of candidate $\underline{V}_R$ and all constant speed velocity vectors for $\|\underline{V}_F\|$ can now be determined as:

$$(h + V_{TX})^2 + (\beta h.\cos(\phi) + V_{TY})^2 + (\beta h.\sin(\phi))^2 = \|\underline{V}_F\|^2 \qquad (15)$$

$$\Rightarrow \qquad h^2(1+\beta^2) + 2h(V_{TX} + V_{TY}\cos(\phi)) + (\|\underline{V}_T\|^2 - \|\underline{V}_F\|^2) = 0$$

Evaluation of β in terms of miss distance, R_{MD}, and the initial line of sight distance, R_0, is given through the instantiation of the scalar product $R_0\underline{U}_{LOS} \bullet \underline{V}_R$ to yield,

$$(R_0\underline{U}_{LOS} \bullet \underline{V}_R)^2 = (R_0\underline{U}_{LOS} \bullet \|\underline{V}_R\|\underline{U}_R)^2 = (R_0\underline{U}_{LOS} \bullet \underline{U}_R)^2\|\underline{V}_R\|^2 = \|\underline{R}_0^2 - R_{MD}^2\|.\|\underline{V}_R\|^2$$

A simple expression for β may be obtained on expansion to components:

$$\beta = \sqrt{\frac{R_{MD}^2}{R_0^2 - R_{MD}^2}} \qquad (16)$$

In summary, assumptions underpinning existing approaches have been clarified, the partitioning of the problem domain has been made explicit in terms of characteristic parameters, and several derivations contained in recently published results (Gates et al, 2008) have been interpreted further and refined for robust implementation in safety critical systems.

3. AIRCRAFT UNIFORM STRAIGHT MOTION IN 3D INERTIAL SPACE

The analysis of proximity between the aircraft is now cast in an Earth referenced frame. This particular approach provides a deterministic method for finding the location of the CPA that can be readily applied in an operational context where earth referenced navigation systems such as Global Positioning System (GPS) are available. The proposed use of the method employed is to more accurately specify aircraft proximity management functions when developing algorithms either for aircraft avionics or for air traffic management systems. This paper contributes towards a generalised concept of CPA where aircraft may have either linear or turning motion.

The square of the norm of the relative range, $\|\underline{R}\|^2$, is used to define the problem. Fermat's equation for stationary points (Sanford, 1930; Ball, 1960; Maak, 1963) is then applied to this function leading to a complex transcendental equation. On differentiation of $\|\underline{R}\|^2$ one might expect a linear relationship to be embedded in the solution. Such linearity is evident and can be extracted by subtle use of a determinant structure when used to express the condition for the collinearity of three points. This approach leads to the identification of a fixed point that lies on the line that is the common perpendicular to the two flight-paths. The fixed point is then used to determine, through geometric construction, the location of the CPA between the aircraft being the minimum miss-distance in relative range. Simple procedures for calculating a closed form solution result and are presented.

3.1. Formulation as an optimisation problem

The situation of aircraft moving with 3D straight motion can be posed in an earth referenced frame with the aircraft positions presented in earth coordinates, as shown in Figure 1.

Clearly, $O_F = (0,0,0)$, $O_T = (d,0,0)$, $P_F = (0, y_F, 0)$, $P_T = (d, y_T, z_T)$, $d = \|\underline{d}\|$ and $R = \|\underline{R}\|$;

where $\underline{d} = \overrightarrow{O_F O_T}$ and $\underline{R} = \overrightarrow{P_F P_T}$. Thus, the equations for straight motion of aircraft are given by:

$$\begin{cases} y_F = V_F t + y_0 & (\text{the straight motion on } \Gamma) \\ \delta = V_T t + \delta_0 & (\text{the straight motion on } \Delta) \end{cases} \qquad (17)$$

where V_F, V_T, y_0 and δ_0 are constants.

$$\Rightarrow \qquad \begin{cases} y_T = \left(V_T \cos\gamma\right) t + \delta_0 \cos\gamma \\ z_T = y_T \tan\gamma \end{cases} \tag{18}$$

From $\quad R^2 = \|\underline{R}\|^2 \qquad \Rightarrow \qquad R^2 = d^2 + \left(y_F - y_T\right)^2 + y_T^2 \tan^2\gamma$ (19)

Thus, the miss-distance of aircraft can be determined if we can find values of y_F and y_T so as to minimise R^2. Clearly, this is a classical nonlinear and unconstrained optimisation problem.

3.2. The determination of the Closest Point of Approach (CPA) for aircraft with 3D straight motion

In this section, CPA is presented as a geometrical relationship between aircraft each with straight motion (flight paths). A classification of the possible proximity termination conditions is also presented. Further, the interpretation provided presents a generalisation of the CPA concept that can be extended to facilitate solutions for other flight manoeuvres such as turning flight or combinations of such manoeuvres. The total derivative of R^2 can be described as:

$$\frac{dR^2}{dt} = \frac{\partial R^2}{\partial y_F}\frac{dy_F}{dt} + \frac{\partial R^2}{\partial y_T}\frac{dy_T}{dt}$$

$$\text{where} \quad \begin{cases} \dfrac{\partial R^2}{\partial y_F} = 2\left(y_F - y_T\right) \\ \dfrac{\partial R^2}{\partial y_T} = -2\left(y_F - y_T\right) + 2y_T \tan^2\gamma \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dy_F}{dt} = V_F \\ \dfrac{dy_T}{dt} = V_T \cos\gamma \end{cases} \tag{20}$$

$$\Rightarrow \qquad \frac{dR^2}{dt} = 2V_F\left(y_F - y_T\right) - 2V_T \cos\gamma\left(y_F - y_T - y_T \tan^2\gamma\right) \tag{21}$$

Fermat’s method for stationary points (Sanford, 1930; Ball, 1960; Maak, 1963) states that stationary points of a function (in this case, R^2) can be found as the roots of the following equation:

$$\frac{d(R^2)}{dt} = 0 \tag{22}$$

$$\Rightarrow \qquad V_F\left(y_F - y_T\right) - V_T \cos\gamma\left(y_F - y_T - y_T \tan^2\gamma\right) = 0$$

Let $V = \dfrac{V_F}{V_T}$ and use the identity $\dfrac{1}{\cos^2\gamma} = 1 + \tan^2\gamma$, to obtain:

$$V\left(y_F - y_T\right) - \cos\gamma\left(y_F - \frac{y_T}{\cos^2\gamma}\right) = 0 \qquad \text{(provided } \cos\gamma \neq 0\text{)}$$

$$\Rightarrow \qquad \left(V - \cos\gamma\right) y_F - \left(V - \frac{1}{\cos\gamma}\right) y_T = 0 \tag{23}$$

This yields Fermat’s equation for stationary points when both aircraft have constant velocities.

$$\Rightarrow \qquad \left(V\cos\gamma - \cos^2\gamma\right) y_F + \left(1 - V\cos\gamma\right) y_T = 0 \tag{24}$$

$$\Rightarrow \qquad \left(1 - \frac{1 - V\cos\gamma}{\sin^2\gamma}\right) y_F + \left(\frac{1 - V\cos\gamma}{\sin^2\gamma}\right) y_T = 0 \qquad \text{(provided } \sin\gamma \neq 0\text{)} \tag{25}$$

Fermat’s equation can be cast in the form of a determinant that defines twice the size of the (signed) area of a planar triangle specified via its vertices: P_F, Q and $P_T^{'}$. The vertices are collinear if the triangle area is zero.
Thus,

$$\begin{vmatrix} 0 & y_F & 1 \\ \dfrac{d\left(1-V\cos\gamma\right)}{\sin^2\gamma} & 0 & 1 \\ d & y_T & 1 \end{vmatrix} = 0 \qquad \text{(by inspection)} \tag{26}$$

Clearly, for $Q = \left(\dfrac{d\left(1-V\cos\gamma\right)}{\sin^2\gamma},\ 0,\ 0\right)$ and $P_T' = \left(d, y_T, 0\right)$ then, P_F, Q and P_T' are collinear. Further, the point Q is called the fixed reference point for stationary states. This relationship shows that if one of the points P_F or P_T is known, then the remaining point can be determined such that R^2 is minimised**.** For example, if point P_F is given, then point P_T' is the intersection of the line QP_F with the line Δ'. Clearly, P_T' is the foot of the Z-Projection of point P_T onto the XY-Plane. Otherwise, if point P_T is given, then point P_T' is known as the foot of P_T by the Z-Projection. Thus, point P_F can be determined by the intersection of the line QP_T' with the line Γ. The geometrical solution is shown in Figure 1. This is the general solution for most straight motion cases. Three exceptions to the general solution that lead to special cases need to be considered separately by a similar approach as above. They are:

i. Orthogonal motion $\Gamma \perp \Delta$: $\gamma = \pm\dfrac{\pi}{2}$,

ii. Parallel motion $\Gamma \parallel \Delta$: $\gamma = 0$ *or* π, and

iii. Intersecting flight paths $\Gamma \cap \Delta = O_F$.

This paper provides a link between legacy approaches based on relative range and newer approaches that are based on navigational systems providing 3D position in an earth referenced frame. The mathematical approach presented here has been shown elsewhere to provide analytic solutions for aircraft with a mix of straight and turning flight, thus this approach can form the basis on which the concept of CPA can be generalised for practical applications.

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