Some new results for the optimal impulse control of Brownian motion

A. S. Hurn,

School of Economics and Finance,
Queensland University of Technology

K. A. Lindsay,

Department of Mathematics,
University of Glasgow

1 Introduction

Stochastic optimal control has become increasingly popular in economics and finance as a tool for modelling optimising behaviour within an environment of ongoing uncertainty. Its applications have been numerous, some prominent examples being menu cost models of investment and the transactions demand for money (see, for example, Constantinides and Richard 1978, Pindyck 1988, Smith 1989, Dixit 1991a and Dixit and Pindyck 1994). This article considers the optimal impulse control of a Brownian motion in the presence of holding costs and discrete costs of adjustment.

The contribution of the paper is threefold. *First*, the construction of the expected cost function for the impulse control function is derived by treating the problem as a first-stopping (or absorption) problem, followed by instantaneous restoration to an internal state. It is demonstrated that the expected costs satisfy the Hamilton-Jacobi-Bellman equation in keeping with traditional methods of impulse control. *Second*, the problem addressed by Dixit (1991b) is extended to deal with a more general class of Brownian motion and control costs. The growing and decaying exponential solutions that arise in Brownian motion with constant drift and noise are replaced by more general functions but the analysis underlying the optimal control problem is still applicable. *Third*, a numerical algorithm is developed to compute the optimal bounds for a general Brownian motion that is restored to a fixed state. The procedure is tested with reference to Dixit's (1991a) analytical approximation for Brownian motion with zero drift and constant variance, constant barrier costs and holding costs proportional to the square of the state. Results are provided for non-zero drift and other specifications of barrier and holding costs.

2 Preliminaries

Consider a general Brownian motion x_t that evolves according to the stochastic differential equation (SDE)

$$dx_t = a(x;\theta)\,dt + b(x;\theta)\,dW_t \qquad x(0) = X\,. \tag{1}$$

where dW_t is the increment in the Weiner process W_t. This unrestricted process is now controlled by barriers at $x = l$ and $x = u$ where $l < u$. When the process impinges on the lower barrier $x = l$, it is instantaneously restored to the interior point $x = L$ at cost $C_l(l, L)$. Similarly, when the upper boundary $x = u$ is encountered, the process is instantaneously restored to the interior point $x = U$ at cost $C_u(u, U)$. The situation is illustrated in Figure 1. Additional costs of holding resource x are incurred at rate $f(x)$. The problem is now to choose the parameters l, L, U and u so as to minimise the expected present value of the total cost starting with resource X given a discount rate ρ.

A solution of this problem was given by Dixit (1991b) who uniformly discretised the state space $[l, u]$ and treated the SDE (1) as a discrete random walk defined on the discretised states. The costs incurred at the boundaries $x = l$ and $x = u$ as result of the restoration to $x = L$ and $x = U$ entered the analysis through the forms for the transitional probabilities at the states corresponding to these points. The probability distribution across states imposed a corresponding distribution in costs from which the expected present value of all future costs could be determined. By refining the discretisation, Dixit showed that the expected cost $F(x)$ of starting at state $x = X$ satisfied the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{1}{2}\,b^2(x)\,\frac{d^2F(x;\theta)}{dx^2} + a(x)\frac{dF(x;\theta)}{dx} - \rho F(x;\theta) + f(x) = 0\,, \tag{2}$$

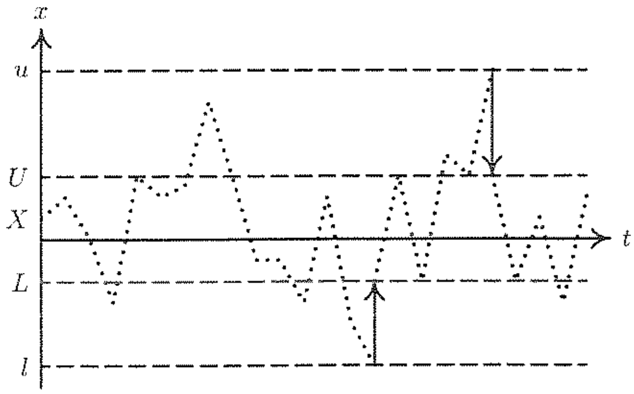

Figure 1: The Brownian motion $dx_t = a(x;\theta)\,dt + b(x;\theta)\,dW_t$ starting from $x = X$ is controlled by the upper barrier $x = u$ at which it is instantaneously restored to $x = U$ and the lower barrier $x = l$ at which it is instantaneously restored to $x = L$.

subject to suitable boundary conditions. Dixit (1991a) also derived this equation by using Ito's lemma, thus circumventing the need to discretise the state space.

It is now proposed to derive this equation by treating the impulse control of the Brownian motion as a first-stopping or absorption problem. The construction and subsequent analysis of the cost function, however, will require some preliminary results from the theory underlying stochastic differential equations of type (1). It is well known that the transitional probability density function $p(x,t \mid X,0)$ of equation (1) satisfies the forward Kolmogorov equation

$$\frac{\partial p(x,t\,|\,X,0)}{\partial t} = \frac{1}{2}\frac{\partial^2\left[b^2(x)p(x,t\,|\,X,0)\right]}{\partial x^2} - \frac{\partial\left[a(x)p(x,t\,|\,X,0)\right]}{\partial x}\,. \tag{3}$$

In subsequent analysis, it is useful to recognise that this equation can be re-expressed in the conservation form

$$\frac{\partial p(x,t \mid X,0)}{\partial t} + \frac{\partial q(x,t \mid X,0)}{\partial x} = 0\,. \tag{4}$$

Self evidently, $q(x,t \mid X,0)$ is defined in terms of $p(x,t \mid X,0)$ by

$$q(x,t) = -\frac{1}{2}\frac{\partial}{\partial x}\left[b^2(x)p(x,t\,|\,X,0)\right] + a(x)p(x,t\,|\,X,0) \tag{5}$$

and is the flux of probability flowing in the positive x direction at time t and state x. To appreciate this definition, it is sufficient to observe that if $[x_{\mathrm{L}}, x_{\mathrm{R}}]$ is a fixed interval of state space then the mass of probability contained within this interval evolves in time according to the rule

$$\frac{d}{dt}\int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} p(x,t \mid X,0)\,dx = q(x_{\mathrm{L}},t \mid X,0) - q(x_{\mathrm{R}},t \mid X,0)\,.$$

Closely related to the forward Kolmogorov equation (3) is the backward Kolmogorov equation

$$\frac{\partial p(x,t\,|\,X,0)}{\partial t} = \frac{1}{2}\,b^2(X)\,\frac{\partial^2 p(x,t\,|\,X,0)}{\partial X^2} + a(X)\frac{\partial p(x,t\,|\,X,0)}{\partial X} \tag{6}$$

describing the behaviour of the transitional density function with respect to the initial state X. In terms of the backward spatial operator

$$\mathcal{L}\left[\psi\right] = \frac{1}{2}\,b^2(X)\,\frac{\partial^2\psi}{\partial X^2} + a(X)\frac{\partial\psi}{\partial X}\,, \tag{7}$$

the backward Kolmogorov equation and the probability flux satisfy respectively

$$\mathcal{L}\left[p(x,t\,|\,X,0)\right] = \frac{\partial p(x,t\,|\,X,0)}{\partial t}\,, \qquad \mathcal{L}\left[q(x,t\,|\,X,0)\right] = \frac{\partial q(x,t\,|\,X,0)}{\partial t}\,. \tag{8}$$

3 The cost function

Consider now the situation in which the process described by SDE (1) is controlled by two absorbing barriers at $x = l$ and $x = u$ where $l < u$. The transitional density function $p(x,t\,|\,X,0)$ for this absorption process, conditional on $x(0) = X$, may be obtained by solving Kolmogorov's equation (3) in the region $(x, t) \in (l, u) \times (0, \infty)$ with boundary conditions

$$p(l, t \mid X, 0) = 0 , \qquad p(u, t \mid X, 0) = 0 \tag{9}$$

and initial condition

$$p(x, 0 \mid X, 0) = \delta(x - X) . \tag{10}$$

The differential equation satisfied by the expected cost function will now be determined by partitioning costs as follows:

(a) expected discounted holding costs for processes prior to their absorption;

(b) expected discounted cost of exercising control at the upper barrier $x = u$, including all subsequent costs, $F(U)$, which necessarily follow as a consequence of this action;

(c) expected discounted cost of exercising control at the lower barrier $x = l$, including all subsequent costs, $F(L)$.

The expected cost of starting at state X is therefore given by

$$\begin{aligned} F(X) &= \int_0^\infty e^{-\rho t} \left(\int_l^u f(\xi) p(\xi, t \mid X, 0) \, d\xi \right) dt \\ &\quad + \int_0^\infty e^{-\rho t} \Big[C_u(u, U) + F(U) \Big] q(u, t \mid X, 0) \, dt \\ &\quad - \int_0^\infty e^{-\rho t} \Big[C_l(l, L) + F(L) \Big] q(l, t \mid X, 0) \, dt . \end{aligned} \tag{11}$$

It is now demonstrated that $F(X)$ satisfies the HJB equation (2). Applying the operator $\mathcal{L}$ to equation (11) yields

$$\begin{aligned} \mathcal{L}\left[F(X)\right] &= \int_0^\infty e^{-\rho t} \left(\int_l^u f(\xi) \mathcal{L}\left[p(\xi, t \mid X, 0)\right] d\xi \right) dt \\ &\quad + \Big[C_u(u, U) + F(U) \Big] \int_0^\infty e^{-\rho t} \, \mathcal{L}\left[q(u, t \mid X, 0)\right] dt \\ &\quad - \Big[C_l(l, L) + F(L) \Big] \int_0^\infty e^{-\rho t} \, \mathcal{L}\left[q(l, t \mid X, 0)\right] dt , \end{aligned} \tag{12}$$

where it is understood that all derivatives in $\mathcal{L}$ are total when applied to $F(X)$. Since

$$\begin{gathered} \mathcal{L}\left[p(\xi, t \mid X, 0)\right] = \frac{\partial p(\xi, t \mid X, 0)}{\partial t} , \\ \mathcal{L}\left[q(u, t \mid X, 0)\right] = \frac{\partial q(u, t \mid X, 0)}{\partial t} , \quad \mathcal{L}\left[q(l, t \mid X, 0)\right] = \frac{\partial q(l, t \mid X, 0)}{\partial t} \end{gathered} \tag{13}$$

then

$$\begin{aligned} \mathcal{L}\left[F(X)\right] &= \int_0^\infty e^{-\rho t} \left(\int_l^u f(\xi) \frac{\partial p(\xi, t \mid X, 0)}{\partial t} \, d\xi \right) dt \\ &\quad + \Big[C_u(u, U) + F(U) \Big] \int_0^\infty e^{-\rho t} \, \frac{\partial q(u, t \mid X, 0)}{\partial t} \, dt \\ &\quad - \Big[C_l(l, L) + F(L) \Big] \int_0^\infty e^{-\rho t} \, \frac{\partial q(l, t \mid X, 0)}{\partial t} \, dt . \end{aligned} \tag{14}$$

Using integration by parts, equation (14) can be reworked into the format

$$\begin{aligned} \mathcal{L}\left[F(X)\right] &= -\int_l^u f(\xi) p(\xi, 0 \mid X, 0) \, d\xi - \Big[C_u(u, U) + F(U) \Big] q(u, 0 \mid X, 0) \\ &\quad + \Big[C_l(l, L) + F(L) \Big] q(l, 0 \mid X, 0) + \rho F(X) . \end{aligned} \tag{15}$$

Since $p(\xi, 0 \mid X, 0) = \delta(\xi - X)$ and $q(u, 0 \mid X, 0) = q(l, 0 \mid X, 0) = 0$ then $F(X)$ is seen to satisfy the HJB equation

$$\frac{1}{2}\, b^2(X)\, \frac{d^2F}{dX^2} + a(X)\frac{dF}{dX} - \rho F + f(X) = 0\,. \tag{16}$$

The boundary conditions to be satisfied by F are obtained directly from equation (11), applied at the upper and lower barriers. These are the familiar value-matching conditions

$$F(u) = C_u(u, U) + F(U)\,, \qquad\qquad F(l) = C_l(l, L) + F(L)\,. \tag{17}$$

Therefore the final formulation of the problem based on absorption methods is identical to that which arises using the traditional methods of impulse control.

4 Optimal choice of parameters

It is evident from (16) and conditions (17) that the final solution for $F(X)$ contains X and four parameters l, L, U and u which must be chosen to minimise F. Let $V(x)$ be the cost function, in the absence of controls, for all processes starting at state x then

$$V(x) = \mathrm{E}\left[\int_0^\infty e^{-\rho t} f(x(t))\, dt\right]\,. \tag{18}$$

The general solution of (16) may be expressed in the form

$$F(x) = D_1 v(x) + D_2 w(x) + V(x) \tag{19}$$

where D_1 and D_2 are arbitrary constants to be determined from the boundary conditions (17) and v and w are two independent solutions of the homogeneous equation

$$\frac{1}{2}\, b^2(x)\, \frac{d^2\phi(x)}{dx^2} + a(x)\frac{d\phi(x)}{dx} - \rho\phi(x) = 0\,. \tag{20}$$

In particular, $v(x)$ and $w(x)$ can be found so that

(a) $v(x)$ is a strictly increasing solution of (20) satisfying $v(l) = 1$;

(b) $w(x)$ is a strictly decreasing positive solution of (20) satisfying $w(l) = 1$.

These claims will be discussed later but are consistent with the increasing and decreasing exponential solutions found by Dixit (1991b) for equation (20) when $a(x)$ and $b(x)$ are constant functions.

4.1 Derivation of the smooth-pasting conditions

The conditions (17) now indicate that D_1 and D_2 are to be found by solving the simultaneous equations

$$\begin{aligned} D_1\big[v(l) - v(L)\big] + D_2\big[w(l) - w(L)\big] &= C_l(l, L) - V(l) + V(L)\,, \\ D_1\big[v(u) - v(U)\big] + D_2\big[w(u) - w(U)\big] &= C_u(u, U) - V(u) + V(U)\,. \end{aligned} \tag{21}$$

Let auxiliary variables Z_1 and Z_2 be defined by

$$Z_1 = C_l(l, L) - V(l) + V(L)\,, \qquad Z_2 = C_u(u, U) - V(u) + V(U) \tag{22}$$

then equations (21) can be solved for D_1 and D_2 to give

$$\begin{aligned} D_1 &= \frac{1}{|A|}\Big(Z_1[w(u) - w(U)] - Z_2[w(l) - w(L)]\Big) \\ D_2 &= \frac{1}{|A|}\Big(-Z_1[v(u) - v(U)] + Z_2[v(l) - v(L)]\Big) \end{aligned} \tag{23}$$

where $|A| = [v(l) - v(L)][w(u) - w(U)] - [v(u) - v(U)][w(l) - w(L)]$ is the determinant of the matrix of the linear system (21). Knowledge of D_1 and D_2 allows the cost function F to be expressed in terms of the initial state x and the parameters l, L, U and u. It is required to optimise F by judicious choice of

l, L, U and u. To illustrate the general procedure, consider the variation of F with respect to u with x, l, L and U held constant. Clearly

$$\frac{\partial F}{\partial u} = \frac{\partial D_1}{\partial u}\, v(x) + \frac{\partial D_2}{\partial u}\, w(x)\,. \tag{24}$$

By differentiating the first of equations (21) with respect to u, it is clear that

$$\frac{\partial D_1}{\partial u}\,[v(l) - v(L)] + \frac{\partial D_2}{\partial u}\,[w(l) - w(L)] = 0$$

from which it now follows that

$$[w(l) - w(L)]\,\frac{\partial F}{\partial u} = \frac{\partial D_1}{\partial u}\left(v(x)[w(l) - w(L)] - w(x)[v(l) - v(L)]\right).$$

Since v is a positive increasing function and w is a positive decreasing function over $[l, u]$, then the products $v(x)[w(l) - w(L)]$ and $-w(x)[v(l) - v(L)]$ are both positive. Therefore $\partial F/\partial u = 0$ if and only if $\partial D_1/\partial u = 0$. Consequently the stationary points of F with respect to variations in u may be determined by solving $\partial D_1/\partial u = 0$ (or equivalently, $\partial D_2/\partial u = 0$). Straightforward calculation gives

$$\frac{\partial D_1}{\partial u} = \frac{w(l) - w(L)}{|A|}\left(D_1\frac{dv(u)}{du} + D_2\frac{dw(u)}{du} - \frac{\partial C_u(u,U)}{\partial u} + \frac{dV(u)}{du}\right)$$

so that

$$\frac{\partial D_1}{\partial u} = 0 \quad\Longleftrightarrow\quad D_1\frac{dv(u)}{du} + D_2\frac{dw(u)}{du} - \frac{\partial C_u(u,U)}{\partial u} + \frac{dV(u)}{du} = 0\,.$$

However

$$\frac{\partial F(x; l, L, U, u)}{\partial x} = D_1\frac{dv(x)}{dx} + D_2\frac{dw(x)}{dx} + \frac{dV(x)}{dx}$$

and therefore

$$\frac{\partial D_1}{\partial u} = 0 \quad\Longleftrightarrow\quad \left.\frac{\partial F(x; l, L, U, u)}{\partial x}\right|_{x=u} = \frac{\partial C_u(u,U)}{\partial u}\,. \tag{25}$$

This completes the formal derivation of one of the familiar smooth-pasting conditions from the value matching conditions (17). A similar analysis reveals that the three remaining conditions are:

$$\frac{\partial D_1}{\partial U} = 0 \quad\Longleftrightarrow\quad \left.\frac{\partial F(x; l, L, U, u)}{\partial x}\right|_{x=U} = -\frac{\partial C_u(u,U)}{\partial U}\,, \tag{26}$$

$$\frac{\partial D_1}{\partial l} = 0 \quad\Longleftrightarrow\quad \left.\frac{\partial F(x; l, L, U, u)}{\partial x}\right|_{x=l} = \frac{\partial C_l(l,L)}{\partial l}\,, \tag{27}$$

$$\frac{\partial D_1}{\partial L} = 0 \quad\Longleftrightarrow\quad \left.\frac{\partial F(x; l, L, U, u)}{\partial x}\right|_{x=L} = -\frac{\partial C_l(l,L)}{\partial L}\,. \tag{28}$$

4.2 Properties of $v(x)$ and $w(x)$

The general proof of optimality relies on the facts that $v(x)$ and $w(x)$ are positive solutions of (20), the first being an increasing function of x and the second being a decreasing function of x. Consider $v(x)$ first. It starts at $v(l) = 1$ with a positive gradient. Suppose that $x = \eta > l$ is the first point at which the gradient of v is zero, then $v''(\eta) \leq 0$ since the gradient is decreasing in the vicinity of η. But $v(\eta) > 0$ and therefore the requirement that $v(x)$ satisfies (20) at $x = \eta$ leads immediately to a contradiction since

$$0 = \frac{1}{2}\,b^2(\eta)\,\frac{\partial^2 v(\eta)}{\partial \eta^2} + a(\eta)\frac{\partial v(\eta)}{\partial \eta} - \rho v(\eta) = \frac{1}{2}\,b^2(\eta)\,\frac{\partial^2 v(\eta)}{\partial \eta^2} - \rho v(\eta) < 0\,.$$

Therefore $v(x)$ is a positive increasing function of x.

The argument for $w(x)$ is more subtle. It is first recognised that every function $w(x)$ satisfying $w(l) = 1$ with $w'(l) < 0$ remains positive under all circumstances. To appreciate this fact, suppose that $x = \eta$ is the first point at which $w'(\eta) = 0$ so that $w(x)$ has decreased monotonically prior to $x = \eta$ and therefore $w''(\eta) \geq 0$. However $w(x)$ satisfies (20) at $x = \eta$ and so

$$0 = \frac{1}{2}\,b^2(\eta)\,\frac{\partial^2 w(\eta)}{\partial \eta^2} + a(\eta)\frac{\partial w(\eta)}{\partial \eta} - \rho w(\eta) \Rightarrow w(\eta) = \frac{b^2(\eta)}{2\rho}\,\frac{\partial^2 v(\eta)}{\partial \eta^2} \geq 0\,.$$

If $w(\eta) > 0$ then $w''(\eta) > 0$ and $w'(x)$ is increasing at $x = \eta$. The argument applied to v now indicates that $w(x)$ increases monotonically for $x \geq \eta$. This establishes the fact that $w(x)$ is always positive. The reason why the function increases is that the particular solution contains a component of $v(x)$ which dominates for large x. There will, however, be a unique negative gradient for which this component is inactive and in this case $w(x)$ will decrease monotonically to 0 as $x \to \infty$; in effect $\eta = \infty$.

5 Numerical computation of optimal barriers

To illustrate the procedure by which the optimal barriers may be obtained, it is convenient to consider the simplest situation in which restoration is to a fixed point $x = R$. It is now required to determine the lower boundary $x = l$ and the upper boundary $x = u$, so as to minimise expected total costs $F(x)$. The optimal barriers are found by solving

$$\frac{b^2(x)}{2}\frac{d^2F}{dx^2} + a(x)\frac{dF}{dx} - \rho F + f(x) = 0 \tag{29}$$

subject to the requirements

$$F(l) - F(R) = C_l(l, R) \qquad \left.\frac{dF}{dl}\right|_{x=l} = \frac{\partial C_l(l,R)}{\partial l} \tag{30}$$

on the lower boundary and

$$F(u) - F(R) = C_u(u, R) \qquad \left.\frac{dF}{du}\right|_{x=u} = \frac{\partial C_u(u,R)}{\partial u}\,, \tag{31}$$

on the upper boundary. The numerical algorithm proceeds as follows. Equation (29) is converted into the first-order system

$$\frac{dy_1}{dx} = y_2\,, \qquad \frac{dy_2}{dx} = \frac{2(\rho y_1 - f(x) - a y_2)}{b^2} \tag{32}$$

where y_1 and y_2 are defined in terms of F by

$$y_1 = F(x)\,, \qquad y_2 = \frac{dF}{dx}\,. \tag{33}$$

Now notice that (30) and (31) specify four conditions, two of which would be sufficient to solve (29); the other two are satisfied only by the optimal choice of l and u. Essentially the problem may be regarded as one in which two functions (in this instance defined by two boundary conditions) in two unknowns (l and u) are to be solved. In general, this is an awkward problem. The proposed methodology uses the boundary conditions to construct an analytic function, $T(z)$, of the complex variable $z = l + iu$. Figure 2 describes the numerical algorithm when boundary conditions on $x = u$ are used to construct $T(z)$. To be specific

$$T(z) = \Big(F(u) - F(R) - C_u(u,R)\Big) + i\Big(\frac{dF(u)}{du} - \frac{\partial C_u(u,R)}{\partial u}\Big)\,. \tag{34}$$

The computation of $T(z)$ is done in two stages. In the first stage of the calculation, equations (32) are integrated from the lower boundary $x = l$ to $x = R$, the point of restoration. In order to satisfy conditions (30), $F(l)$ is guessed initially and this guess is taken as the starting value for y_1. The starting value for y_2 is already known from (30). By systematically changing $F(l)$ (for example, using the secant algorithm described below in the complex case), a value for $F(l)$ is found for which the integrated solution to equations (32) satisfies condition (30). On the successful completion of this first stage, $F(R)$ and dF/dR are determined and become the initial conditions for the integration of the differential equations (32) from $x = R$ to the upper barrier $x = u$. On completion of this second integration, the resulting values of $F(u)$ and dF/du are used to calculate $T(z)$ from (34). Recall that only the optimal boundaries satisfy $T(z) = 0$; other choices of z make $T(z)$ non-zero.

The solution of $T(z) = 0$ is found using the secant algorithm but applied to a complex function. Suppose that z_1 and z_2 are two choices for the optimal barriers, then the secant algorithm gives

$$z_{\text{new}} = z_2 - \frac{T_2(z_2 - z_1)}{T_2 - T_1} \tag{35}$$

as an improved estimate for the solution of $T(z) = 0$ where $T_1 = T(z_1)$ and $T_2 = T(z_2)$. This estimate is accepted as optimal provided two sequential estimates z_1 and z_2 are sufficiently close, or the algorithm

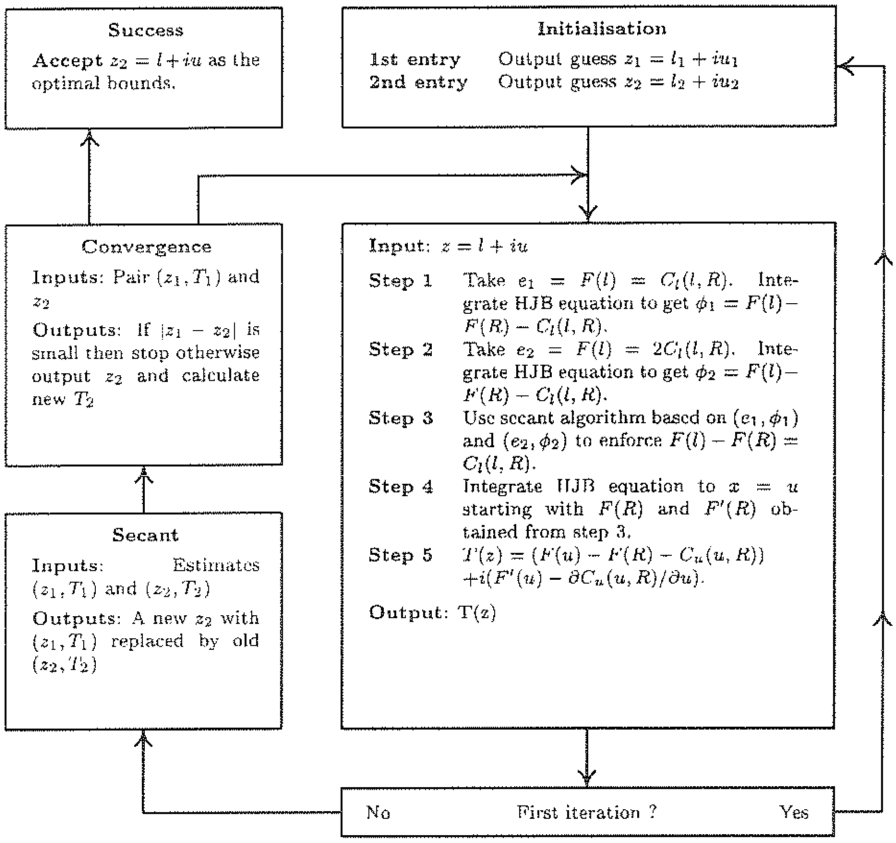

Figure 2: The global structure of the numerical procedure to estimate the optimal barriers $x = l$ and $x = u$ for a controlled Brownian motion that is restored instantaneously from the barriers $x = l$ or $x = u$ to $x = R$.

continues. In the latter case, z_1 and T_1 are discarded and replaced by z_2 and T_2. The old value of z_2 is now replaced by z_{new} and the corresponding value for T_2 is recomputed. The algorithm converges effectively and delivers the optimal boundaries l and u efficiently.

Specimen numerical calculations were carried out for the Brownian motion

$$dx_t = \mu\, dt + \sigma\, dW_t$$

in which μ and σ take constant values. All calculations assume that future costs are discounted at rate $\rho = 0.05$. Barrier costs $C_u(u, U)$ and $C_l(l, L)$ were chosen to be

$$C_u(u, U) = G + B(u - U)\,, \qquad C_l(l, L) = G + B(L - l)$$

with $G = 0.01$ and $B = 0.01$ while holding costs were modelled by $f_1(x) = kx^2$ in the first application and $f_2(x) = k|x|$ in the second application with $k = 0.1$ in both instances. As an initial check of the numerical algorithm, the optimal bounds for the restore-to-zero problem in the absence of drift, μ, and proportional transaction costs, B, was calculated. These barriers were compared with Dixit's (1991b) well known estimate $h = (6\sigma^2 G/k)^{1/4}$ based on the premise that the optimal barriers are symmetrical about zero, the point of restoration. For $\sigma = 0.05$ and $\sigma = 0.1$, the numerical estimates are 0.2045 and 0.2838 respectively. Although not constrained to be symmetrical in the numerical computation, the optimal barriers do in fact exhibit symmetry. The numerical estimates compare very favourably with 0.1968 and 0.2783 from Dixit's approximate formula.

Figure 3 illustrates the behaviour of the optimal estimates for $\mu \in [0.0, 0.1]$ and all other parameters taking the values specified previously. Clearly the optimal barriers in the presence of drift are no longer symmetrical. Both examples indicate that, in the presence of positive drift, the lower barrier is substantially relaxed while the upper barrier is relatively more tight. At first sight this latter result may appear counter intuitive, in the respect that the upper barrier is encountered more frequently resulting

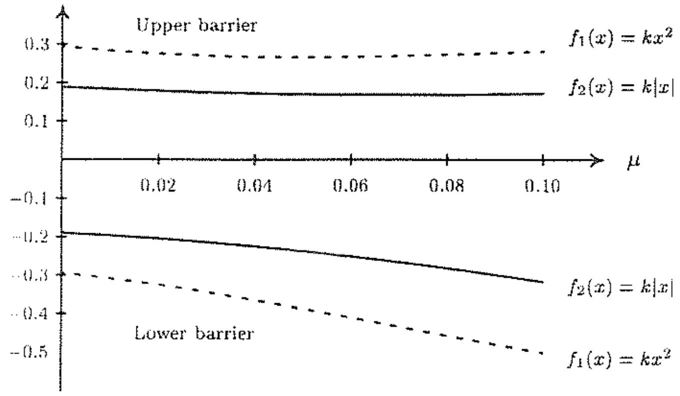

Figure 3: Optimal upper and lower barriers are shown for the holding costs $f_1(x) = kx^2$ and $f_2(x) = k|x|$ as μ ranges through $[0.0, 0.1]$. These calculations assume $G = B = 0.01$, $\sigma = 0.1$ and $\rho = 0.05$.

in increased control costs. However, the positive drift will imply rapidly rising holding costs if the upper barrier were relaxed. There is also evidence to suggest that the dependence of the upper barrier on μ is consistent with the familiar polynomial-like behaviour found by Baumol (1952) and Miller and Orr (1966). Although not illustrated in Figure 3, the effect becomes more pronounced as drift increases. This is to be expected since the problem becomes more deterministic for large values of μ. Finally, the choice of $f_2(x) = k|x|$ is more punitive a specification of holding cost than $f_1(x) = kx^2$ for small x. Consequently, the barriers for f_2 are tighter than those for f_1.

6 Conclusion

This article has described how the optimal impulse control of Brownian motion can be treated as an absorption problem. It is demonstrated that the expected cost function obtained by this means satisfies the familiar Hamilton-Jacobi-Bellman equation. Furthermore, the value-matching conditions are shown to lead naturally to the smooth-pasting conditions for optimal control of the class of Brownian motion considered here. A numerical procedure is proposed to investigate the behaviour of optimal barriers in general problems where symmetry cannot be assumed. The results obtained from the application of the algorithm are in agreement with those derived from existing analytical solutions. In the reported results for problems where such solutions are difficult to obtain or non-existent, the numerical solutions behave sensibly and accord with existing intuition.

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