SOLVING THE DYNAMICS OF A NON-LINEAR REPRESENTATIVE AGENT MODEL

by

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ABSTRACT

This paper describes two alternative approaches (modified reverse shooting and forward shooting) for solving the time-path of a representative agent model following an exogenous shock. In particular, reverse shooting is demonstrably better for solving part of the model but must be modified before it can be used to solve the full model. On the other hand, an unmodified form of forward shooting can be used to solve both part and full models

1. INTRODUCTION

In this paper we consider the computational aspects of solving a well-known representative agent model (Matsuyama, 1987), which has a number of important dynamic properties. These properties have significant implications, common to a range of macroeconomic models, for computing the model solution.

Firstly the model has a number of stable and unstable trajectories so that it is likely to be complicated to solve the model for a stable solution. The economy is initially at a stable steady-state equilibrium, and when shocked by, say, an exogenous change in world interest rates, then it moves to a stable trajectory leading to a new steady-state equilibrium. The movement to the new equilibrium is assumed to come about as a consequence of optimising behaviour of the agents in the model. In the model, certain variables 'jump' instantaneously after the shock, and force the economy onto the trajectory leading to the stable equilibrium.

A second property of the model is that it is nonlinear with nonlinearities arising as a direct consequence of optimising behaviour by the representative agents. The usual approach is to linearise the model in the neighbourhood of the steady-state and then solve the linearised model. This approach can be particularly unreliable if the 'jumps' required to bring the economy back onto a stable path are particularly large.

These properties, especially the property of 'jumps' to the stable path, introduce some interesting challenges to solving the model.

2. THE MODEL

The chosen model is a real model of a small open economy in a one-product world. The economy is assumed to be so small in the international market for tradeable goods that it is a price-taker in the market for foreign exchange. Agents in the economy also face perfect capital markets and a given world interest rate, r.

There are four sectors in this economy: the corporate sector, the household sector, the government sector and the external sector. These four sectors can be aggregated to yield a model of a small open economy given by the following set of equations:

$$\dot{q} = \{r - b[\Lambda(q)^2]\}q - F_K \quad (1a)$$

$$\frac{\dot{K}}{K} = \Lambda(q)[1 - b\Lambda(q)] \quad (1b)$$

$$\dot{C} = (r-\theta)C - p(p+\theta)[qK - D + B] \quad (1c)$$

$$\dot{D} = C - F(K,1) + G + K\Lambda(q) + rD \quad (1d)$$

where
F = F(K,L) = $aK^{\alpha}L^{1-\alpha}$ = output of the firm;
K = real capital stock;
L = 1 = demand for labour;
q = average (and marginal) Tobin's q;
$\Lambda(q) = \dfrac{q-1}{2bq}$;
C = real aggregate consumption;
r = real world interest rate (assumed exogenous);
p = instantaneous probability of death per unit time for representative consumers;
6 = consumer's rate of time preference;
D = overseas debt;
B = domestic holdings of government bonds;
G = government expenditure (assumed exogenous and fixed).

There are four endogenous variables in the model, given by q, K, C and D. The other parameters and variables given by p, 6, r, b, G and B are exogenously fixed. All variables and parameters as well as the functional form of $\Lambda(q)$ have been defined above.

Throughout this paper it will be assumed that the model has been calibrated using plausible parameter values. The objective of this paper is then to find a suitable solution approach that will define the trajectory of a stable No-Ponzi game solution.

Dynamic properties of linearised model

Linearising the model in the neighbourhood of the steady state yields the following fourth-order linear dynamic system with an asterisk indicating a corresponding steady-state value.

$$\begin{bmatrix} \dot{q} \\ \dot{K} \\ \dot{C} \\ \dot{D} \end{bmatrix} = \begin{bmatrix} r & -F_{KK} & 0 & 0 \\ \dfrac{K^*}{2b} & 0 & 0 & 0 \\ -p(p+\theta)K^* & -p(p+\theta) & r-\theta & p(p+\theta) \\ \dfrac{K^*}{2b} & -r & 1 & r \end{bmatrix} \begin{bmatrix} q-q^* \\ K-K^* \\ C-C^* \\ D-D^* \end{bmatrix} \tag{2}$$

Expressing the linearised model in this way clearly indicates that the model has a block-recursive structure, where the dynamics of q and K can be solved independently of the dynamics of C and D. This means that the dynamic model can be solved in two steps, first solving the investment sub-model, which defines a second-order system in q and K. The full model can then be solved by substituting solutions for q and K into the $\dot{C}$ and $\dot{D}$ equations and then solving for the second-order system in C and D. This two-step solution approach can also be applied to the original (non-linear) model given by equations 1a – 1d.

A general idea about the stability properties of the original (non-linear) model can be obtained by examining the stability properties of the linearised system. The eigenvalues for this system are given by:

$$\lambda_1, \lambda_2 = \frac{r \pm \sqrt{r^2 - \dfrac{2F_{KK}K^*}{b}}}{2} \tag{3a}$$

$$\lambda_3 = r + p \tag{3b}$$

$$\lambda_4 = r - 6 - p \tag{3c}$$

Since $F_{KK} < 0$, equation 3a defines two real-valued eigenvalues, one positive and one negative. Henceforth, it is assumed that $\lambda_1 > r > 0 > \lambda_2$. Also, if it is assumed that $6 < r < 6 + p$ then $\lambda_3 > 0 > \lambda_4$. Then, assuming that $\lambda_2 \neq \lambda_4$, there are two real-valued positive eigenvalues given by λ_1 and λ_3, and two distinct real-valued negative eigenvalues, given by λ_2 and λ_4.

Thus both the investment sub-model and the C and D components of the full model will have one positive and one negative eigenvalue, thereby exhibiting the property of saddle-path instability. As a consequence, following an exogenous shock to the system, it will be necessary for one of the K and q variables and one of the C and D variables to jump instantaneously so as to ensure stability of the solution. Since K and D are stock variables, which cannot jump instantaneously in this model, it is appropriate that q and C should be the jump variables. These properties of the linearised model carry over to the original (non-linear) model.

3. SOLVING THE INVESTMENT SUB-MODEL

The investment sub-model is given by equations 1a and 1b, with the corresponding linearised model being given by:

$$\begin{bmatrix} \dot{q} \\ \dot{K} \end{bmatrix} = \begin{bmatrix} r & -F_{KK} \\ \dfrac{K^*}{2b} & 0 \end{bmatrix} \begin{bmatrix} q-q^* \\ K-K^* \end{bmatrix} \tag{4}$$

As demonstrated above, this second-order dynamic system has two eigenvalues, given by equation 3a. Hence the linearised system has two real-valued eigenvalues given by $\lambda_1 > r > 0 > \lambda_2$ thereby exhibiting the property of saddle-path instability.

Solutions to the investment sub-model starting from a range of initial conditions can be used to derive a phase diagram for the dynamics of the investment sub-model of the original (non-linear) model. The differential equations 1a and 1b that define the investment sub-model are non-linear. For this reason, at each set of initial conditions, a variable step size Runge-Kutta algorithm provides an appropriate solution method.

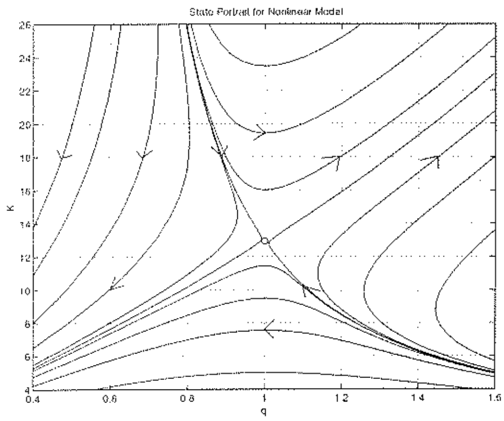

FIGURE 1
SAMPLE PHASE DIAGRAM
INVESTMENT SUB-MODEL

Figure 1 shows saddle-path dynamics derived in this way using parameter values as detailed in the Appendix. These dynamics have similar saddle-path stability properties to those derived above for the linearised model.

Solving the model using reverse shooting

As a consequence of the saddle-path property, the numeric problem is to find the initial conditions for q, given that both the initial condition for K and the terminal conditions for both variables are known. Solving the investment sub-model is then equivalent to solving the following problem.

Find q(0) subject to:

$$\dot{q} = f(q, K) \quad (5a)$$

$$\dot{K} = g(q, K) \quad (5b)$$

$$K(0) = K_0 \quad (5c)$$

$$q(T) = q^* + \varepsilon_q \quad (5d)$$

$$K(T) = K^* + \varepsilon_K \quad (5e)$$

where T is some (exogenously given) large number representing the terminal point for time and ε_q and ε_K are small error terms that are 'close enough' to zero.

This problem can be solved using a solution approach, which is referred to in this paper as reverse shooting. The aim of this approach is to find the stable trajectories of the model and generate the stable arms in (q, K) phase space. This approach makes use of the feature that time can be abstracted from the solution of the model. The stable arms forwards in time will become the unstable arms with time going backwards. The same will apply for the unstable arms, with reverse time making them the stable arms. This approach finds the forward-stable arms by finding the unstable arms in reverse time (backward-unstable arms). This motivates the word reverse in the name for the approach.

The approach also makes use of the separatrix property of saddles (Khalil, 1996). The stable trajectories from a saddle form a separatrix so that the phase plane of the model is divided into four separate regions. Solutions always remain in one and only one region. Choosing a solution close to the boundary of one of these regions will ensure that the solution will remain close to the boundary. Choosing a backward-unstable solution close to the boundary will provide a time-path for the forward-stable solution (stable arm).

Using this property and the fact that any solution that is close to the steady-state equilibrium is close to all four boundaries, linearisation can be helpful in the generation of the stable trajectories for a non-linear model. From the linearisation of the investment sub-model at the steady state, the eigenvalues are such that $\lambda_1 > 0 > \lambda_2$. The corresponding eigenvectors are denoted by $\mathrm{v}(\lambda_1)$ and $\mathrm{v}(\lambda_2)$. The forward-stable trajectories of the non-linear model will be tangent to the forward-stable eigenvector, $\mathrm{v}(\lambda_2)$, as the trajectories approach the steady state. Similarily, the forward-unstable trajectories will be tangent to the forward-unstable eigenvector, $\mathrm{v}(\lambda_1)$, as they approach the steady state. These properties allow a approach for finding the forward-stable arms of the investment sub-model by using reverse time and choosing initial conditions so that ε_q and ε_K are close to zero and tangent to the forward-stable eigenvector.

Figure 2 shows the stable arms for the linearised and the original (non-linear) model. These stable arms have been derived using the reverse shooting approach and the same parameter values as Figure 1. Once the stable arm (or forward-stable trajectory) has been determined in this manner, initial values for q(0) can be obtained by reading the corresponding value of q(0) along the stable arm for the initial condition K(0).

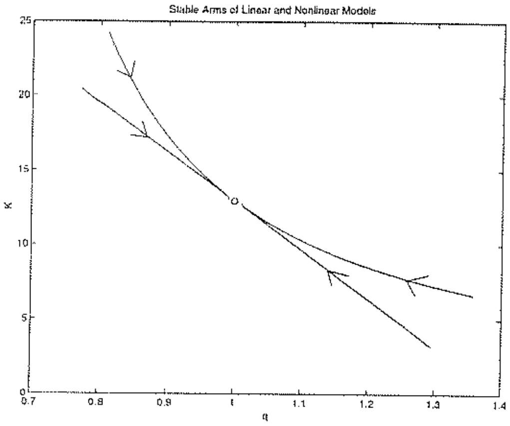

FIGURE 2
SAMPLE STABLE ARMS
LINEARISED AND ORIGINAL (NON-LINEAR) MODELS
INVESTMENT SUB-MODEL

Solving the model using forward shooting

Another approach is to use forward shooting (Burden and Faires, 1993; Judd, 1998). The general approach with forward shooting is to guess the unknown initial condition, solve the model as an initial value problem and see if the terminal conditions to the initial value problem are close enough to the steady-state equilibrium.

To solve the investment sub-model, K(0) is given and the shooting approach uses an initial guess for the initial condition of q(0). This turns the problem into an initial value problem, which will generate terminal values, q(T) and K(T). The aim of the shooting approach is to find the particular q(0) such that q(T) and K(T) are 'close enough' to q^* and K^*. Once again, a Runge-Kutta algorithm provides an appropriate solution method for solving the differential equations. A simplex algorithm (or a Newton algorithm) provides a suitable way to search for the appropriate initial conditions.

A problem with this approach is that it is necessary to generate multiple solutions to the underlying differential equation. This is in stark contrast to the reverse shooting approach, which requires only one solution of the differential equation to provide a definitive solution for each stable arm.

4. SOLVING THE FULL MODEL

Solving the full model, given by equations 1a-1d, can be summarised by the following problem. Find q(0) and C(0) subject to:

$$\dot{q} = f(q, K) \quad (6a)$$
$$\dot{K} = g(q, K) \quad (6b)$$
$$\dot{C} = h(q, K, C, D) \quad (6c)$$
$$\dot{D} = j(q, K, C, D) \quad (6d)$$
$$K(0) = K_0 \quad (6e)$$
$$D(0) = D_0 \quad (6f)$$
$$q(T) = q^* + \varepsilon_q \quad (6g)$$
$$K(T) = K^* + \varepsilon_K \quad (6h)$$
$$C(T) = C^* + \varepsilon_C \quad (6i)$$
$$D(T) = D^* + \varepsilon_D \quad (6j)$$

As before, T is some (exogenously given) large number representing the terminal point for time and each ε_i is a small error term that is 'close enough' to zero.

Specification of equations 6a-6d clearly demonstrates the block-recursive structure of the model. As a consequence of this structure the dynamics of q and K can be solved independently of the dynamics of C and D.

Solving the model using a modified version of reverse shooting

It is generally not possible to solve the full model using reverse shooting. The primary reason for this is that there is no way of ensuring that the path derived using this approach will pass through K_0 and D_0 at the same point in time. However, at least for the chosen model, it is possible to modify the reverse shooting approach so that an appropriate solution is derived.

The modified approach uses the block recursive structure of the chosen model, which allows for a separation of the model into two sub-models. These sub-models can be solved sequentially. Each sub-model has two endogenous variables, and each sub-model has the saddle-path property. Numeric solutions of the first sub-model (the investment sub-model) are initially used to estimate the stable arm of its saddle-path. Solutions along this stable arm are then used as exogenous in the solution of the second sub-model (defining the dynamics of C and D), allowing calculation of the stable arm for the full model.

Once solutions of q and K have been determined, the solutions for these variables can be taken as exogenous, thus reducing the full model to the second-order dynamical system in C and D, given by equations 1c and 1d. This can be further reduced to a system of equations of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t) \qquad (7)$$

where $\tilde{q}$ and $\tilde{K}$ denote solutions for q and K derived from the investment sub-model, and

$$\mathbf{x}(t) = \begin{bmatrix} C(t) - C^* \\ D(t) - D^* \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} r - \theta & p(p+\theta) \\ 1 & r \end{bmatrix},$$

$$\mathbf{b}(t) = \begin{bmatrix} -p(p+\theta)[\tilde{q}(t)\tilde{K}(t) - q^* K^*] \\ -a[\tilde{K}(t)]^\alpha + \tilde{K}(t)\Lambda(\tilde{q}(t)) + a[K^*]^\alpha \end{bmatrix}.$$

The solution to this system of equations is then given by:

$$\mathbf{x}(t) = \mathbf{P}\mathbf{e}^{\Phi t}\left(\begin{bmatrix} 0 \\ Z \end{bmatrix} - \int_t^\infty \mathbf{e}^{-\Phi s}\mathbf{P}^{-1}\mathbf{b}(s)ds\right) \qquad (8)$$

where

$\lambda_3 = r + p > 0$, $\lambda_4 = r - 6 - p < 0$

and $\quad \Phi = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix}$,

$$\mathbf{e}^{\Phi t} = \begin{bmatrix} e^{\lambda_3 t} & 0 \\ 0 & e^{\lambda_4 t} \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} p & -\theta - p \\ 1 & 1 \end{bmatrix}.$$

The solution to equation 8 can be derived using an appropriate numeric quadrature algorithm such as the trapezoidal rule or Simpson's rule. The constant, Z, can then be chosen consistent with the initial value for D, thus ensuring an appropriate initial jump in C.

This modified reverse shooting approach, like the reverse shooting approach for the investment sub-model, has the advantage that the solution can be derived with one pass of the data. This is unlike the forward shooting approach, which requires the generation of multiple solutions. The disadvantage of the modified approach is that, because it depends on a particular property of the model (the block recursive structure), it does not have the universal applicability of the forward shooting approach discussed below.

Solving the model using forward shooting

Unlike reverse shooting approach, which requires significant changes to solve the full model, the forward shooting approach can be extended to the full model without significant modification. The general approach with forward shooting is substantially unchanged: guess the unknown initial conditions, solve the model as an initial value problem and see if the terminal conditions to the initial value problem are close enough to the steady-state equilibrium. In the full model, K(0) and D(0) are given and the forward shooting approach uses initial guesses for both q(0) and C(0). This turns the problem into an initial value problem, which will generate corresponding terminal values, q(T), K(T), C(T) and D(T). In this case, the aim is to find particular values for q(0) and C(0) so that the terminal values are 'close enough' to q^*, K^*, C^* and D^*. This can be achieved by searching over a grid of initial values for q and C using a simplex algorithm or a Newton algorithm.

5. NUMERICAL ISSUES

One of the central issues that arise in evaluating the two approaches is the numerical accuracy of each approach. The reverse shooting approach uses numerical techniques to solve an initial value problem for the investment sub-model and then uses a numeric quadrature algorithm to solve the larger model. The forward shooting approach uses numeric search procedures to find the unknown initial conditions with an initial value problem for the full (four endogenous variable) model within the search.

In both approaches, initial value problems for a nonlinear ordinary differential equation must be solved. In the reverse shooting approach, the initial value problem is solved twice for the investment sub-model, with initial values close to the steady-state and using reverse time. One solution gives the trajectory of the stable arm on one side of the steady-state and the the other solution gives the stable arm on the other side of the steady-state.

When each of these is implemented in a numerical initial value problem solver, the solver is solving an initial value problem that explodes over (reverse) time. Truncation and round-off errors will be magnified by this process. To minimise these errors a variable step-size Runge-Kutta can be used.

The forward shooting approach involves multiple solutions to initial value problems for each stable arm. In this case the initial value problems are forward in time. The basic underlying approach is to find the unknown initial conditions such that the difference between the terminal conditions for the initial value problem are and the known terminal conditions are 'close enough' to zero. This involves a numerical minimisation procedure (such as Newton's method) to find the unknown initial conditions with the initial value problem embedded within it.

One issue with the forward shooting approach is that the separatrix property of the model will make it difficult to get close to the steady-state as trajectories will tend to diverge from the steady-state. This means that small differences in the initial conditions can make large differences in the terminal conditions, which increases the difficulty of the numeric search procedure. Numeric errors introduced by the search procedure and by the initial value problem solver will be magnified by this process. To minimise errors introduced by the numeric methods, a good initial value problem solver (such as a Runge Kutta) and a robust search routine should be used.

The modified reverse shooting approach also incorporates a numeric quadrature routine to solve the integral in equation 8. This routine will introduce truncation errors. The integral will also include errors introduced from the use of an initial value problem to solve the investment sub-model. Again, an accurate numeric integration technique will help to reduce these errors.

While numeric issues are important in the solution of a model such as this, the primary interest is to obtain approximations that indicate how the economy evolves following an exogenous shock. For such questions, it is not necessary to derive numerically exact solutions. Under these circumstances, precise numerical accuracy of the solution approaches is not of primary concern.

6. CONCLUSION

This paper has described two alternative techniques (modified reverse shooting and forward shooting) for solving the time-path of a representative agent model following an exogenous shock.

The modified reverse shooting approach requires the solution of a single initial value problem followed by numerical integration. This will require less computational resources than the forward shooting approach, which requires a (difficult) search incorporating the solution of an initial value problem at each step in the search. For ease of computation the reverse shooting approach is clearly preferable.

However, the modified reverse shooting approach makes use of a specific feature of the model - the fact that the model is block recursive so that the investment sub-model can be solved independently of the full model. This approach shows how using a property of the model can significantly improve the computational effort required to solve the model.

On the other hand, the forward shooting approach is a more general procedure. It can be used for a wider range of models than the reverse shooting approach.

APPENDIX

The parameter values underlying Figures 1 and 2 are given in the following table.

Parameter	Value
r	0.05
a	1
b	5
α	0.3

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