An application of the continuous time replicator dynamic to economics

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Abstract: The present analysis is an application of the continuous time replicator dynamic to economics. Two types of problems are considered under conditions of a normalized constraint and non-negative constrains. The first model considers a quadratic programming problem and the second considers a nonlinear programming model. The story of the following models is: there are three or more corporations in an oligopoly market. They behave so as to maximize their profits defined by the difference between their sales and cost functions with conjectural variations. In order to solve the problem, we apply the variable metric gradient projection method (VMGPM). In economics there are many models concerning conjectural variation and Nash equilibrium. Though economic models are relatively primitive, the present approach may be useful to examine the process of reaching equilibrium.

1. Introduction

The present analysis is an application of the continuous time replicator dynamic to economics. Recently, evolutionary economics has developed especially to chart the dynamic path. Economists are interested in the path that leads to stationary points.

When the model is complex and has conjectures among behavioral units, the solution is obtained by a simulation method. In the present analysis, we consider the profit maximization behavior of corporations with their conjectures.

In section 2 the numerical models are explained. Section 3 denotes the maximization method for the numerical models. Section 4 considers the relationship between the replicator dynamic process and the present method. Section 5 reports the simulation results, and section 6 presents concluding remarks.

2. Models

Two types of problems are considered under conditions of a normalized constraint and non-negative constrains. The first model considers a quadratic programming problem and the second considers a nonlinear programming model.

The story of the following models is: there are three or more corporations in an oligopoly market. They behave so as to maximize their profits defined by the difference between their sales and cost functions with conjectural variations.

2.1 Quadratic programming problem with three corporations

$$\begin{aligned}\max E(\mathbf{x}) = &(-16.5x_1^2 + 30x_1 - 0.2)\\ &+ (-22x_2^2 + 40x_2 - 0.5)\\ &+ (-11x_3^2 + 20x_3 - 0.2)\\ &- \theta(x_1x_2 + x_2x_3 + x_3x_1)\end{aligned}$$

$$\text{s.t. } \sum_i x_i = 1,\ x_i \geq 0\ (i=1,2,3) \qquad (1)$$

2.2 Nonlinear programming problem with three corporations

$$\begin{aligned}\max E(\mathbf{x}) = &\log 2(x_1 + 1\\ &- 0.1(x_1x_2 + x_2x_3 + x_3x_1))\\ &- \exp(x_1) + 1\\ &+ \log 3(x_2 + 1\\ &- 0.1(x_1x_2 + x_2x_3 + x_3x_1))\\ &- \exp(1.5x_2) + 1\\ &+ \log 5(x_2 + 1\\ &- 0.3(x_1x_2 + x_2x_3 + x_3x_1))\\ &- \exp(2x_3) + 1\end{aligned}$$

$$\text{s.t. } \sum_i x_i = 1,\ x_i \geq 0\ (i=1,2,3) \qquad (2)$$

2.3 Quadratic programming problem with one hundred corporations

$$\begin{aligned}\max E(\mathbf{x}) = &\sum_i (a_i x_i(2 - x_i) - b_i)\\ &- \theta \sum_i \sum_{j\neq i} x_i x_j\end{aligned}$$

$$\text{s.t. } \sum_i x_i = 1,\ x_i \geq 0\ (i=1,\dots,100) \qquad (3)$$

where a_i's $\in [19,20]$ are generated randomly from the uniform distribution, and b_i's $\in$ [0.10,0.12] are generated randomly from the uniform distribution.

3. Optimizing Method

We consider the optimizing problem with a normalized constraint and non-negative constraints

$$\max E(\mathbf{x})$$

$$\text{s.t. } \sum_i x_i = 1,\ x_i \ge 0\ (i=1,\dots,n) \qquad (4)$$

In order to solve the problem, we apply the variable metric gradient projection method (VMGPM) which was developed by Goldfarb (1969), Karmarker (1984,1990), Bayer and Lagarias (1989) and Vidyasagar (1995). The solution is obtained as a stationary point of the following differential equation,

$$d\mathbf{x}(t)/dt = -P_M(\mathbf{x}(t))M^{-1}(\mathbf{x}(t))\nabla E(\mathbf{x}(t))^T \qquad (5)$$

with initial condition,

$$\mathbf{x}(0) \in \{\mathbf{x}|\ \sum_i x_i = 1\} \cap \{\mathbf{x}|\ x_i \ge 0\ (i=1,\dots,n)\} \quad (6)$$

where the projection matrix is defined as:

$$P_M = I - M^{-1}l^T(lM^{-1}l^T)^{-1}l. \qquad (7)$$

The metric matrix, M, is the diagonal matrix whose i-th diagonal element is $1/x_i$, and the projection matrix, P_M, is obtained as:

$$P_M = I - A \qquad (8)$$

where the i-th row of A is all x_is. Then, we can write the concrete form of the VMGPM for the problem as:

$$dx_i(t)/dt = -x_i(t)\{\partial E(\mathbf{x}(t))/\partial x_i - \sum_j x_j(t)\partial E(\mathbf{x}(t))/\partial x_j\} \qquad (9)$$

The stationary point satisfies the Kuhn-Tucker condition when the dynamic of the equation converges to the stationary point.

4. An Interpretation based on the Replicator Equation

The replicator equation has been studied in the field of population genetics as a mathematical model for studying evolutionary phenomena. The general form of the replicator equation is:

$$dx_i(t)/dt = x_i\{f_i(\mathbf{x}(t)) - g(\mathbf{x}(t))\} \qquad (10)$$

$$(i=1,\dots,n)$$

$$g(\mathbf{x}) = \sum_j x_j f_j \qquad (11)$$

$$\mathbf{x} \in \{\mathbf{x}|\ \sum_i x_i = 1\} \cap \{\mathbf{x}|\ x_i \ge 0\ (i=1,\dots,n)\} \qquad (12)$$

where x_i denotes the frequency of the i-th element in the population, $f_i(\mathbf{x})$ denotes the fitness of the i-th element depending on the state of the population $\mathbf{x}$ and $g(\mathbf{x})$ is called the mean fitness. The trajectory of the vector field from the initial state is restricted on the simplex S_n, and it denotes the evolution of the frequencies in the population.

Fisher's equation is one class of the replicator equation in which the fitness is defined as:

$$f_i(\mathbf{x}) = \sum_j w_{ij}x_j \qquad (13)$$

$$(i=1,\dots,n)$$

$$w_{ij} = w_{ji} \qquad (14)$$

$$(i,j=1,\dots,n)$$

Shahshahni (1979), Sigmund (1985,1987) and Akin (1990) showed that the Fisher's equation can be regarded as the gradient projection system of the mean fitness

$$g(\mathbf{x}) = \sum_i \sum_j w_{ij}x_i x_j \qquad (15)$$

under the Riemannian metric on the tangent space of the simplex S_n called the Shahshahni metric.

In this paper, we mention the class of the replicator equation in which the fitness is defined as the inverse direction of the Euclidean gradient

$$f_i(\mathbf{x}) = -\partial E(\mathbf{x})/\partial x_i, \qquad (16)$$

$$(i,j=1,\dots,n)$$

where $E(\mathbf{x})$: $R^n \to R^1$ is differentiable. This is a generalization of the Fisher's equation.

The objective of this paper is to propose a new dynamic searching model for optimization problems with equality and inequality constraints and indicate that the replicator equation can be employed in this manner.

5. Simulation Results

We obtain the searching trajectories of the replicator equation by the fourth order modified Runge-Kutta method with the allowable margin of error equal to 10^{-10}

At first, we consider the simulation results of the model 2.1. We made up seven initial states, namely (0.33,0.33,0,33), (0.8,0.1,0.1), (0.1,0.8,0.1), (0.1,0.1,0.8), (0.05,0.475,0.475), (0.475,0.05,0.475), and (0.475,0.475,0.05). Figure 1 shows the variation of the profit E(x) by the analytical solution depending on the changes of the parameter, θ, (see mathematical appendix). Depending on the value of θ, indicating by the horizontal axis in Figure 1, the stationary points move from interior to corner solutions.

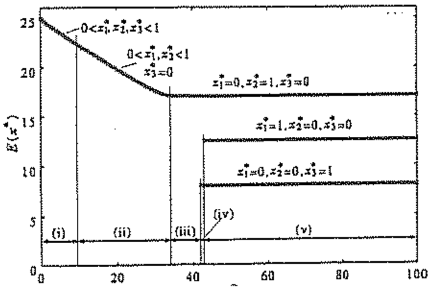
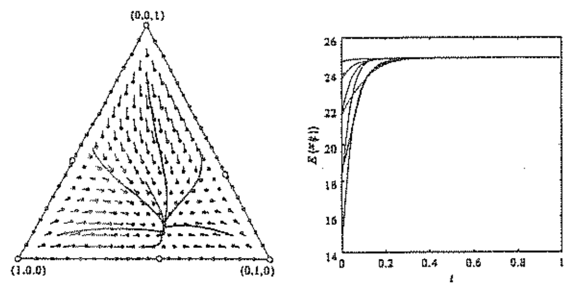


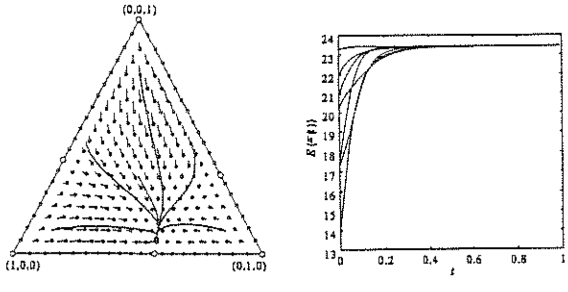
Figure 1 The Variation of the Profit E(x) by the Analytical Solution depending on the Changes of the Parameter, θ,

From Figures 2 though 8, the stationary path and the time to reach the stationary points are indicated. Figure 2(a) indicates that, in the case of the value of θ as 0, the stationary point is obtained as the interior solution, and the convergence takes place at about 0.4 time units as indicated in Figure 2(b). In the case of θ as 20, the boundary solution is obtained in Figure 4(a), and it took about 0.8 time units to reach the stationary point in Figure 4(b). The variation of the convergence process due to changes in θ is easily understood in looking at Figures 2(b) through 8(b). The bigger the value of θ, the longer it takes to reach the stationary point. However, the value of θ exceeds 50, it takes little time to reach the stationary points because only one corporation remains in the market.



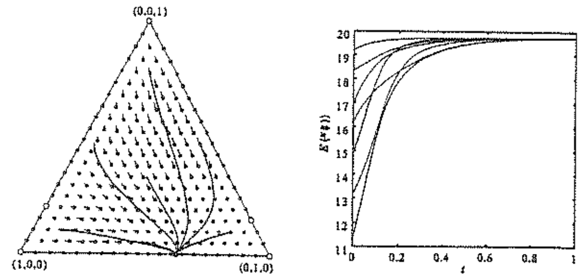
(a) (b)

Figure 2(a) The Stationary Point (θ = 0), (b) Convergence Time to Reach the Stationary Point (θ = 0)



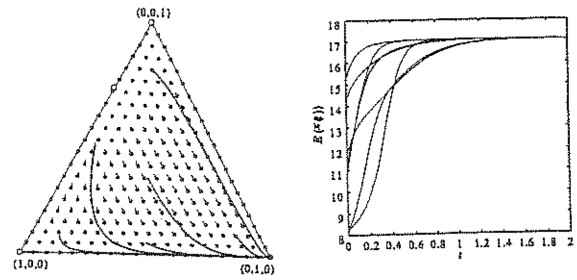
(a) (b)

Figure 3(a) The Stationary Point (θ = 5), (b) Convergence Time to Reach the Stationary Point (θ = 5)



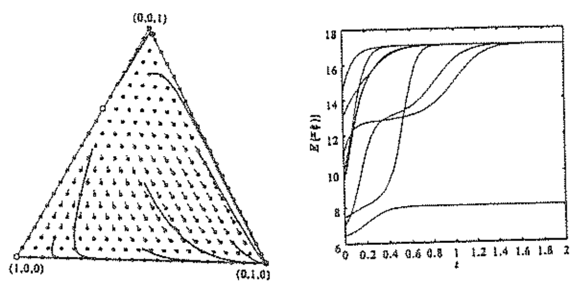
(a) (b)

Figure 4(a) The Stationary Point (θ = 20), (b) Convergence Time to Reach the Stationary Point (θ = 20)



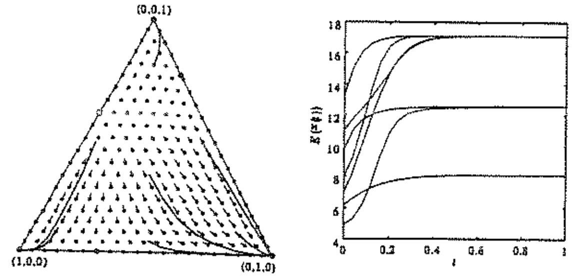
(a) (b)

Figure 5(a) The Stationary Point (θ = 38), (b) Convergence Time to Reach the Stationary Point (θ = 38)



(a) (b)

Figure 6(a) The Stationary Point (θ =42.50), (b) Convergence Time to Reach the Stationary Point (θ = 42.5)



(a) (b)

Figure 7(a) The Stationary Point (θ = 50), (b) Convergence Time to Reach the Stationary Point (θ =50)

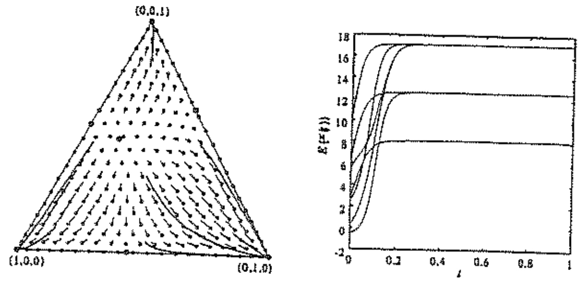


Figure 8(a) The Stationary Point (θ = 70), (b) Convergence Time to Reach the Stationary Point (θ = 70)

For the model 2.2 we set up seven initial states, namely (0.33,0.33,0,33), (0.8,0.1,0.1), (0.1,0.8,0.1), (0.1,0.1,0.8), (0.05,0.475,0.475), (0.475,0.05,0.475), and (0.475,0.475,0.05). Figure 9 indicates the interior solution and convergence path.

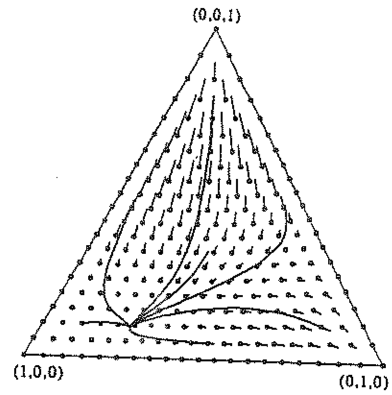


Figure 9 The Stationary Point

For the model2.3 we obtained the stationary points for one hundred corporations. A hundred initial states are set up such that x_i's $\in [0,1]$ are generated randomly from the uniform distribution under the condition that $\sum_i x_i = 1$. The bigger the value of θ, the longer it takes to reach the stationary point. However, the value of θ exceeds 40, it takes little time to reach the stationary points because a few or only one corporation remains in the market indicated in Figures 10 and 11.

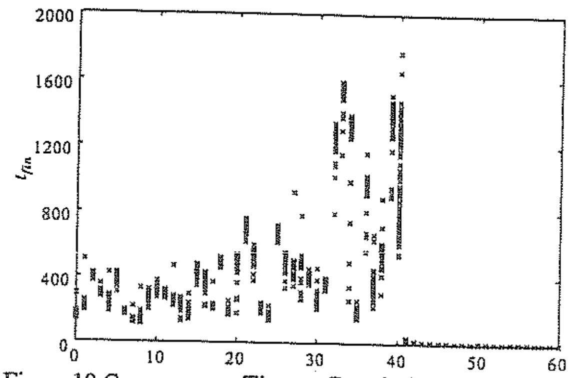


Figure 10 Convergence Time to Reach the Stationary Point

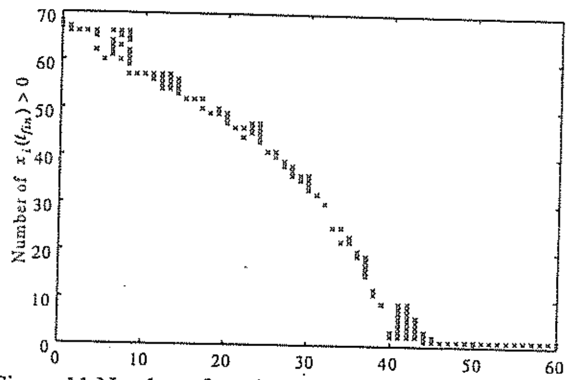


Figure 11 Number of x_i whose value is positive

Figure 12 (a),(b) and (c) depict the solutions of the stationary point from a set of an initial state in the case that θ's are 0, 30 and 50, respectively. According to increase in θ, the number of corporations decreases in the market. The movement of standard deviation indicates that there is a catastrophe at the level of θ being 40 in Figure 12(d). Before 40 their standard deviation is nearly zero, but after 40 it suddenly jumps due to realization of the corner solutions.

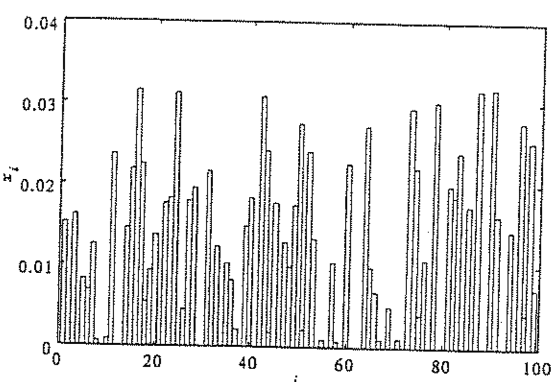


Figure 12(a) The Solution of the Stationary point (θ = 0)

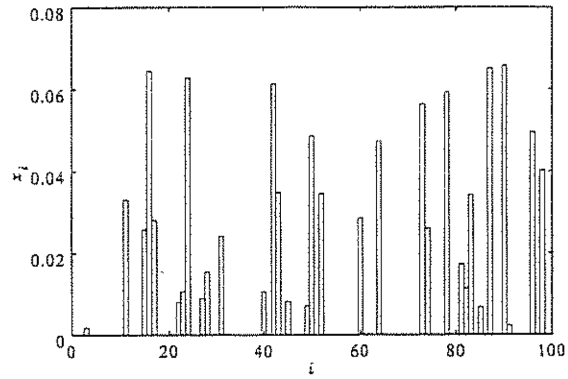


Figure 12(b) The Solution of the Stationary point (θ = 30)

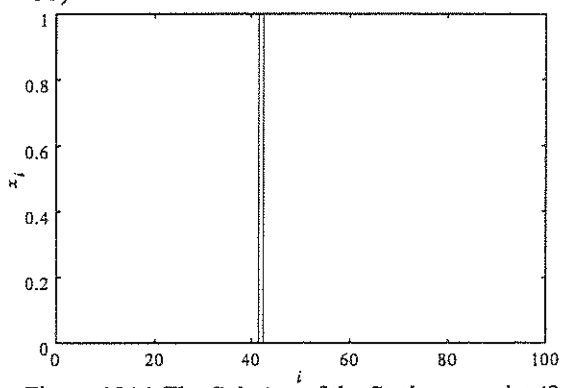


Figure 12(c) The Solution of the Stationary point (θ = 50)

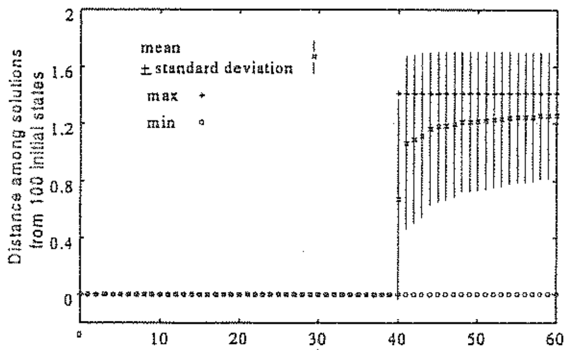


Figure 12(d) The Distance among Solutions

6. Conclusion

We have proposed a new dynamic searching model to solve optimization problems with equality and inequality constraints named VMGPM, and indicate that the replicator equation can be regarded as the proposed VMGPM for the problem with normalized and non-negative constraints.

In economics there are many models concerning conjectural variation and Nash equilibrium. Though economic models are relatively primitive, this approach may be useful to examine the process of reaching equilibrium.

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Mathematical Appendix: Analytical solution of the quadratic problem

The quadratic problem is:

$$\max E(x) = \sum_i (-a_i x_i^2 + b_i x_i) + \theta(x_1 x_2 + x_2 x_3 + x_3 x_1)$$

s.t. $\sum_i x_i = 1$, $x_i \geq 0$ (i=1,2,3)

The following necessary conditions for the maximum problem with the condition of equality and inequality constraints are obtained using the Lagrange function of:

$$L(\mathbf{x}, \boldsymbol{\lambda}, \phi) = -E(\mathbf{x}) + \sum \lambda_i (-x_i) + \phi(\sum_i x_i - 1)$$

We have analytical solutions for profit maximization depending on the value of θ whose values are from 0 to 100 increasing by increments of 0.5:

(1) the unique solution at the interior of $0 < x_1^*, x_2^*, x_3^* < 1$;
$x_i^* = ((\theta - 2a_j)(\theta - 2a_k) - (\theta - 2a_j)(b_i - b_k) - (\theta - 2a_k)(b_i - b_j))/(3\theta^2 - 4(a_i + a_j + a_k)\theta + 4(a_i a_j + a_j a_k + a_k a_i)$;
$0 \leq \theta \leq 9$;

(2) the unique solution on the side of $0 < x_1^*, x_2^* < 1$, $x_3^* = 0$,
$x_i^* = (b_i + 2a_j - b_j - \theta)/(2((a_i + a_j) - \theta))$, $x_j = (b_j + 2a_i - b_i - \theta)/(2((a_i + a_j) - \theta))$, $(-\theta^2 + (2(a_i + a_j) - (b_i + b_j) + 2b_k)\theta + 2(a_i b_j + a_j b_i - (a_i + a_j)b_k) - 4a_i a_j)/(2((a_i + a_j) - \theta)) \geq 0$;
$9.5 \leq \theta \leq 33.5$;

(3) the unique solution on the corner of $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 0$;
$x_i^* = (b_i + 2a_j - b_j - \theta)/(2((a_i + a_j) - \theta))$, $x_j = (b_j + 2a_i - b_i - \theta)/(2((a_i + a_j) - \theta))$, $(-\theta^2 + (2(a_i + a_j) - (b_i + b_j) + 2b_k)\theta + 2(a_i b_j + a_j b_i - (a_i + a_j)b_k) - 4a_i a_j)/(2((a_i + a_j) - \theta)) \geq 0$;
$34 \leq \theta \leq 41.5$;

(4) the two solutions on the corner of $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 0$; and $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 1$;
$\theta \geq 2a_i - b_i + \max\{b_j, b_k\}$;
$42 \leq \theta \leq 42.5$;

(5) the three solutions on the corner of $x_1^* = 1$, $x_2^* = 0$, $x_3^* = 0$; $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 0$;, and $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 1$;
$\theta \geq 2a_i - b_i + \max\{b_j, b_k\}$;
$43 \leq \theta$;