Mutually Dependent Rainfall Input and Its Stochastic Response

Gaku Tanak: Graduate School of Engineering, Hokkaido University, Japan
Mutsuhiro Fujita: Graduate School of Engineering, Hokkaido University, Japan
Masaru Kaido: Obayashi Corporation, Japan
Susumu Kumagai: Sapporo Business Office, Nissan Construction Co. Ltd., Japan

Abstract: This paper proposes a new method for estimating a design flood based on stochastic analysis of rainfall-runoff phenomena. Using a storage function runoff model, we derived differential equations that provide the first four moments of discharge even if rainfall input is a non-stationary and mutually dependent random variable. The validity of the proposed differential equations was cross-checked by a simulation. The results showed that it is possible to obtain the probability density function of discharge from the calculated first four moments of discharge.

1. Introduction

The planning of flood-control projects requires a design flood. In Japan, the design flood is conventionally estimated by (1) determination of the recurrence interval or the return period after analyzing hydroeconomic problems, (2) estimation of a design rainfall depth whose duration time is 2 or 3 days corresponding to the return period, (3) estimation of a hyetograph (in which total rainfall equals the above design rainfall depth), (4) calculation of a hydrograph through the estimated hyetograph and a runoff model, and (5) determination of a design flood from the hydrograph. The procedure for steps (1) and (2) is based on stochastic theory. In step (3), past rainfall patterns that have caused major flood events are taken into account. However, the procedure for step (3) includes empirical methods. This paper proposes a new method for estimating a probability density function of discharge. Generally, runoff models are described by non-linear differential equations. If rainfall input is a random process, these differential equations are attributed to random differential equations. Even if the stochastic properties of rainfall input are known, it is not easy to solve these random differential equations and estimate the probability density function of discharge because of the non-linearity of differential equations. Furthermore, observed rainfall is a non-stationary random process and mutually dependent random variable. In this study, we adopted a storage function runoff model and derived differential equations that provide the first four moments of discharge under the condition that the rainfall input is a non-stationary random process and mutually dependent random variable.

2. Fundamental Theory

The runoff model considered here is a storage function runoff model derived by Hoshi and Yamaoka [1]. This type of runoff model is widely used in Japan.

$$\frac{ds_h}{dt}+q_h=r \quad (1) \qquad s_h=k_1 q_h^{p_1}+k_2\frac{dq_h^{p_2}}{dt} \quad (2)$$

s_h: storage; q_h: discharge;

k_1 and k_2: storage coefficients;

p_1 and p_2: storage exponents

By eliminating s_h from Eq. (1) and Eq. (2), the following equation can be obtained.

$$k_1\frac{dq_h^{p_1}}{dt}+k_2\frac{d^2q_h^{p_2}}{dt^2}+q_h=r \qquad (3)$$

If the rainfall input, r, is a random variable, the discharge output, q_h, is also described by a random process. This paper's direct aim is to calculate the first four moments ($\bar{q}_h$, $\sigma_{q_h}^2$, μ_{q_h3}, μ_{q_h4}) of q_h in Eq. (3). Let us assume that these random variables, r and q_h, both consist of a mean and a deviation from its mean. The signs "‾" and "~" show the mean and the deviation from its mean, respectively.

$$r=\bar{r}+\tilde{r},\ E(\tilde{r})=0 \quad (4) \quad q_h=\bar{q}_h+\tilde{q}_h,\ E(q_{\tilde{h}})=0 \quad (5)$$

The following approximations are used for the exponential-type random variables, $q_h^{p_1}$ and $q_h^{p_2}$.

$$q_h^{p_1}=\alpha_1\bar{q}_h+\beta_1\tilde{q}_h \quad (6) \quad q_h^{p_2}=\alpha_2\bar{q}_h+\beta_2\tilde{q}_h \quad (7)$$

The four coefficients, α_1, β_1, α_2 and β_2, were proposed by Bras and Georgakakos [2].

$$\alpha_1=\bar{q}_h^{p_1-1}\left\{1+\frac{1}{2}p_1(p_1-1)\frac{E(\tilde{q}_h^2)}{\bar{q}_h^2}+\frac{1}{6}p_1(p_1-1)(p_1-2)\frac{E(\tilde{q}_h^3)}{\bar{q}_h^3}+\cdots\right\} \quad (8)$$

$$\beta_1=\frac{\bar{q}_h^{p_1+1}}{E(\tilde{q}_h^2)}\left\{p_1\frac{E(\tilde{q}_h^2)}{\bar{q}_h^2}+\frac{1}{2}p_1(p_1-1)\frac{E(\tilde{q}_h^3)}{\bar{q}_h^3}+\frac{1}{6}p_1(p_1-1)(p_1-2)\frac{E(\tilde{q}_h^4)}{\bar{q}_h^4}+\cdots\right\} \quad (9)$$

$$\alpha_2=\bar{q}_h^{p_2-1}\left\{1+\frac{1}{2}p_2(p_2-1)\frac{E(\tilde{q}_h^2)}{\bar{q}_h^2}+\frac{1}{6}p_2(p_2-1)(p_2-2)\frac{E(\tilde{q}_h^3)}{\bar{q}_h^3}+\cdots\right\} \quad (10)$$

$$\beta_2=\frac{\bar{q}_h^{p_2+1}}{E(\tilde{q}_h^2)}\left\{p_2\frac{E(\tilde{q}_h^2)}{\bar{q}_h^2}+\frac{1}{2}p_2(p_2-1)\frac{E(\tilde{q}_h^3)}{\bar{q}_h^3}+\frac{1}{6}p_2(p_2-1)(p_2-2)\frac{E(\tilde{q}_h^4)}{\bar{q}_h^4}+\cdots\right\} \quad (11)$$

Eq. (12) is obtained by substituting Eq. (4) to Eq. (7) into Eq.(3).

$$k_1\frac{d(\alpha_1\bar{q}_h+\beta_1\tilde{q}_h)}{dt}+k_2\frac{d^2(\alpha_2\bar{q}_h+\beta_2\tilde{q}_h)}{dt^2}+\bar{q}_h+\tilde{q}_h=\bar{r}+\tilde{r} \quad (12)$$

The expectation of Eq. (12) gives Eq. (13).

$$k_1\frac{d(\alpha_1\bar{q}_h)}{dt}+k_2\frac{d^2(\alpha_2\bar{q}_h)}{dt^2}+\bar{q}_h=\bar{r} \quad (13)$$

By subtracting Eq. (13) from Eq. (12),

$$k_1\frac{d(\beta_1\tilde{q}_h)}{dt}+k_2\frac{d^2(\beta_2\tilde{q}_h)}{dt^2}+\tilde{q}_h=\tilde{r} \quad (14)$$

Eq. (13) and Eq. (14) are rewritten by

$$\frac{d^2\bar{V}}{dt^2}+f_1(t)\frac{d\bar{V}}{dt}+g_1(t)\bar{V}=\bar{r} \quad (15)$$

$$\frac{d^2\tilde{V}}{dt^2}+f_2(t)\frac{d\tilde{V}}{dt}+g_2(t)\tilde{V}=\tilde{r} \quad (16)$$

Where, $\bar{V}=k_2\alpha_2\bar{q}_h$ (17) $\tilde{V}=k_2\beta_2\tilde{q}_h$ (18)

$$f_1(t)=\frac{k_1\alpha_1}{k_2\alpha_2} \quad (19) \quad g_1(t)=\frac{1}{k_2}\left\{k_1\frac{d}{dt}\left(\frac{\alpha_1}{\alpha_2}\right)+\frac{1}{\alpha_2}\right\} \quad (20)$$

$$f_2(t)=\frac{k_1\beta_1}{k_2\beta_2} \quad (21) \quad g_2(t)=\frac{1}{k_2}\left\{k_1\frac{d}{dt}\left(\frac{\beta_1}{\beta_2}\right)+\frac{1}{\beta_2}\right\} \quad (22)$$

If $\bar{V}$ and the second, third and fourth moments of $\tilde{V}$ is calculated, it is possible to obtain the first four moments of q_h (Eq. (43) to Eq. (45)). Eq. (16) is transformed into simultaneous differential equations by introducing the complex coefficients, $f_3(t)$ and $g_3(t)$, to solve Eq. (16).

$$\frac{d\tilde{V}_1}{dt}+f_3(t)\tilde{V}_1=\tilde{r} \quad (23) \quad \frac{d\tilde{V}}{dt}+g_3(t)\tilde{V}=\tilde{V}_1 \quad (24)$$

$$f_3(t)=F(t)+jG(t) \quad (25) \quad g_3(t)=H(t)+jI(t) \quad (26)$$

j : imaginary unit

The following equations are obtained by comparing Eq. (16) with Eq. (23) and Eq. (24).

$$\frac{dH(t)}{dt}+\{f_2(t)-H(t)\}H(t)=g_2(t)-I(t)^2 \quad (27)$$

$$\frac{dI(t)}{dt}+\{f_2(t)-2H(t)\}I(t)=0 \quad (28)$$

$$F(t)=f_2(t)-H(t) \quad (29) \quad G(t)=-I(t) \quad (30)$$

$\tilde{V}$ and $\tilde{V}_1$ in Eq. (23) and Eq. (24) are defined as complex-valued functions. On the other hand, $\tilde{V}$ in Eq. (16) is a real-valued function. Only the real parts of $\tilde{V}$ and $\tilde{V}_1$ in Eq. (23) and Eq. (24) therefore need to be considered. The symbol "R_e" is used to represent the real part. As a result, the following equations are obtained from Eq. (23) and Eq. (24).

$$R_e\{\tilde{V}\}=C_1(t)W_1(t)+S_1(t)W_2(t) \quad (31)$$

$$\frac{dW_1}{dt}+HW_1=W_5W_3+W_6W_4 \quad (32)$$

$$\frac{dW_2}{dt}+HW_2=W_6W_3-W_5W_4 \quad (33)$$

$$\frac{dW_3}{dt}+FW_3=C_1(t)\tilde{r}(t) \quad (34)$$

$$\frac{dW_4}{dt}+FW_4=S_1(t)\tilde{r}(t) \quad (35)$$

$$\frac{dC_1}{dt}=-IS_1,\ C_1(0)=1 \quad (36) \quad \frac{dS_1}{dt}=IC_1,\ S_1(0)=0 \quad (37)$$

$$W_5(t)=C_1(t)^2-S_1(t)^2 \quad (38) \quad W_6(t)=2C_1(t)S_1(t) \quad (39)$$

After raising Eq. (31) to the second, third and fourth power and taking the expectation of each of them, the second, third and fourth order moments of V, σ_V^2, μ_{V3} and μ_{V4}, can be derived.

$$\sigma_V^2=E[\{C_1(t)W_1(t)+S_1(t)W_2(t)\}^2] \qquad (40)$$

$$\mu_{V3}=E[\{C_1(t)W_1(t)+S_1(t)W_2(t)\}^3] \qquad (41)$$

$$\mu_{V4}=E[\{C_1(t)W_1(t)+S_1(t)W_2(t)\}^4] \qquad (42)$$

The relationship between the stochastic properties of q_h and V are derived by Eq. (18).

$$\sigma_V^2=(k_2\beta_2)^2\sigma_{q_h}^2 \quad (43) \qquad \mu_{V3}=(k_2\beta_2)^3\mu_{q_h3} \quad (44)$$

$$\mu_{V4}=(k_2\beta_2)^4\mu_{q_h4} \quad (45)$$

If only the first term is adopted to define α_1, α_2, β_1 and β_2 in Eq. (8) to Eq. (11), $F(t)$, $H(t)$, $I(t)$, $C_1(t)$, $S_1(t)$, $W_5(t)$ and $W_6(t)$ are deterministic functions and can be obtained from Eq. (15) and Eq. (19) to Eq. (22). W_1, W_2, W_3 and W_4 are random functions, and the higher order cumulant function of $r(t)$ must be know to obtain $E\{W_1^mW_2^n\}$ (n, m: integer, $n+m=4$). On the other hand, observed rainfall is a discrete time series and is calculated from continuous rainfall, $r(t)$.

$$r_{d_i}=\frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} r(\tau)d\tau \qquad (46)$$

Δt: time interval of observed rainfall

r_{d_i} consists of the mean of r_{d_i}, $\bar{r}_{d_i}$, and the deviation from $\bar{r}_{d_i}$, $\tilde{r}_{d_i}$.

$$\bar{r}_{d_i}=\frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} \bar{r}(\tau)d\tau \qquad (47)$$

$$\tilde{r}_{d_i}=\frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} \tilde{r}(\tau)d\tau \qquad (48)$$

Tanaka et al.[3] suggested that observed rainfall can be described by a first order auto-regressive process.

$$\tilde{r}_{d_i}=\rho\tilde{r}_{d_{i-1}}+N_i \qquad (49)$$

$\tilde{r}_{d_i}$: deviation from a mean, $\bar{r}_d$;
N_i: noise component;
i : integer; ρ: regression coefficient

It is possible to estimate the higher order cumulant function of $\tilde{r}_{d_i}$ such as $E\{\tilde{r}_{d_i}\tilde{r}_{d_j}\}$, $E\{\tilde{r}_{d_i}\tilde{r}_{d_j}\tilde{r}_{d_k}\}$ and $E\{\tilde{r}_{d_i}\tilde{r}_{d_j}\tilde{r}_{d_k}\tilde{r}_{d_l}\}$ from Eq. (49). If we focus on $E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\}$, we can derive

$$E\{\tilde{r}_{d_i}\tilde{r}_{d_j}\}=\frac{1}{\Delta t^2}\int_{(i-1)\Delta t}^{i\Delta t}\int_{(j-1)\Delta t}^{j\Delta t} E\{\tilde{r}(\tau)\tilde{r}(\tau_2)\}d\tau_2 d\tau_1 \quad (50)$$

The left side of Eq. (50) is known, and Eq. (50) must be solved to obtain $E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\}$. Tanaka et al.[3] presented the following equations.

$$\bar{r}=\bar{r}_d \qquad (51)$$

$$E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\}=\frac{\Delta t}{C}\sum_{n=0}^{\infty}\sigma_{r_d}^2 \times A\delta(\tau_1-\tau_2-n\Delta t)e^{-\gamma(\tau_1-\tau_2)} \qquad (52)$$

$$E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\tilde{r}(\tau_3)\}=\frac{\Delta t^2}{C^2}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\mu_{r_d3} \times A\delta(\tau_1-\tau_2-n\Delta t) \times\delta(\tau_2-\tau_3-(m-n)\Delta t)e^{-\gamma(\tau_1+\tau_2-2\tau_3)} \qquad (53)$$

$$\begin{aligned}E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\tilde{r}(\tau_3)\tilde{r}(\tau_4)\}=&\frac{\Delta t^3}{C^3}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{n=0}^{m}B\\ &\times\delta(\tau_1-\tau_2-n\Delta t)\delta(\tau_2-\tau_3-(m-n)\Delta t)\\ &\times\delta(\tau_3-\tau_4-(l-m)\Delta t)\\ &\times\Big\{(\mu_{r_d4}-3\sigma_{r_d}^4)e^{-\gamma(\tau_1+\tau_2+\tau_3-3\tau_4)}\\ &+2\sigma_{r_d}^4e^{-\gamma(\tau_1+\tau_2-\tau_3-\tau_4)}+\sigma_{r_d}^4e^{-\gamma(\tau_1-\tau_2+\tau_3-\tau_4)}\Big\}\end{aligned} \qquad (54)$$

Where,

$$A=\begin{cases}1 & (\rho>0)\\(-1)^n & (\rho<0)\end{cases}, \qquad B=\begin{cases}1 & (\rho>0)\\(-1)^{l+m+n} & (\rho<0)\end{cases}$$

$$\gamma=-\frac{\log|\rho|}{\Delta t}, \qquad \tau_1\geq\tau_2\geq\tau_3\geq\tau_4$$

c: unit with dimension, t ; $\delta(t)$: delta function

$\bar{r}_d$, $\sigma_{r_d}^2$, μ_{r_d3} and μ_{r_d4} show the first four moments of observed rainfall. By using Eq. (52) to Eq. (54), the following differential equations for $E\{W_1^mW_2^n\}$ are obtained.

$$\frac{dU_i}{dt}+2FU_i=\Delta t\sigma_{r_d}^2(C_1^{3-i}S_1^{i-1}+U_f^2Y_i), \quad i=1\sim3 \quad (55)$$

$$Z_1(n,m)=\begin{cases}0 & [t/\Delta t]<1\\ \sum_{i=1}^{[t/\Delta t]}\rho^iu_n(t)u_m(t-i\Delta t) & elsewhere\end{cases}$$

$$Y_1=2Z_1(1,1),\ Y_2=Z_1(1,2)+Z_1(2,1),\ Y_3=2Z_1(2,2)$$

$$u_1(t)=\frac{C_1(t)}{U_f(t)}, \qquad u_2(t)=\frac{S_1(t)}{U_f(t)}$$

$$\begin{cases}\dfrac{dU_{i+3}}{dt}+(F+H)U_{i+3}=W_5U_i+W_6U_{i+1}+\Delta t\sigma_{r_d}^2U_hY_{i+3}\\ \dfrac{dU_{i+5}}{dt}+(F+H)U_{i+5}=W_6U_i-W_5U_{i+1}+\Delta t\sigma_{r_d}^2U_hY_{i+5}\end{cases}, \quad i=1,2 \qquad (56)$$

$$Z_2(n,m)=\begin{cases}0 & [t/\Delta t]<1\\ \sum_{i=1}^{[t/\Delta t]}\rho^iX_n(t,i)u_m(t-i\Delta t) & elsewhere\end{cases},$$

$$\begin{cases} Y_{i+3}=C_1^{\,2-i}S_1^{\,i-1}\{Z_2(1,1)+Z_2(2,2)\} \\ Y_{i+5}=C_1^{\,2-i}S_1^{\,i-1}\{Z_2(2,1)-Z_2(1,2)\}' \end{cases} \quad i=1,2$$

$$X_n(t,i)=U_{a,n}(t)-U_{a,n}(t-i\Delta t),\ \ i=1,2$$

$$\frac{dE\{W_1^{\,2}\}}{dt}+2HE\{W_1^{\,2}\}=2W_5U_4+2W_6U_5 \tag{57}$$

$$\frac{dE\{W_1W_2\}}{dt}+2HE\{W_1W_2\}=W_5(U_6-U_5)+W_6(U_7+U_4) \tag{58}$$

$$\frac{dE\{W_2^{\,2}\}}{dt}+2HE\{W_2^{\,2}\}=2W_6U_6-2W_5U_7 \tag{59}$$

$$\frac{dU_f}{dt}+FU_f=0,\qquad U_f(0)=1 \tag{60}$$

$$\frac{dU_h}{dt}+HU_h=0,\qquad U_h(0)=1 \tag{61}$$

$$\frac{dU_{a,i}}{dt}=W_{i+4}\frac{U_f}{U_h},\ \ i=1,2 \tag{62}$$

$$\frac{dU_{i+7}}{dt}+3FU_{i+7}=\Delta t^2\mu_{r_d3}\Big\{C_1^{\,4-i}S_1^{\,i-1}+U_f^3(Y_{i+7}+Y_{i+11}+Y_{i+15})\Big\},\ i=1\sim4 \tag{63}$$

$$Z_3(n,m,l)=\begin{cases}0 & [t/\Delta t]<1\\ \sum_{i=1}^{[t/\Delta t]}\rho^{2i}u_n(t)u_m(t)u_l(t-i\Delta t), & \text{elsewhere}\end{cases}$$

$$Z_4(n,m,l)=\begin{cases}0 & [t/\Delta t]<1\\ \sum_{i=1}^{[t/\Delta t]}\rho^{i}u_n(t)u_m(t-i\Delta t)u_l(t-i\Delta t), & \text{elsewhere}\end{cases}$$

$$Z_5(n,m,l)=\begin{cases}0 & [t/\Delta t]<2\\ \sum_{j=2}^{[t/\Delta t]}\sum_{i=1}^{j-1}\rho^{2j-i}u_n(t)u_m(t-i\Delta t)u_l(t-j\Delta t), & \text{elsewhere}\end{cases}$$

$Y_8=3Z_3(1,1,1)$, $Y_9=Z_3(1,1,2)+2Z_3(1,2,1)$,

$Y_{10}=2Z_3(1,2,2)+Z_3(2,2,1)$, $Y_{11}=3Z_3(2,2,2)$,

$Y_{12}=3Z_4(1,1,1)$, $Y_{13}=Z_4(2,1,1)+2Z_4(1,1,2)$,

$Y_{14}=2Z_4(2,1,2)+Z_4(1,2,2)$, $Y_{15}=3Z_4(2,2,2)$,

$Y_{16}=6Z_5(1,1,1)$,

$Y_{17}=2\{Z_5(1,1,2)+Z_5(1,2,1)+Z_5(2,1,1)\}$,

$Y_{18}=2\{Z_5(2,2,1)+Z_5(2,1,2)+Z_5(1,2,2)\}$,

$Y_{19}=6Z_5(2,2,2)$,

$$\begin{cases}\dfrac{dU_{i+11}}{dt}+(2F+H)U_{i+11}=W_5U_{i+7}+W_6U_{i+8}+\Delta t^2\mu_{r_d3}U_fU_hY_{i+19}\\ \dfrac{dU_{i+14}}{dt}+(2F+H)U_{i+14}=W_6U_{i+7}-W_5U_{i+8}+\Delta t^2\mu_{r_d3}U_fU_hY_{i+22}\end{cases},\ i=1\sim3 \tag{64}$$

$$Z_6(n,m,l)=\begin{cases}0 & [t/\Delta t]<1\\ \sum_{i=1}^{[t/\Delta t]}\rho^{2i}X_n(t,i)u_m(t)u_l(t-i\Delta t), & \text{elsewhere}\end{cases}$$

$$Z_7(n,m,l)=\begin{cases}0 & [t/\Delta t]<1\\ \sum_{i=1}^{[t/\Delta t]}\rho^{i}X_n(t,i)u_m(t-i\Delta t)u_l(t-i\Delta t), & \text{elsewhere}\end{cases}$$

$$Z_8(n,m,l)=\begin{cases}0 & [t/\Delta t]<2\\ \sum_{j=2}^{[t/\Delta t]}\sum_{i=1}^{j-1}\rho^{2j-i}X_n(t,i)u_m(t-i\Delta t)u_l(t-j\Delta t), & \text{elsewhere}\end{cases}$$

$$Z_9(n,m,l)=\begin{cases}0 & [t/\Delta t]<2\\ \sum_{j=2}^{[t/\Delta t]}\sum_{i=1}^{j-1}\rho^{2j-i}X_n(t,j)u_m(t-i\Delta t)u_l(t-j\Delta t), & \text{elsewhere}\end{cases}$$

$$y_1=Z_6(1,1,1)/2+Z_7(1,1,1)+Z_8(1,1,1)+Z_9(1,1,1)+Z_6(2,1,2)/2+Z_7(2,1,2)+Z_8(2,2,1)+Z_9(2,1,2)'$$

$$y_2=Z_6(1,2,1)/2+Z_7(1,2,1)+Z_8(1,1,2)+Z_9(1,2,1)+Z_6(2,2,2)/2+Z_7(2,2,2)+Z_8(2,2,2)+Z_9(2,2,2)'$$

$$y_3=Z_6(2,1,1)/2+Z_7(2,1,1)+Z_8(2,1,1)+Z_9(2,1,1)-Z_6(1,1,2)/2-Z_7(1,1,2)-Z_8(1,2,1)-Z_9(1,1,2)'$$

$$y_4=Z_6(2,2,1)/2+Z_7(2,2,1)+Z_8(2,1,2)+Z_9(2,2,1)-Z_6(1,2,2)/2-Z_7(1,2,2)-Z_8(1,2,2)-Z_9(1,2,2)'$$

$$\begin{cases}Y_{3i+17}=2C_1y_{2i-1}\\ Y_{3i+18}=C_1y_{2i}+S_1y_{2i-1},\ \ i=1,2\\ Y_{3i+19}=2S_1y_{2i}\end{cases}$$

$$\begin{cases}\dfrac{dU_{i+17}}{dt}+(F+2H)U_{i+17}=2W_5U_{i+11}+2W_6U_{i+12}+\Delta t^2\mu_{r_d3}U_h^{\,2}Y_{i+25}\\ \dfrac{dU_{i+19}}{dt}+(F+2H)U_{i+19}=2W_6U_{i+14}-2W_5U_{i+15}+\Delta t^2\mu_{r_d3}U_h^{\,2}Y_{i+27}\\ \dfrac{dU_{i+21}}{dt}+(F+2H)U_{i+21}=W_5(U_{i+14}-U_{i+12})+W_6(U_{i+15}+U_{i+11})+\Delta t^2\mu_{r_d3}U_h^{\,2}Y_{i+29}\end{cases},\ i=1,2 \tag{65}$$

$$Z_{10}(n_1,n_2,m,l)=\begin{cases}0 & [t/\Delta t]<1\\ \sum_{i=1}^{[t/\Delta t]}\rho^i X_{n_1,n_2}(t,i,i)\\ \quad\times u_m(t-i\Delta t)u_l(t-i\Delta t) & elsewhere\end{cases},$$

$$Z_{11}(n_1,n_2,m,l)=\begin{cases}0 & [t/\Delta t]<2\\ \sum_{j=2}^{[t/\Delta t]}\sum_{i=1}^{j-1}\rho^{2j-i}X_{n_1,n_2}(t,i,i)\\ \quad\times u_m(t-i\Delta t)u_l(t-j\Delta t) & elsewhere\end{cases},$$

$$Z_{12}(n_1,n_2,m,l)=\begin{cases}0 & [t/\Delta t]<2\\ \sum_{j=2}^{[t/\Delta t]}\sum_{i=1}^{j-1}\rho^{2j-i}X_{n_1,n_2}(t,i,j)\\ \quad\times u_m(t-i\Delta t)u_l(t-j\Delta t) & elsewhere\end{cases},$$

$$X_{n_1,n_2}(t,i,j)=U_{a,n_1,n_2}(t)-U_{a,n_1,n_2}(t-i\Delta t)\\ -\{U_{a,n_1}(t)-U_{a,n_1}(t-i\Delta t)\}U_{a,n_2}(t-j\Delta t)$$

$$y_5=2\{Z_{10}(1,1,1,1)+Z_{11}(1,1,1,1)+Z_{12}(1,1,1,1)\\ +Z_{10}(1,2,1,2)+Z_{11}(1,2,2,1)+Z_{12}(1,2,1,2)\\ +Z_{10}(2,1,2,1)+Z_{11}(2,1,1,2)+Z_{12}(2,1,2,1)\\ +Z_{10}(2,2,2,2)+Z_{11}(2,2,2,2)+Z_{12}(2,2,2,2)\},$$

$$y_6=2\{Z_{10}(2,2,1,1)+Z_{11}(2,2,1,1)+Z_{12}(2,2,1,1)\\ -Z_{10}(2,1,1,2)-Z_{11}(2,1,2,1)-Z_{12}(2,1,1,2)\\ -Z_{10}(1,2,2,1)-Z_{11}(1,2,1,2)-Z_{12}(1,2,2,1)\\ +Z_{10}(1,1,2,2)+Z_{11}(1,1,2,2)+Z_{12}(1,1,2,2)\},$$

$$y_7=Z_{10}(1,2,1,1)+Z_{11}(1,2,1,1)+Z_{12}(1,2,1,1)\\ -Z_{10}(1,1,1,2)-Z_{11}(1,1,2,1)-Z_{12}(1,1,1,2)\\ +Z_{10}(2,2,2,1)+Z_{11}(2,2,1,2)+Z_{12}(2,2,2,1)\\ -Z_{10}(2,1,2,2)-Z_{11}(2,1,2,2)-Z_{12}(2,1,2,2)\\ +Z_{10}(2,1,1,1)+Z_{11}(2,1,1,1)+Z_{12}(2,1,1,1),\\ +Z_{10}(2,2,1,2)+Z_{11}(2,2,2,1)+Z_{12}(2,2,1,2)\\ -Z_{10}(1,1,2,1)-Z_{11}(1,1,1,2)-Z_{12}(1,1,2,1)\\ -Z_{10}(1,2,2,2)-Z_{11}(1,2,2,2)-Z_{12}(1,2,2,2)$$

$$\begin{cases}Y_{2i+24}=C_1 y_{i+4}\\ Y_{2i+25}=S_1 y_{i+4}\end{cases},\ i=1\sim3$$

$$\frac{dU_{a,i,j}}{dt}=W_{i+4}\frac{U_f}{U_h}U_{a,j},\ \ i=1,2,\ j=1,2 \tag{66}$$

$$\frac{dE\{W_1^{\ 3}\}}{dt}+3HE\{W_1^{\ 3}\}=3W_5U_{18}+3W_6U_{19} \tag{67}$$

$$\frac{dE\{W_1^{\ 2}W_2\}}{dT}+3H_hE\{W_1^{\ 2}W_2\}=W_5(2U_{22}-U_{19})\\ +W_6(2U_{23}+U_{18}) \tag{68}$$

$$\frac{dE\{W_1W_2^{\ 2}\}}{dt}+3HE\{W_1W_2^{\ 2}\}=W_5(U_{20}-2U_{23})\\ +W_6(U_{21}+2U_{22}) \tag{69}$$

$$\frac{dE\{W_2^{\ 3}\}}{dt}+3HE\{W_2^{\ 3}\}=3W_6U_{20}-3W_5U_{21} \tag{70}$$

The differential equations for $E\{W_1^{\ m}W_2^{\ n}\}$ (n, m: integer, $n+m=4$) are not expressed for a space problem.

3. Examination Based on Simulation

The derived theoretical equations use two approximations: one is Eq. (6) and Eq. (7) for exponential-type random variables ($q_h^{\ p_1}$ and $q_h^{\ p_2}$), and the other is Eq. (52) to Eq. (54) for the cumulant functions of rainfall, $r(t)$. The validity of the derived theoretical equations is cross-checked by simulation. The simulation is carried out by directly and numerically solving the following dimensionless form of Eq. (3).

$$K_1\frac{dQ^{p_1}}{dT}+K_2\frac{d^2Q^{p_2}}{dT^2}+Q=R \tag{71}$$

Hoshi and Yamaoka [1] presented the parameters involved in the dimensionless Eq. (71).

$$p_1=0.6,\qquad K_1=\frac{1}{1+p_1},\qquad \Delta T=0.1,$$

$$\sigma_{R_d}^{\ 2}=\frac{\sigma_N^{\ 2}}{1-\rho^2}=\begin{cases}0.25 & 0\le T\le 2\\ 0 & elsewhere\end{cases},$$

$$f(N)=\begin{cases}\lambda e^{-\lambda(N-\frac{1}{\lambda})} & -\frac{1}{\lambda}\le N \quad \lambda{:}const.\\ 0 & elsewhere\end{cases} \tag{72}$$

$$E(N)=0,\quad \sigma_N^{\ 2}=\frac{1}{\lambda^2},\quad \mu_{N3}=\frac{2}{\lambda^3},\quad \mu_{N4}=\frac{9}{\lambda^4}$$

(rectangular rainfall input)

$$\bar{R}=\begin{cases}1 & 0\le T\le 2\\ 0 & elsewhere\end{cases},\ p_2=p_1^{\ 1.5},\ K_2=0.1p_1^{\ -0.2}$$

(triangular rainfall input)

$$\bar{R}=\begin{cases}0.5+2T & 0\le T\le 1\\ 4.5-2T & 1\le T\le 2\\ 0 & elsewhere\end{cases},\ p_2=0.4509,\ K_2=0.09608$$

$f(N)$ shows the probability density function of the noise component in Eq. (49). **Figure 1 and Figure 2** show the computed results. Solid lines and dotted lines show the simulated results and the solutions of the proposed differential equations, respectively. These figures show good agreement with each other.

4. Estimation of Probability Density Function of Discharge and Conclusions

It is possible to estimate the probability density function by obtaining the first four moments of discharge. **Figure 3** shows the relationship (black lines) between β_x and β_y for the triangular rainfall input. The standardized parameters, β_x and β_y, are expressed by Eq. (73).

$$(\beta_x\ ,\ \beta_y)=\left(\mu_{Q3}^2/\sigma_Q^6\ ,\ \mu_{Q4}/\sigma_Q^4\right) \qquad (73)$$

Black dots show the locations of β_x and β_y at the peak discharge. **Figure 3** indicates that the discharge from the storage function runoff model described by Eq. (1) and Eq. (2) belongs to the gamma distribution or the log-normal distribution.

References

1) K. Hoshi & I. Yamaoka: A Relationship between Kinematic Wave and Storage Routing Model, Proceedings of the 26th Japanese Conference on Hydraulics, pp. 273-278, 1982. (in Japanese)

2) R. L. Bras and K. P. Georgakakos: Real time nonlinear filtering techniques in streamflow forecasting -A statistical linearization approach, Proceedings of the Third International Symposium on Stochastic Hydraulics, pp. 95-105, 1980.

3) G. Tanaka, M. Fujita and S. Kumagai: Estimating of the probability density function for flood runoff, 2nd International Conference on Water Resources, Handbook and Proceedings, Vol. 2, pp. 1160-1166, 1999.

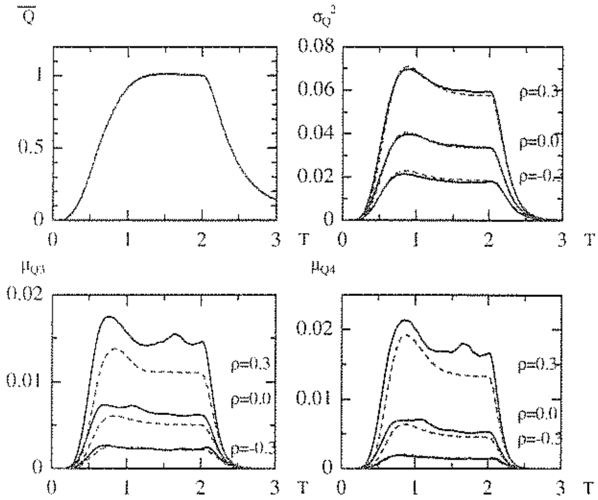

Figure 1 Computed results (rectangular rainfall input)

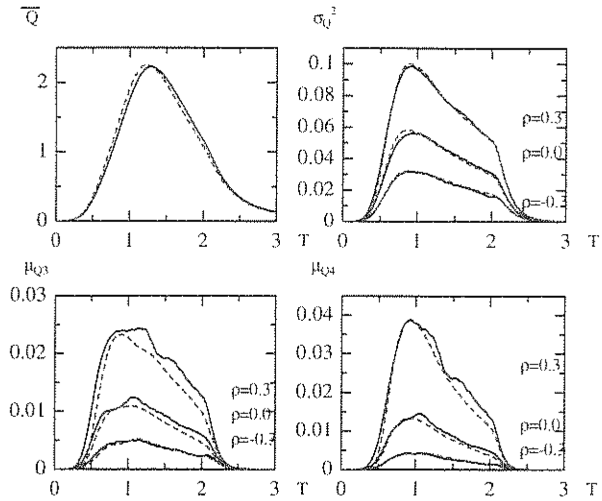

Figure 2 Computed results (triangular rainfall input)

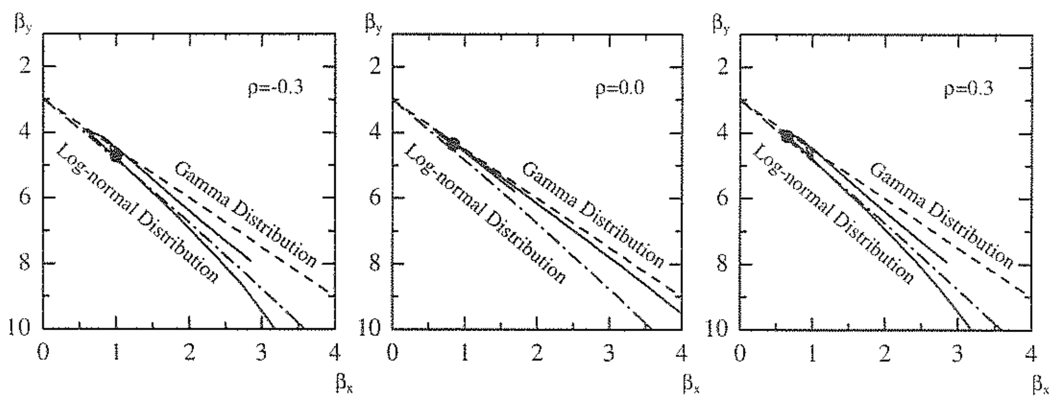

Figure 3 Estimating of the probability density function of discharge