Sensitivity and Conflict Analysis in Multi-criteria Decision Making

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Abstract The concept of distance among vectors is considered in this paper. In a decisional procedure of choice between different alternatives the definition of such a distance assumes a particular relevance to perform both a sensitivity analysis and a conflict analysis. In fact the sensitivity analysis can be analytically formulated as a minimisation problem consisting in finding the minimum distance from the reference vector of parameters at which a rank reversal occurs. In conflict analysis one of the key points is to measure the distance between the positions of different decision makers. Since frequently a way of describing preferences consists of giving a ranking vector concerning either the objectives or the alternatives directly, such problem is again a problem of distances between vectors.

Of course the distance among vectors is not univocally determined, and different definitions are possible. Many proposals have been made in the literature, which are briefly reviewed in this paper, and a new very simple definition of distance, namely the angle between the two vectors in the space of their components, is given. The advantage of this "angle distance" is that it is invariant with respect to an eventual normalization of the parameter vectors. This can be very important, for example, for a sensitivity analysis on a vector of weights which are defined with an arbitrary normalization.

1. INTRODUCTION

Multi-criteria decision analysis is always characterized by uncertainty and conflictuality, that may regard:

- a single value, e.g. the impact estimate of a certain action on a criterion;
- a function, e.g. a value function (making reference to the classical Keeney and Raiffa multi-attribute value theory [1976]);
- a vector, e.g. the vector of weights showing the relative importance of the different criteria;
- all the possible combinations of the previous elements, including matrices.

In this paper we refer to the vector case. This case is very important since it includes the vector of weights, which reflects the subjective value system of the decision maker and is one of the key points of a multi-criteria analysis, as well as the vector of ranking of the alternatives, on which most frequently conflict analysis focuses.

When the estimate of a reference vector is uncertain the aim of sensitivity analysis is to check the robustness of the found solution (typically a ranking of the alternatives), to point out the vector elements that are critical because a small change in their value may cause a rank reversal, and to show which are the alternatives that are really competitive, since they classify in the first position for feasible values of the vector. The sensitivity analysis can be analytically formulated as a minimisation problem (Rios Insua and French [1991]), consisting in finding the minimum distance from the reference vector at which a rank reversal occurs. Such minimum distance is a measure of the robustness of the solution corresponding to the reference vector. For each alternative the vector at minimum distance from the reference one at which it classifies in the first position can be computed, being a measure of the competitiveness of such alternative.

When many decision makers or interest groups are present, conflictuality is likely to occur. Conflict is characterised by at least two reference vectors, each supported by one or more actors. Attempts to reduce or solve the conflict can be done through negotiation. The analytical support to negotiation consists mainly in giving a clear information (see Kacprzyk and Fedrizzi [1988] and Lewandowski et al. [1980] for discussions on conflict indicators). Conflict indicators can be obtained based on the distances between the vectors: all the distances between pairs of decision makers can be computed and ordered in a matrix D, whose generic element d_{ij} is the distance between the vectors of decision maker i and decision maker j. The matrix D synthesises the main elements which determine a higher or lower degree of conflict:

- the number of vectors which are different is the number of elements $d_{ij} \neq 0$ (the higher the number the higher the conflict);
- the uniformity of distribution of the decision makers judgements (it is more conflictual a situation in which the decision makers distribute their judgements uniformly than one in which almost the all of them agree on a judgement and only a few

support different ones) can be measured by the standard deviation of the matrix elements (the higher the standard deviation the lower the conflict);
- the average distance among the decision makers vectors is the average of all the elements of D (the higher the average the higher the conflict). The average distance can also be computed on any squared submatrix of D obtained considering only the rows and columns corresponding to a decision makers subset.

These three indexes are global conflict indicators since they give information about the overall conflict without specifying anything on the single decision makers. Individual conflict indexes can be useful in the negotiation, to determine which are the most critical decision makers. Again the matrix D gives the opportunity to compute individual conflict indexes: the sum of the elements of a row gives an indication of the distance from the row decision maker of all the others, while the sum of the elements of a column gives an indication of the distance of the column decision maker from all the others.

The concept of distance among vectors is therefore central in both sensitivity and conflict analysis. Many different measures of such a distance are possible, leading, in general, to different results. In the next Section the main classes of distances are briefly reviewed and in Section 3 a new definition, the "angle distance", is introduced.

2. DISTANCE AMONG VECTORS

The distance among vectors is not defined univocally. We give here a general classification and some examples of specific definitions (see also Bogart [1973] and [1975]), considering both ranking (ordinal) vectors and score (numerical) ones. Of course the distances defined for ordinal vectors can be used for numerical ones. The opposite can also be done, but introducing some arbitrariety, for example by attributing to each element of ordinal vectors a score which corresponds to its position in the ranking.

Main classes of distances:
- *symmetric/asymmetric distances*: the distance d(A,B) among the two vectors A and B is symmetric if it is always equal to the distance d(B,A) among B and A; the distance is asymmetric in the opposite case;
- *lexicographic / compensative / not purely compensative distances*: a distance among two vectors is lexicographic if, taken one of the two as reference, a difference in the position of one element cannot be compensated by any difference of positions of inferior elements. A compensative distance allows such a compensation, since it considers the number and the magnitude of differences in the elements positions without giving importance to the reference element position. A not purely compensative distance is somehow intermediate: there can be a compensation but differences in the positions of superior elements are weighted more than differences in the positions of inferior ones. Lexicographic and not purely compensative distances are always asymmetric. Lexicographic and not purely compensative distances are particularly used for the ranking vectors, since a difference in the first positions of the ranking is usually considered much more important than a difference in the lower positions, or, more generally, the k-th position is more important than the (k+1)-th.

2.1 Ordinal vectors

When dealing with ordinal vectors, one should distinguish between strong and weak ranking vectors, as well as between complete or partial ones. To fix the ideas, we refer in the following examples to strong and complete ranking vectors.

Some examples of distances defined for ordinal vectors are given here:

1) Kendall, symmetric and compensative distance:

$$d(A,B)=\frac{n_{acc}-n_{dis}}{n_{acc}+n_{dis}}$$

with n_{acc} and n_{dis} number of accordances and of disaccordances between the two vectors. If d(A,B)=1 there are no disaccordances between the two vectors, while if d(A,B)=-1 there are no accordances; intermediate values describe intermediate situations. If, for example, given three elements α, β and γ, the two ranking vectors are

$$A=[\alpha\ \beta\ \gamma]^T \qquad B=[\gamma\ \beta\ \alpha]^T$$

then

$$d(A,B)=d(B,A)=\frac{1-2}{1+2}=-0.333$$

2) Symmetric and compensative distance:

$$d(A,B)=\sum_{i,j,i\neq j} D(i,j)$$

with D(i,j)=0 if the ranking order between element i and j is the same in the two vectors, D(i,j)=1 otherwise. Considering the (n-1)+(n-2)+...+1 disequations which are needed to describe the ranking of an n-dimensional vector, this distance is equal to the number of disequations which are different in the description of the two rankings. If for example

$A = [\alpha \;\; \beta \;\; \gamma]^T \qquad B = [\alpha \; \gamma \;\; \beta]^T$

then

$d(A,B) = d(B,A) = 0 + 0 + 1 = 1$

3) Lexicographic and asymmetric distance with base h:

$$d(A,B) = \sum_{i=1}^{n} |i - b_i| \cdot h^{n-i}$$

where b_i is the position in the B vector of the alternative which is in position i in the A vector, n is the number of elements of each vector, h is an arbitrary whole number such that $h \geq n$, which is called base of the distance. Fixed $h = 10$ and given the two vectors

$A = [\alpha \;\; \beta \;\; \gamma]^T \qquad B = [\beta \;\; \gamma \;\; \alpha]^T$

then

$d(A,B) = 2 \cdot 10^2 + 1 \cdot 10^1 + 1 \cdot 10^0 = 211$

$d(B,A) = 1 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0 = 112$.

2.2 Numerical vectors

The same classes of distances can be used to classify the distances among numerical vectors. A further distinction can be done between distances based on differences and on ratios of values.

Examples

1) Euclidean squared, compensative and symmetric, based on differences, distance:

$$d(A,B) = \sum_{i=1}^{n} (a_i - b_i)^2$$

where n is the number of elements of the vectors and a_i and b_i are the scores of the i-th element in vector A and B respectively. If, for example:

$A = [2 \;\; 5 \;\; 3]^T \qquad B = [2 \;\; 6 \;\; 2]^T$

then

$d(A,B) = d(B,A) = (2\text{-}2)^2 + (5\text{-}6)^2 + (3\text{-}2)^2 = 2$

2) Tchebycheff, compensative and symmetric, based on differences, distance:

$$d(A,B) = \max_i |a_i - b_i|$$

where a_i and b_i are the scores of the i-th element in vector A and B respectively. If, for example:

$A = [2 \;\; 5 \;\; 3]^T \qquad B = [3 \;\; 6 \;\; 1]^T$

then

$d(A,B) = d(B,A) = \max \;\; \{ 1, 1, 2 \} = 2$

3) Klafszky et al. [1989], asymmetric and not purely compensative, based on differences and ratios, distance, for vectors normalised so that $\Sigma\, a_i = \Sigma\, b_i = 1$:

$$d(A,B) = \sum_{i=1}^{n} \frac{|a_i - b_i|}{1 - a_i}$$

where n is the number of elements of the vectors and a_i and b_i are the scores of the i-th element in vector A and B respectively. If, for example:

$A = [.2 \;\; .5 \;\; .3]^T \qquad B = [.2 \;\; .6 \;\; .2]^T$

then

$$d(A,B) = \frac{0}{0.8} + \frac{0.1}{0.5} + \frac{0.1}{0.7} = 0.343$$

$$d(B,A) = \frac{0}{0.8} + \frac{0.1}{0.4} + \frac{0.1}{0.8} = 0.375$$

4) Logarithmic, compensative and symmetric, based on ratios, distance, for vectors with no zero elements:

$$d(A,B) = \sum_{i=1}^{n} \left| \ln \frac{a_i}{b_i} \right|$$

with a_i and b_i scores of the i-th element in vector A and B respectively. If, for example:

$A = [2 \;\; 5 \;\; 3]^T \qquad B = [2 \;\; 6 \;\; 2]^T$

then

$d(A,B) = d(B,A) = 0 + 0.18 + 0.41 = 0.59$

3. THE DISTANCE BETWEEN NORMALISED VECTORS

3.1 Problems Connected to the Use of Normalised Vectors

A problem arises for numerical vectors which are defined via a linear normalisation maintaining the ratios between the vector elements, as, for example, the vector of weights determining the relative importance of the different criteria. Chosen a definition of distance and given three vectors A, B, C, the distance d(A,B) can result greater or smaller then d(A,C) depending on the normalisation of the vectors. Two simple examples will show this fact for both distances based on differences and ratios.

Let's consider two different normalisations for three vectors A, B, and C: with the first normalisation (vectors A', B', and C') the sum of the elements of each vector is set equal to 1, while with the second (vectors A", B", and C") the first element of each vector is set equal to one:

$$A' = \begin{vmatrix} 1/2 \\ 1/2 \end{vmatrix} \qquad B' = \begin{vmatrix} 4/11 \\ 7/11 \end{vmatrix} \qquad C' = \begin{vmatrix} 2/3 \\ 1/3 \end{vmatrix}$$

$$A'' = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \qquad B'' = \begin{vmatrix} 1 \\ 7/4 \end{vmatrix} \qquad C'' = \begin{vmatrix} 1 \\ 1/2 \end{vmatrix}$$

Using the most common Euclidean squared distance (based on differences) it results:

d(A', B') = 0.037   d(A', C') = 0.056

d(A", B") = 0.563   d(A", C") = 0.250 .

With the first normalisation B looks closer to A than C, while with the second C looks closer to A than B.

Since the normalisation has to maintain the ratios between the vectors elements one could think that distances based on ratios do not present this problem, but that's not the case, as shown by the following example of two different normalisations for three vectors A, B, and C: with the first normalisation (vectors A', B', and C') the first element of each vector is set equal to 1, while with the second (vectors A", B", and C") the last element of each vector is set equal to one:

$$A' = \begin{vmatrix} 1 \\ 4 \\ 1 \end{vmatrix} \qquad B' = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} \qquad C' = \begin{vmatrix} 1 \\ 4/6 \\ 1 \end{vmatrix}$$

$$A'' = \begin{vmatrix} 1 \\ 4 \\ 1 \end{vmatrix} \qquad B'' = \begin{vmatrix} 1/2 \\ 1 \\ 1 \end{vmatrix} \qquad C'' = \begin{vmatrix} 1 \\ 4/6 \\ 1 \end{vmatrix}$$

Using the logarithmic ratio distance (based on ratios) it results:

d(A', B') = 1.386   d(A', C') = 1.791

d(A", B") = 2.079   d(A", C") = 1.791 .

With the first normalisation B looks closer to A than C, while with the second C looks closer to A than B.

This fact is of course unacceptable, because it makes any result of both a sensitivity or a conflict analysis for normalised vectors completely unreliable.

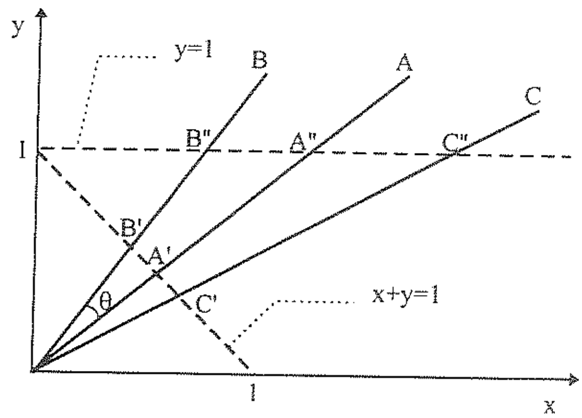

Figure 1: The bi-dimensional vectors A and B represented in the space of their components x and y. The two dotted lines determine two different normalisations of the vectors.

A simple geometrical interpretation (see Figure 1) can explain the reason behind such behaviour and help finding a solution. Let us consider three bi-dimensional vectors A, B, and C. In the x-y space of their components the three vectors are represented by three lines. Each normalisation of a vector determine a point in the plane. For instance if we consider vector A, and we normalise it in such a way that the sum of its components is equal to one, we obtain point A'; if we normalise it in such a way that its second component is equal to one, we obtain point A". If we want to know the distance between vectors A and B (A and C), we cannot measure it as the (Euclidean) distance A'B' (A'C') between points A' and B' (A' and C') or as the distance A"B" (A"C") between points A" and B" (A" and C"). In fact it can happen, as shown in Figure 1 and depending on the relative slopes of the vectors and of the lines determining the normalisation, that A'B' > A'C' but A"B" < A"C". What is invariant with respect to any normalisation is the angle that two vectors form in the space of their components: obviously the angle θ formed by points A' and B' with respect to the origin coincide with the angle formed by points A" and B" with respect to the origin.

Synthesising we can affirm that the information that the normalised vectors give is the direction they define in the space of their components and not a precise point in such a space. A correct way of measuring distances between vectors is therefore to measure the distance between their directions. A very simple distance which satisfy this requirement is the "angle distance" proposed in the next paragraph.

3.1 The Angle Distance

As seen in the previous paragraph, the distance between two vectors A and B can be defined as a measure of the angle θ they form in the space of their components. We propose here to use an "angle distance" which is the cosine of the angle θ, and has the following formula:

$$d(A,B) = \cos\vartheta = \sum_{i=1}^{n} \frac{a_i}{\rho_A} \cdot \frac{b_i}{\rho_B}$$

where n is the dimension of the vectors, a_i and b_i are the scores of the i-th element in vector A and B respectively, and

$$\rho_A = \sqrt{\sum_{j=1}^{n} a_j^{\,2}} \quad , \qquad \rho_B = \sqrt{\sum_{j=1}^{n} b_j^{\,2}} \quad .$$

Note that $\dfrac{a_i}{\rho_A}$ $\left(\dfrac{b_i}{\rho_B}\right)$ is the director cosine of vector A (B) with respect to the i-th ax.

If you want to minimise θ, as in the case of sensitivity analysis, you will have to maximise the angle distance (cosθ).

This angle distance is invariant with respect to the normalisation of the vectors.

4. CONCLUDING REMARKS

The problem of measuring distances among vectors is central in both sensitivity and conflict analysis. A synthetic review of the main classes and definitions of distances for both ordinal and score vectors has been given. It has been shown that for numerical vectors which undergo an arbitrary normalization, with the only requirement that the ratios between the vector elements is maintained, both distances based on differences and based on ratios are unacceptable. In fact using such distances the results of a sensitivity or a conflict analysis depend on the chosen arbitrary normalization. This means that such results are completely unreliable, and furthermore that they could be manipulated by the analyst to obtain a desired result. At this point the reason to perform a sensitivity or a conflict analysis could be only to confuse the actors of the decisional process. However, by means of a geometrical interpretation, a new definition of distance among vectors, the "angle distance", has been given. Such angle distance is independent from the normalization, so that it can be used for reliable sensitivity and conflict analysis.

Two directions of further development are sketched here. The first consists in exploring all the family of invariant distances and their geometric interpretations. For example the angle distance defined in this paper could be considered the angle-analogue to the Euclidean distance, but the angle-analogue to the Tchebycheff distance or to other distances, eventually non-compensative and asymmetric, could also be defined. The second direction of development consists in individuating for both sensitivity and conflict analysis meaningful indexes based specifically on the angle distance.

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