ASYMPTOTIC PROPERTIES OF THE ESTIMATED LONG-RUN MPC IN A DYNAMIC MODEL WITH AN INTEGRATED REGRESSOR

Koichi Maekawa, Michael McAleer*and Zonglu He

Hiroshima University, University of Western Australia, Hiroshima University

ABSTRACT: This paper investigates the asymptotic properties of the OLS-based estimator of the long-run MPC in a dynamic consumption function, $y_t = c+\alpha y_{t-1}+\beta z_t+u_t$, where z_t is integrated of order one, I(1). It is known that the estimated long-run MPC, $\hat{\delta} = \hat{\beta}/(1-\hat{\alpha})$, where $\hat{\alpha}$ and $\hat{\beta}$ are OLS estimators, is $\sqrt{T}-$consistent and asymptotically distributed as normal when z_t is fixed or is a stationary random variable and u_t is serially uncorrelated. It is well known that $\hat{\alpha}$ and $\hat{\beta}$ are inconsistent estimators if u_t is serially correlated. In this paper it is shown that, in such circumstances, $\hat{\delta}$ is super consistent and has a non-standard asymptotic distribution when z_t is $I(1)$ and u_t is serially correlated. The effects of serial correlation in u_t on the asymptotic distributions of the short-run and long-run estimators are also examined. The theoretical results were supported by Monte Carlo simlations, which also examined the implications for statistical inference when the integrated regressor is misspecified as stationary.

1 Introduction

In this paper we study the asymptotic properties of the estimated long-run marginal propensity to consume (MPC) in a dynamic consumption function given by

$$y_t = c + \alpha y_{t-1} + \beta z_t + u_t$$

where y and z are, respectively, consumption and income, c is a constant term, and u_t is assumed to be a stationary AR(1) process. The long-run MPC is defined as $\delta(\alpha, \beta) = \beta/(1-\alpha)$ (simply denoted as δ hereafter) and we consider a non-linear estimator of the long-run MPC, $\hat{\delta} = \hat{\beta}/(1-\hat{\alpha})$, where $\hat{\alpha}$ and $\hat{\beta}$ are the OLS estimators.

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Maddala and Rao (1973) studied a similar model in the absence of a constant term and derived the limiting bias when z_t is a stationary AR(1) process, but did not examine any distributional properties of the estimators. Maekawa et al. (1996) (henceforth MYTH) analysed the same model, showed that the OLS estimators are $\sqrt{T}-$inconsistent, and that $\sqrt{T}(\hat{\alpha}-\alpha)$ and $\sqrt{T}(\hat{\beta}-\beta)$ are distributed as asymptotic normal even if z_t is integrated of order 1.

The model examined in this paper is a special case of Park and Phillips(1989). However, the analysis differs in that the long-run implications of the estimated coefficients, as well as their asymptotic distributions, are examined and derived in the present paper. We consider two models, with and without a constant c, and derive the asymptotic distributions of $\hat{\delta}$ in both cases. As $\hat{\alpha}$ and $\hat{\beta}$ are inconsistent when u_t is serially correlated, it might be anticipated that $\hat{\delta}$ is also inconsistent. Somewhat strikingly, it is shown that $\hat{\delta}$ is superconsistent.

The plan of this paper is as follows. We specify the model and estimator in detail in section 2, and derive the asymptotic distribution of $\hat{\delta}$ in section 3. In section 4 we present the results of a Monte Carlo study to illustrate the theoretical results and extract the implications for statistical inference when z_t is misspecified as stationary. (The detailed derivations and proofs are available on request from the authors.) In what follows, $\Rightarrow$ signifies weak convergence in distribution and the summation operator $\sum$ denotes $\sum_{t=1}^{T}$.

2 Model and Estimator

We deal with the following dynamic model:

$$
\begin{aligned}
y_t &= c + \alpha y_{t-1} + \beta z_t + u_t, \ |\alpha| < 1 \\
u_t &= \rho u_{t-1} + v_t, |\rho| < 1 \\
z_t &= \lambda z_{t-1} + \varepsilon_t, \ |\lambda| \leq 1 \\
t &= 1, 2, \cdots, T
\end{aligned} \tag{1}
$$

where $v_t \sim NID(0, \sigma_v^2)$ and $\varepsilon_t \sim NID(0, \sigma_\varepsilon^2)$ are assumed to be independent. Since z_t is non-stationary when $\lambda = 1$, y_t is also non-stationary.

The standardized OLS estimator of $(c, \alpha, \beta)'$ is given as

$$
\begin{pmatrix} \hat{c} - c \\ T(\hat{\alpha} - \alpha) \\ T(\hat{\beta} - \beta) \end{pmatrix} = A^{-1}B \tag{2}
$$

where

$$
A = \begin{pmatrix} 1 & \frac{1}{T^2}\sum y_{t-1} & \frac{1}{T^2}\sum z_t \\ \frac{1}{T^2}\sum y_{t-1} & \frac{1}{T^3}\sum y_{t-1}^2 & \frac{1}{T^3}\sum y_{t-1}z_t \\ \frac{1}{T^2}\sum z_t & \frac{1}{T^3}\sum y_{t-1}z_t & \frac{1}{T^3}\sum z_t^2 \end{pmatrix}
$$

$$
B = \begin{pmatrix} \frac{1}{T}\sum u_t \\ \frac{1}{T^2}\sum u_t y_{t-1} \\ \frac{1}{T^2}\sum u_t z_t \end{pmatrix}.
$$

Defining $\delta = \beta/(1-\alpha)$, it is straightforward to show that the estimation error $T(\hat{\delta} - \delta)$ can be written as

$$
\begin{aligned}
T(\hat{\delta} - \delta) &= T\left(\tfrac{\hat{\beta}}{1-\hat{\alpha}} - \tfrac{\beta}{1-\alpha}\right) \\
&= \frac{\frac{\beta}{1-\alpha}T(\hat{\alpha} - \alpha) + T(\hat{\beta} - \beta)}{1 - \hat{\alpha}}.
\end{aligned} \tag{3}
$$

The last term is more convenient for obtaining the asymptotic distribution of $T(\hat{\delta} - \delta)$ than is the second term. To do so, it is necessary to evaluate each element in (2) asymptotically. It follows from (1) that, for $\lambda = 1$:

$$
y_{t-1} = \frac{\beta}{1-\alpha} z_t + \frac{c}{1-\alpha} + a_t \tag{4}
$$

where

$$
a_t = -\tfrac{\beta}{1-\alpha}\sum_{s=0}^{\infty}\alpha^s \varepsilon_{t-s} + \sum_{s=0}^{\infty}\alpha^s u_{t-s-1}
$$

(note that y_{t-1} and z_t are cointegrated - see MYTH for details).

Applying (3), the functional central limit theorem, and the continuous mapping theorem (see Phillips (1987)), and performing algebra similar to that in MYTH, we can derive the following Lemma.

Lemma 1.

Let B_v and B_ε be Brownian motions obtained by

$\frac{1}{\sqrt{T}}\sum v_t \Rightarrow B_v$ *and* $\frac{1}{\sqrt{T}}\sum \varepsilon_t \Rightarrow B_\varepsilon$.

Then we have the following asymptotic distributions.

(i) $\lambda = 1$:

(a) $\frac{1}{\sqrt{T}}\sum u_t \Rightarrow \frac{1}{1-\rho}B_v(1)$

(b) $\frac{1}{T}\sum z_t u_t \Rightarrow \frac{1}{1-\rho}\int B_\varepsilon(r)dB_v(r)$

(c) $\frac{1}{T}\sum y_{t-1}u_t$

$\Rightarrow (\frac{\beta}{1-\alpha})(\frac{1}{1-\rho})\int B_\varepsilon(r)dB_v(r) + P$,
where $P = \frac{\rho\sigma_v^2}{(1-\alpha\rho)(1-\rho^2)}$

(d) $\frac{1}{T^{3/2}}\sum y_{t-1} \Rightarrow \frac{\beta}{1-\alpha}\int B_\varepsilon(r)dr$

(e) $\frac{1}{T^2}\sum y_{t-1}z_t \Rightarrow \frac{\beta}{1-\alpha}\int B_\varepsilon^2(r)dr$

(f) $\frac{1}{T^2}\sum z_t^2 \Rightarrow \int B_\varepsilon^2(r)dr$

(g) $\frac{1}{T}\sum z_t\varepsilon_t \Rightarrow \frac{1}{2}[B_\varepsilon^2(1) + \sigma_\varepsilon^2]$

$(h)\ \frac{1}{T}\sum\sum_{s=0}^{\infty}\alpha^s\varepsilon_{t-s}z_t \Rightarrow$
$\frac{1}{2}(\frac{1}{1-\alpha})[B_\varepsilon^2(1)+\sigma_\varepsilon^2]$

$(i)\ \frac{1}{T^{3/2}}\sum z_t \Rightarrow \int B_\varepsilon(r)dr$

$(j)\ \frac{1}{\sqrt{T}}\sum a_t \Rightarrow$
$\frac{1}{1-\alpha}[-\frac{\beta}{1-\alpha}B_\varepsilon(1)+\frac{1}{1-\rho}B_v(1)] \equiv Q_1$

$(k)\ \frac{1}{T}\sum a_t z_t \Rightarrow$
$(\frac{1}{1-\alpha})(\frac{1}{1-\rho})\int B_\varepsilon(r)dB_v(r)$
$-\frac{1}{2}(\frac{\beta}{1-\alpha})(\frac{1}{1-\alpha})[B_\varepsilon^2(1)+\sigma_\varepsilon^2] \equiv Q_2$

$(l)\ \frac{1}{T}\sum a_t^2 \xrightarrow{p} (\frac{1}{1-\alpha^2})(\frac{\beta}{1-\alpha})^2\sigma_\varepsilon^2$
$+\frac{(1+\alpha\rho)\sigma_v^2}{(1-\alpha^2)(1-\alpha\rho)(1-\rho^2)}$
$\equiv Q_3$

$(m)\ \frac{1}{T}\sum a_t y_{t-1} \Rightarrow \frac{\beta}{1-\alpha}Q_2 + Q_3 \equiv Q_4.$

(ii) $|\lambda| < 1$:

$(n)\ \frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t \Rightarrow \frac{1}{1-\rho}B_v(1)$

$(o)\ \frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}u_t \Rightarrow \frac{c}{1-\alpha}\frac{1}{1-\rho}B_v(1)$
$+\frac{\beta}{1-\alpha}\frac{1}{(1-\lambda)(1-\rho)}N(0,\sigma_\varepsilon^2\sigma_v^2)+\sqrt{T}P$

$(p)\ \frac{1}{\sqrt{T}}\sum_{t=1}^{T} z_t u_t \Rightarrow \frac{1}{(1-\lambda)(1-\rho)}N(0,\sigma_\varepsilon^2\sigma_v^2)$

$(q)\ \frac{1}{T}\sum_{t=1}^{T} y_{t-1}z_t \xrightarrow{p} \frac{\lambda}{1-\lambda^2}\frac{\beta}{1-\alpha\lambda}\sigma_\varepsilon^2$

$(r)\ \frac{1}{T}\sum_{t=1}^{T} b_t^2 \xrightarrow{p} \frac{\beta^2(1+\alpha\lambda)}{(1-\alpha\lambda)(1-\lambda^2)(1-\alpha^2)}\sigma_\varepsilon^2$
$+\frac{(1+\alpha\rho)}{(1-\alpha\rho)(1-\rho^2)(1-\alpha^2)}\sigma_v^2 \equiv \tilde{Q}$

$(s)\ \frac{1}{T}\sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{p} \tilde{Q} + (\frac{c}{1-\alpha})^2$

$(t)\ \frac{1}{T}\sum_{t=1}^{T} y_{t-1} \xrightarrow{p} \frac{c}{1-\alpha}$

$(u)\ \frac{1}{T}\sum_{t=1}^{T} z_t^2 \xrightarrow{p} \frac{1}{1-\lambda^2}\sigma_\varepsilon^2.$

Proof. See Appendix A.

This leads to the following Theorems, Corollary and Remarks.

Theorem 1

(i) When $\lambda = 1$ in equation (1), we have

$$\text{plim}(\hat{\alpha}-\alpha) = \gamma^*$$
$$\text{plim}(\hat{\beta}-\beta) = \delta^*$$
$$\sqrt{T}(\hat{\alpha}-\alpha-\gamma^*) \Rightarrow N(0,\sigma^2)$$
$$\sqrt{T}(\hat{\beta}-\beta-\delta^*) \Rightarrow \frac{\beta}{1-\alpha}N(0,\sigma^2)$$
$$where\ \sigma^2 = \frac{1}{(1-\alpha)^2(1-\rho)^2Q_3^2}(\frac{\beta}{1-\alpha})^2\sigma_\varepsilon^2\sigma_v^2$$
$$\gamma^* = \frac{\rho\sigma_v^2}{(1-\alpha\rho)(1-\rho^2)Q_3}\ and\ \delta^* = -\frac{\beta}{1-\alpha}\gamma^*.$$

(ii) When $|\lambda| < 1$ in equation (1), we have

$$\sqrt{T}(\hat{\alpha}-\alpha-\alpha^*) \Rightarrow N(0,\sigma_\alpha^2)$$
$$\sqrt{T}(\hat{\beta}-\beta-\beta^*) \Rightarrow N(0,\sigma_\beta^2)$$
$$\sqrt{T}(\hat{\delta}-\delta-\delta^*) \Rightarrow N(0,\sigma_\delta^2)$$

where $\alpha^* = \frac{P}{D_1}$

$$P = \frac{\rho\sigma_v^2}{(1-\rho^2)(1-\alpha\rho)}$$
$$D_1 = \frac{1}{1-\alpha^2}[\frac{\beta^2\sigma_\varepsilon^2}{(1-\alpha\lambda)^2}+\frac{(1+\alpha\rho)\sigma_v^2}{(1-\rho^2)(1-\alpha\rho)}]$$
$$\beta^* = -\frac{\beta\lambda}{1-\alpha\lambda}\alpha^*$$
$$\sigma_\alpha^2 = [\frac{\beta}{1-\alpha}\frac{1}{(1-\rho)D_1}]^2\sigma_\varepsilon^2\sigma_v^2$$
$$\sigma_\beta^2 = \frac{1}{(1-\alpha^2)^2(1-\rho)^2D_1^2}\times$$
$$(\frac{\beta^2\sigma_\varepsilon^2}{1-\alpha\lambda}+\frac{(1+\lambda)(1+\alpha\rho)\sigma_v^2}{(1-\rho^2)(1-\alpha\rho)})^2\left(\frac{\sigma_v}{\sigma_\varepsilon}\right)^2$$
$$\delta^* = \frac{(\frac{\beta}{1-\alpha}-\frac{\beta\lambda}{1-\alpha\lambda})\alpha^*}{1-\alpha-\alpha^*}\ \text{and}$$
$$\sigma_\delta^2 = (\frac{\frac{\beta}{1-\alpha}\sigma_\alpha+\sigma_\beta}{1-\alpha-\alpha^*})^2.$$

Proof. See Appendix B.

Applying Lemma 1 to (2), and performing lengthy algebraic manipulations, yields the following main result.

Theorem 2

When $\lambda = 1$ in equation (1), we have

$$T(\hat{\delta}-\delta) \Rightarrow \frac{\frac{1}{1-\rho}\int B_\varepsilon(r)dB_v(r) + F}{(1-\alpha)\{\int B_\varepsilon^2(r)dr - (\int B_\varepsilon(r)dr)^2\}} \quad (5)$$

where

$F =$

$-\{\frac{1}{1-\rho}B_v(1) + \frac{\beta}{1-\alpha}\mu B_\varepsilon(1)\}\int B_\varepsilon(r)dr +$

$\frac{1}{2}\frac{\beta}{1-\alpha}\mu\{B_\varepsilon^2(1)+\sigma_\varepsilon^2\}$

$\mu = \frac{(1+\alpha)\rho\sigma_v^2}{(1-\alpha\rho)(1-\rho^2)(\frac{\beta}{1-\alpha})^2\sigma_\varepsilon^2 + (1-\rho)\sigma_v^2}$.

Proof. See Appendix B.

Corollary 1. When the constant term $c = 0$ and $\lambda = 1$ in (1), we have

$$T(\hat{\delta}-\delta) \Rightarrow \frac{\frac{1}{1-\rho}\int B_\varepsilon(r)dB_v(r) + \frac{1}{2}\frac{\beta}{1-\alpha}\mu\{B_\varepsilon^2(1)+\sigma_\varepsilon^2\}}{(1-\alpha)\int B_\varepsilon^2(r)dr}. \quad (6)$$

Proof. The distribution in (5) includes terms associated with $\int B_\varepsilon(r)dr$, which arises from the limits of $\sum y_{t-1}$ and $\sum z_{t-1}$ in the case with a constant term. Therefore, those terms vanish in (5) when the constant term is zero, and so (6) follows.

Remark 1 *$\hat{\delta}$ is $\sqrt{T}-$inconsistent for $\rho \neq 0$ and $|\lambda| < 1$ (it is $\sqrt{T}$ $-$consistent only if $\rho = 0$ and $|\lambda| < 1$). Since $\frac{\sqrt{T}(\hat{\delta}-\delta)}{\sigma_\delta} \Rightarrow N(\frac{\sqrt{T}\delta^*}{\sigma_\delta}, 1)$ because of Theorem 1(ii), it is invalid to draw inferences based on the asymptotic normality of the standardized $\hat{\delta}$, i.e., $\frac{\hat{\delta}-\delta}{s_{\hat{\delta}}}$, where $s_{\hat{\delta}}$ is an estimator of $\sqrt{var(\hat{\delta})}$, the standard error of $\hat{\delta}$, which is approximated by $(\frac{\partial\delta(\theta)}{\partial\theta})'Var(\hat{\theta})(\frac{\partial\delta(\theta)}{\partial\theta})|_{\theta=\hat{\theta}}$, with $\theta = (c, \alpha, \beta)'$.*

Remark 2 *Comparing (5) and (6), the asymptotic distributions depend on the existence of the constant term c but not on the level of c itself.*

Remark 3 *The OLS estimators $\hat{\alpha}$ and $\hat{\beta}$ are $\sqrt{T}-$inconsistent but $\hat{\delta}$ is $T-$consistent because the two asymptotic biases cancel out each other in $\hat{\delta}$. From (3) it is not difficult to see the cancellation. Furthermore, the distributions of $\sqrt{T}(\hat{\alpha}-\alpha)$ and $\sqrt{T}(\hat{\beta}-\beta)$ are asymptotically normal, but $T(\hat{\delta}-\delta)$ has a non-standard asymptotic distribution.*

Remark 4 *Maddala and Rao (1973) showed for the case $|\lambda| < 1$ and $c = 0$:*

$$p\lim(\hat{\alpha}-\alpha) = \gamma'$$
$$p\lim(\hat{\beta}-\beta) = -\frac{\beta\lambda}{1-\alpha\lambda}\gamma'$$

where

$$\gamma' = \frac{\frac{\rho\sigma_v^2}{(1-\alpha\rho)(1-\rho^2)}}{\frac{1}{1-\alpha^2}(\frac{\beta^2}{(1-\alpha\lambda)^2}\sigma_\varepsilon^2 + \frac{(1+\alpha\rho)\sigma_v^2}{(1-\alpha\rho)(1-\rho^2)})}.$$

As noted by MYTH, $\gamma' = \gamma^*$ when $\lambda = 1$. However Maddala and Rao did not examine any asymptotic distributional properties of the estimators.

3 A Quasi-Maximum Likelihood Estimator

It is intractable to derive a closed-form solution of the full maximum likelihood estimator for c, α, β, ρ and δ in (1) because of the non-linearity. In what follows, we shall restrict attention to the quasi-maximum likelihood estimator(QMLE) when ρ is given.

The log-likelihood function is given by

$\mathcal{L}(c, \alpha, \beta; \rho) =$

$\frac{1}{2}\ln(2\pi) - \frac{T}{2}\ln\sigma_v^2 - \frac{1}{2\sigma_v^2}\sum_{t=1}^T v_t^2.$

It is straightforward to see that

$v_t = y_t - \rho y_{t-1} - c(1-\rho) - \alpha(y_{t-1} - \rho y_{t-2})$
$-\beta(z_t - \rho z_{t-1})$
$= Y_t - C - \alpha Y_{t-1} - \beta Z_t$

where

$$C = c(1-\rho),$$
$$Y_t = y_t - \rho y_{t-1},$$
$$Z_t = z_t - \rho z_{t-1}.$$

Inserting Y_t and Z_t into $\mathcal{L}(c,\alpha,\beta;\rho)$ and differentiating $\mathcal{L}$ with respect to C,α,β yields the MLE $\tilde{C},\tilde{\alpha},\tilde{\beta}$ for C,α,β as the solution of the three normal equations:

$$\frac{\partial \mathcal{L}}{\partial C} = \sum_{t=2}^{T}(Y_t - \tilde{C} - \tilde{\alpha}Y_{t-1} - \tilde{\beta}Z_t) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \alpha} = -\sum_{t=2}^{T}(Y_t - \tilde{C} - \tilde{\alpha}Y_{t-1} - \tilde{\beta}Z_t)Y_{t-1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \beta} = -\sum_{t=2}^{T}(Y_t - \tilde{C} - \tilde{\alpha}Y_{t-1} - \tilde{\beta}Z_t)Z_t = 0.$$

Notice that if ρ is given, the $\tilde{C},\tilde{\alpha},\tilde{\beta}$ are simply the OLS estimators for the transformed equation

$$Y_t = C + \alpha Y_{t-1} + \beta Z_{t-1} + v_t. \quad (7)$$

It follows immediately that the OLS estimators $\tilde{C},\tilde{\alpha},\tilde{\beta}$ are consistent because the v_t are not serially correlated, and are uncorrelated with Y_{t-1} and Z_{t-1}. A consistent estimator of ρ is obtained in a similar manner to MYTH, as follows. First, rewrite the transformed equation (7) as

$$y_t = C + a' y_{t-1} + b' y_{t-2} + c' z_t + d' z_{t-1} + v_t \quad (8)$$

where $a' = \alpha + \rho$, $b' = -\alpha\rho$, $c' = \beta$, and $d' = -\beta\rho$, so that $-\frac{d'}{c'} = \rho$. As the OLS estimators $\hat{C}$, $\hat{a}'$, $\hat{b}'$, $\hat{c}'$, $\hat{d}'$ are consistent, $-\frac{\hat{d}'}{\hat{c}'} = \hat{\rho}$ is a consistent estimator of ρ. From these considerations, we propose the following iterative procedure:

(i) Obtain the consistent estimator $\hat{\rho}$ above and use it as an initial value of ρ, denoted by $\rho^{(1)}$.

(ii) Transform the model (1) to the above equation (8).

(iii) Calculate $\tilde{C},\tilde{\alpha},\tilde{\beta}$ from (8) by the OLS method and obtain $\tilde{c} = \frac{\tilde{C}}{1-\hat{\rho}}$.

(iv) Calculate the residuals from the estimated original regression (1), $\hat{u}_t = y_t - \tilde{c} - \tilde{\alpha}y_{t-1} - \tilde{\beta}z_t$.

(v) Iterate steps (i)-(iv) until $\tilde{c},\tilde{\alpha},\tilde{\beta}$ converge.

We will call the resulting estimators $\tilde{c}^*,\tilde{\alpha}^*,\tilde{\beta}^*$ the QMLE. MYTH considered the two-step estimator obtained from steps (i) to (iii) in the absence of a constant term.

The asymptotic distributions of $\tilde{\alpha},\tilde{\beta}$ and $\tilde{\delta} = \frac{\tilde{\beta}}{1-\tilde{\alpha}}$ are obtained by setting $\rho = 0$ in Theorems 1 and 2 and in Corollary 1. Concerning the QMLE $\tilde{\alpha}^*,\tilde{\beta}^*$ and $\tilde{\delta}^* = \frac{\tilde{\beta}^*}{1-\tilde{\alpha}^*}$, the asymptotic distributions are the same as those for $\hat{\alpha},\hat{\beta}$ and $\hat{\delta}$ in the section 2.

4 Simulations

Monte Carlo experiments were performed on the following specific model:

$$y_t = 1 + 0.38 y_{t-1} + 0.4 z_t + u_t,$$
$$u_t = \rho u_{t-1} + v_t, \qquad |\rho| < 1, \sigma_v^2 = 0.25$$
$$z_t = \lambda z_{t-1} + \varepsilon_t, \qquad |\lambda| \le 1, \sigma_\varepsilon^2 = 1.0$$
$$t = 1, 2, \cdots, T$$

where the long-run MPC, based on $\alpha = 0.38$ and $\beta = 0.4$, is $\delta = 0.645$. In the experiments, the parameters are specified as $\rho = 0.0, 0.5, 0.8; \lambda = 0.5, 1.0; T = 30, 100, 500, 1000$. We calculated $d = (\hat{\delta} - \delta)/s_{\hat{\delta}}$ 5000 times for each parameter combination, where $s_{\hat{\delta}}$ is the estimated standard error. The asymptotic distribution of d is $N(\frac{\sqrt{T}}{\sigma_\delta}\delta^*, 1)$ for $|\lambda| < 1.0$.

From Figures 1-8, we observe the following:

(1) $|\boldsymbol{\lambda}| < \mathbf{1}$:

1. When $\rho = 0$, the distribution of d is very close to the standard normal, even when T is small, and hence the bias of $\hat{\delta}$ is very small.
2. When $\rho \neq 0$, the mean of d becomes large as the sample size T increases because the mean is proportional to $\sqrt{T}$, as in Remark 1. As ρ increases, the normal approximation deteriorates badly.

(2) $\boldsymbol{\lambda = 1}$:

1. $\hat{\delta}$ is always $T-$consistent.

2. The asymptotic distribution of $\hat{\delta}$ is more concentrated than the normal (the kurtosis is larger than 3, as in the case of the normal).

5 Concluding Remarks

We have investigated the asymptotic properties of the OLS-based estimator of the long-run MPC when the income variable is integrated of order 1. In such a case, standard statistical inference which relies on asymptotic normality is biased and misleading. More importantly, inferences based on asymptotic normality is much worse when the independent variable is stationary ($|\lambda| < 1$) and the disturbances are serially correlated because the standardized estimator of the long-run MPC is $\sqrt{T} - inconsistent$, so that the standardized ratio $d = (\hat{\delta} - \delta)/s_{\hat{\delta}}$ is also $\sqrt{T}$ − *inconsistent.* The theoretical results were supported by Monte Carlo simulations, which also examined the implications for statistical inferences when the integrated regressor is misspecified as stationary.

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Fig. 1

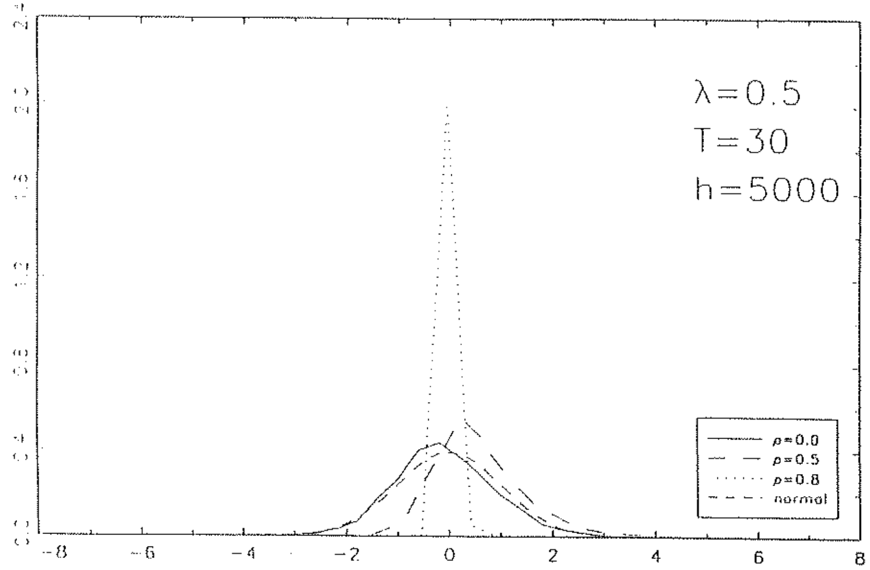

Fig. 3

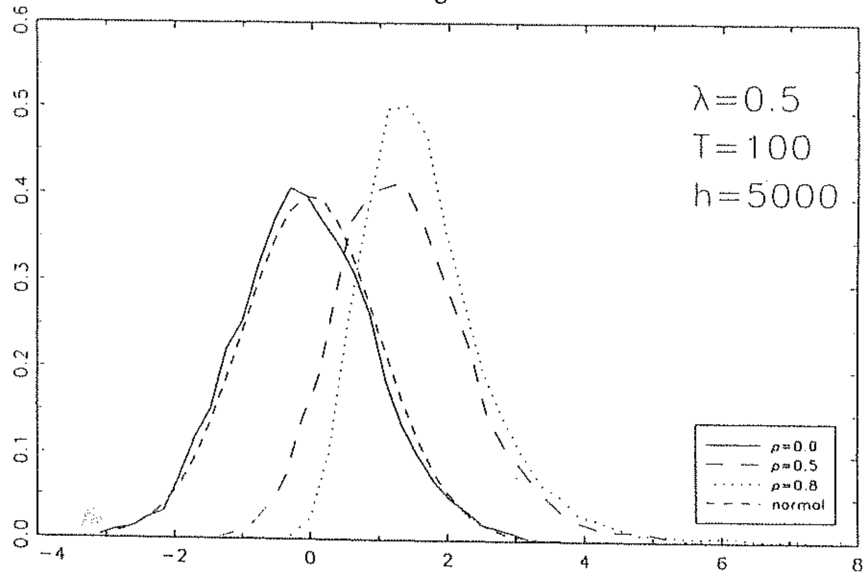

Fig. 2

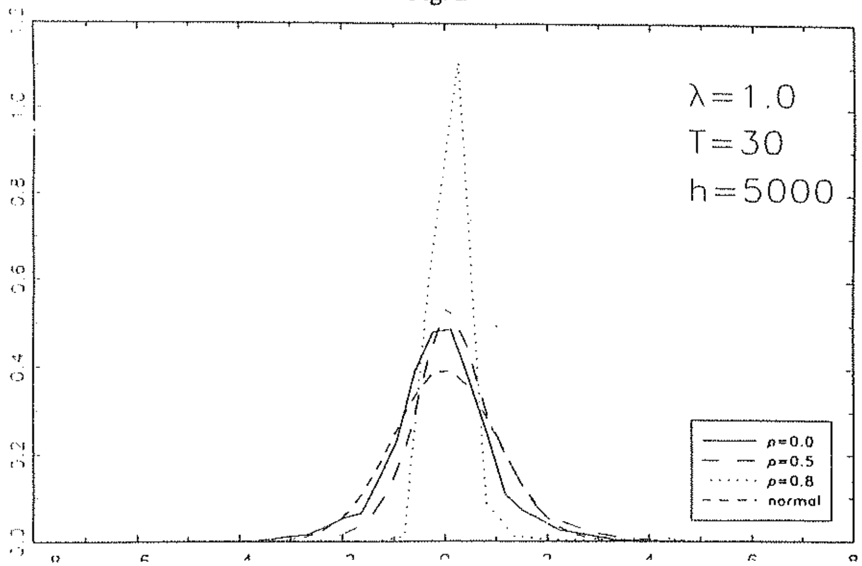

Fig. 4

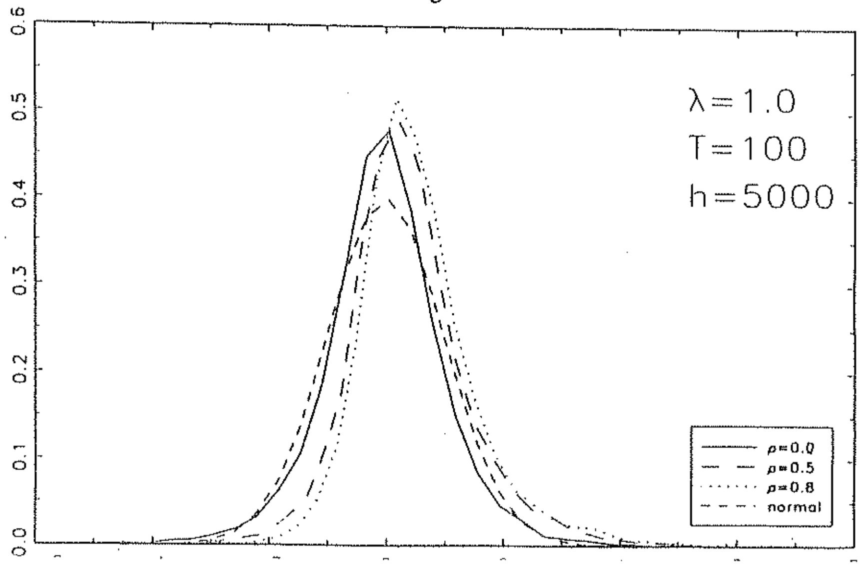

Fig. 5

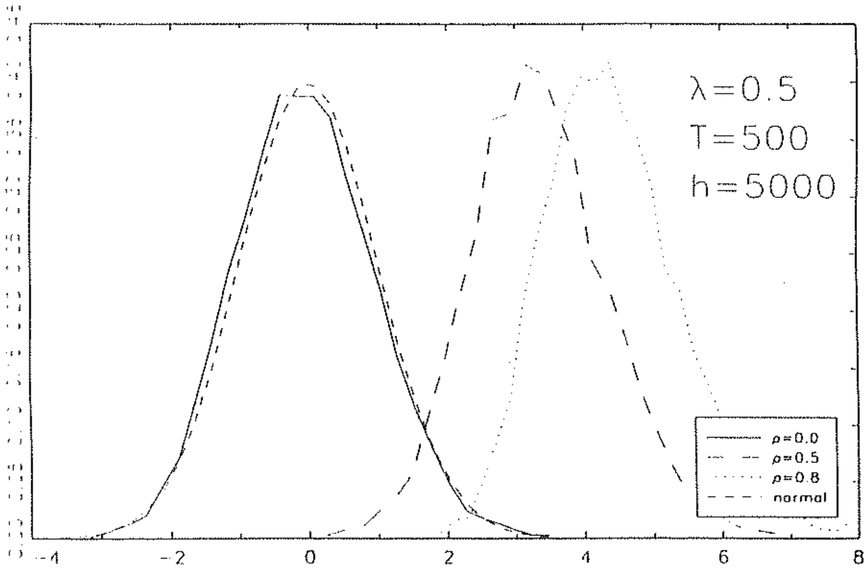

Fig. 7

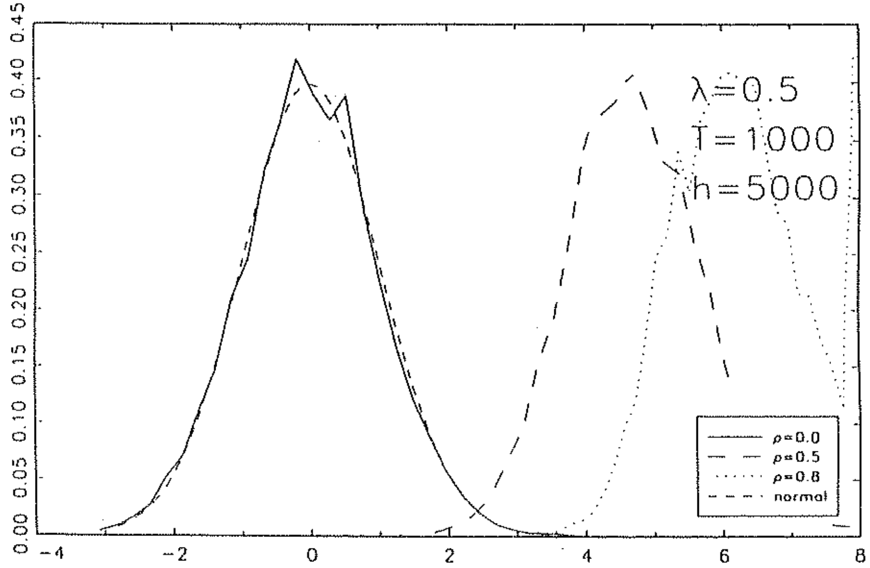

Fig. 6

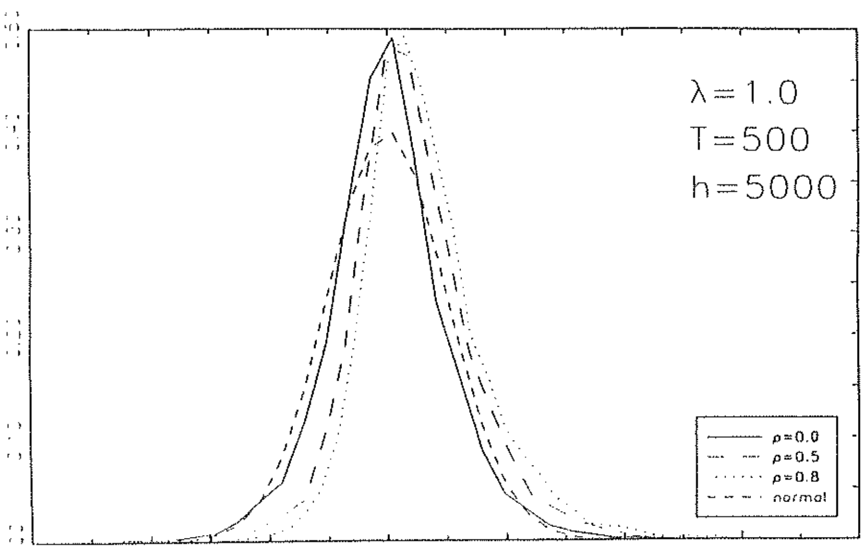

Fig. 8