A Brownian Bridge Bias Correction Method for Simulation of Exotic Option Valuation

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Abstract This paper studies bias correction in Monte Carlo simulation of exotic option valuation. Due to the discretization error of the stochastic differential equation of the diffusion process for the risk-neutralized stock price, the maximum (minimum) of the option price obtained in simulation is underestimated (over-estimated). By applying the Brownian bridge in each discretized time interval, we obtain an analytical expression for the expected value of the bias. The simulation results are improved significantly by adding this bias term. Numerical examples are given to demonstrate the efficient performance of our method.

Key Words: Monte Carlo Simulation, Brownian Bridge, Option Valuation.

1. INTRODUCTION

Monte Carlo simulation is a standard technique in valuing exotic option price over a specified time interval. It simply samples the asset price path at the discrete time intervals. The method gives a bias estimation for the extreme values of the stock price because at many times between the intervals' end points, the maximum (minimum) may occurred and ignored. Shorten the length of the time subintervals can reduce the bias and give a more accurate answer. However, the bias error reduces slowly when decreasing the length of the time (see the numerical examples). In this paper we study the bias correction in Monte Carlo simulation of exotic option valuation by Brownian bridge. We consider the discretization of the stochastic differential equation of the diffusion process for the risk-neutralized stock price:

$$dS(t) = rS(t)dt + \sigma(t, S(t))dW(t). \quad (1)$$

Here let us define some notations which we will be using throughout the discussion.

(i) $S(t)$, the stock price at time t,

(ii) $S(0)$, today's stock price,

(iii) $W(t)$, the standard Wiener process,

(iv) r, the riskless rate,

(v) $\sigma(t, S(t))$, the volatility of the stock,

(iv) T, the maturity time,

(v) E, the strike price.

In the standard simulation approach, we divide the interval $[0, T]$ into n equal intervals as follows:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < T_n = T.$$

The discretized model of (1) is then given by

$$\begin{array}{l} S(t_{i+1}) - S(t_i) = \\ rS(t_i)(T/n) + \sigma(t_i, S(t_i))[W(t_{i+1}) - W(t_i)] \\ \quad i = 1, 2, \cdots, n. \end{array} \quad (2$$

Here we assume that the volatility $\sigma(t_i S(t_i))$ is constant in any sub-interval $[t_i, t_{i+1}]$. We observe that the difference of the Wiener process $[W(t_{i+1}) - W(t_i)]$ is a normal distribution of mean zeros and variance T/n. By using this fact and (2) we can obtain a sample path of the stock price. Let us consider the Max option as an example. The payoff of this option in the discretized time interval is evaluated as follows:

$$X = \max\{0, \max_{o \leq i \leq n} S(t_i) - E\}.$$

However, to avoid the stock price from going to negative, we consider the following revised model (the stock price is lognormal):

$$d(\log(S)) = rdt + \frac{\sigma(t, S(t))}{S(t)} dW(t).$$

The discretized model is then given by

$$\log(S(t_{i+1})) - \log(S(t_i)) = r(T/n) + \frac{\sigma(t_i, S(t_i))}{S(t_i)}(W(t_{i+1}) - W(t_i)). \quad (3)$$

The standard simulation algorithm for the Max option reads:

For $j = 1$ to M
$S(t_0) = S(0)$;
For $i = 1$ to n
Generate $N \sim N(0, 1)$;
$S(t_i) = S(t_{i-1}) exp(r(T/n) + \frac{\sigma(t_i, S(t_i))}{S(t_i)} \sqrt{T/nN})$;
end;
$X_j = \max\{0, \max_{o \leq i \leq n} S(t_i) - E\}$;
end;
$X = e^{-rT} \frac{1}{M} \sum_{j=1}^{M} X_j$.

We remark that in our numerical examples, we assume that $\sigma(t, S(t)) = \sigma S(t)$. Thus in the discretized model algorithm we have the term

$$\frac{\sigma(t_i, S(t_i))}{S(t_i)} = \sigma.$$

Many options such as the Max option, the Knock-out option and the Swing option (they will be discussed afterwards) require the maximum and minimum of the stock price see [1, 2, 3] for instance. Thus the method above will introduce a bias for the these extreme values (maximum and minimum) of the stock price. To deal with this bias, we apply the Brownian bridge method in each discretized interval and obtain an analytical expression for the expected value of the bias. The simulation results are improved significantly by adding this bias term. Numerical examples are given to demonstrate the efficient performance of our method.

The remainder of the paper is organized as follows. In §2, we derive the expression of the bias by using the Brownian bridge method. In §3, we give numerical examples to compare our method with the standard Monte Carlo and the method discussed in [1] for Max, Knock-out and Swing options. Concluding remarks are given in §4 to address some extensions of our method.

2. BIAS CORRECTION BY BROWIAN BRIDGE METHOD

In this section, we discuss bias correction by using Brownian bridge method.

Lemma 1 *Let W_t be a Wiener process in the time interval $[0, \tau]$, then we have the conditional probability*

$$P[max_{0 \leq t \leq \tau} W(t) \geq \beta | W(\tau) = \alpha] = e^{-2\beta(\beta - \alpha)/\tau}.$$

This result is known as the 'tied down Brownian motion' or the 'Brownian bridge'.

Proof: See [5, p. 265]. □

We note that Lemma 1 simply tells us

$$P[max_{0 \leq t \leq \tau} W(t) \leq \beta | W(\tau) = \alpha] = 1 - e^{-2\beta(\beta - \alpha)/\tau}. \quad (4)$$

In [1], the authors apply Lemma 1 to each sub-interval $[t_i, t_{i+1}]$ and generate a random variable B_{max} with cumulative distribution function (4). This can be done by the using following function

$$\frac{[\alpha + \sqrt{\alpha^2 - 2(T/n)\log(1 - U)}]}{2} \text{ where } U \sim U[0, 1]$$

Hence $S(t_i) exp(\sigma(B_{max}))$ (c.f. standard simulation algorithm) is used as an estimation for

$$\max_{t_i < t \leq t_{i+1}} \{S(t)\}.$$

Here we improve the result (see the numerical examples) by using the following proposition.

Proposition 1 *Let W_t be a Wiener process in the time interval $[0, \tau]$, then we have*

$$E(max_{0 \leq t \leq \tau}) = \alpha + e^{\alpha^2/(2\tau)} [\sqrt{\frac{\pi \tau}{8}} - \int_0^{\alpha/2} e^{\frac{-2u^2}{\tau}} du]$$

and

$$E(min_{0\leq t\leq\tau}) = \alpha - e^{\alpha^2/(2\tau)}[\sqrt{\frac{\pi\tau}{8}} - \int_0^{\alpha/2} e^{\frac{-2u^2}{\tau}}du].$$

Proof:

By Lemma 1, we have

$$P[max_{0\leq t\leq\tau}W(t) \leq \beta | W(\tau) = \alpha] = 1 - e^{-2\beta(\beta-\alpha)/\tau}.$$

By differentiation we have the conditional probability density function of $(max_{0\leq t\leq\tau}W(t))$ given by

$$f[max_{0\leq t\leq\tau}W_t = y | W_\tau = \alpha] = \frac{2(2y-\alpha)}{\tau}e^{-2y(y-\alpha)/\tau}.$$

Therefore we have

$$\begin{aligned} E(max_{0\leq t\leq\tau}) &= \int_\alpha^\infty \tfrac{2(2y-\alpha)y}{\tau} e^{-2y(y-\alpha)/\tau}dy \\ &= \alpha + \{e^{\frac{\alpha^2}{2\tau}}[\sqrt{\tfrac{\pi\tau}{8}} - \int_0^{\alpha/2} e^{\frac{-2u^2}{\tau}}du]\}. \end{aligned}$$

The expression in $\{.\}$ is the expected bias. By symmetry we also have

$$E(min_{0\leq t\leq\tau}) = \alpha - e^{\alpha^2/(2\tau)}[\sqrt{\frac{\pi\tau}{8}} - \int_0^{\alpha/2} e^{\frac{-2u^2}{\tau}}du].$$

□

By applying Proposition 1 to each sub-interval $[t_i, t_{i+1}]$, to correct the bias due to discretization, we can get a more accurate valuation of the stock price. In the following section we will demonstrate the efficient performance of our method by some numerical examples.

3. NUMERICAL EXAMPLES FOR SOME EXOTIC OPTION VALUATIONS

In this section, we give some simulation results for the Max, Knock-out and Swing options. We compare our results with the standard Monte Carlo simulation and the bias correction method discussed in [1]. All the simulations are done by using MATLAB in a HP 712 workstation. We denote X_M, X_{max} and X_E to be the results simulated by standard Monte Carlo, bias correction method in [1] and our Brownian bridge bias correction method respectively.

3.1 The Max Call Option

The Max call option is discussed in §1. In the following simulations, we consider Max call option with two different maturities $T = 1/12$ (one month) and $T = 1/4$ (three months) in Table 1 and Table 2 respectively. The initial price (today price) $S(0)$ is 50 and strike price $E = S(0) = 50$. For $T = 1/12$, we consider two different number of discretization intervals: $n = 30$ (one day) and $n = 60$ (half a day) respectively. For $T = 1/4$, we consider two different number of discretization intervals: $n = 90$ (one day) and $n = 180$ (half a day) respectively. The analytic value of the option has the following formula (see [3]):

$$S(0)\{e^{-rT}N(d_1)(1-\frac{\sigma^2}{2r}) + e^{-rT}(1-N(d_2))(1+\frac{\sigma^2}{2r}) -$$

where

(5)
$$d_1 = -(r - \frac{\sigma^2}{2r})\sqrt{\frac{T}{\sigma^2}} \text{ and } d_2 = -(r + \frac{\sigma^2}{2r})\sqrt{\frac{T}{\sigma^2}}$$

and $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}dt$. The following tables give the simulated results and the percentage errors of three methods. Among all the three methods and the simulated examples, our method gives the best performance. The standard Monte Carlo simulation always under-estimate the option values.

	$n = 30$		
m	1000	5000	10000
X_M	1.996 (27%)	2.389 (13%)	2.549 (8%)
X_{max}	2.868 (5%)	3.005 (10%)	2.889 (6%)
X_E	2.608 (5%)	2.847 (4%)	2.838 (4%)

	$n = 30$		
m	1000	5000	10000
X_M	2.472 (10%)	2.532 (7%)	2.625 (4%)
X_{max}	2.988 (9%)	2.933 (7%)	2.877 (2%)
X_E	2.877 (5%)	2.688 (2%)	2.712 (1%)

The maturity $T = 1/12$ (one month), $\sigma = 0.25$, $\mu = r = 0.1$. The true value is 2.732.

Table 1

	$n = 90$		
m	1000	5000	10000
X_M	4.259 (6%)	4.356 (4%)	4.445 (2%)
X_{max}	4.795 (6%)	4.679 (3%)	4.627 (2%)
X_E	4.377 (4%)	4.567 (1%)	4.553 (0.4%)

	$n = 180$		
m	1000	5000	10000
X_M	4.288 (5%)	4.381 (3%)	4.407 (3%)
X_{max}	4.852 (7%)	4.703 (4%)	4.600 (1%)
X_E	4.633 (2%)	4.598 (1%)	4.545 (0.2%)

The maturity $T = 1/4$ (three months), $\sigma = 0.25$, $\mu = r = 0.1$. The true value is 4.537.

Table 2

3.2 The Knock-out Option

In the Knock-out option, the traders believe that the stock price will go down once a certain support level H is penetrated. Traders who have a bullish short-term and long-term outlook on the stock and would like a cheap call would be interested in this kind of down-and-out call. The payoff of a Knock-out option is given as follows (see [2, p. 410] and [6]):

$$B(S(0), E) - [\frac{S(0)}{H}]^{-2r/\sigma} B(\frac{H^2}{S(0)}, \frac{H^2}{E}).$$

Here $B(S, E)$ is the standard Black-Sholes formula for a call option with stock price S and strike price E. For the standard simulation, the payoff is given by

$$X = \begin{cases} S(T) - E & \text{if } X(T) > E \text{ and } \min_{0\le i\le n} S(t_i) > H \\ 0 & \text{otherwise.} \end{cases}$$

In the following numerical examples, we let $H = 45$, $E = S(0) = 50$. We consider Knock-out option with two different maturities $T = 1/4$ (three months) and $T = 1$ (one year) in Table 3 and Table 4 respectively. For $T = 1/4$, we consider two different number of discretization intervals: $n = 30$ (three-day) and $n = 90$ (one-day) respectively. For $T = 1$, we consider two different number of discretization intervals: $n = 54$ (one-week) and $n = 365$ (one-day) respectively. Among all the three methods and almost all the simulated examples, our method gives the best performance.

	$n = 30$		
m	1000	5000	10000
X_M	4.596 (14%)	4.449 (10%)	4.244 (5%)
X_{max}	3.468 (14%)	3.655 (9%)	3.881 (4%)
X_E	3.781 (6%)	3.824 (5%)	3.912 (3%)

	$n = 90$		
m	1000	5000	10000
X_M	4.342 (8%)	4.231 (4%)	4.205 (4%)
X_{max}	3.738 (7%)	3.823 (5%)	3.895 (3%)
X_E	3.811 (5%)	3.998 (1%)	4.012 (1%)

The maturity $T = 1/4$ (three months), $\sigma = 0.5, \mu = 0.15, r = 0.1$. The true value is 4.032.

Table 3

	$n = 54$		
m	1000	5000	10000
X_M	7.825 (43%)	6.533 (19%)	5.977 (9%)
X_{max}	6.545 (19%)	5.757 (5%)	5.732 (5%)
X_E	4.757 (13%)	5.065 (8%)	5.353 (2%)

	$n = 360$		
m	1000	5000	10000
X_M	6.321 (15%)	6.041 (10%)	5.955 (9%)
X_{max}	5.742 (5%)	5.703 (4%)	5.662 (3%)
X_E	4.648 (15%)	5.258 (4%)	5.395 (1%)

The maturity $T = 1$ (one year), $\sigma = 0.5, \mu = 0.15, r = 0.1$. The true value is 5.477.

Table 4

3.3 The Swing Option

The Swing option is useful for investors who expect the stock to have large trading range. The option has no analytic solution and can be made by buying a butterfly spread call option, see [4, p. 180]. The payoff of this option is given by:

$$X = \max\{0, \max_{0\le i\le n} S(t_i) - \min_{0\le i\le n} S(t_i) - E\}.$$

In the following numerical examples, we let $E = S(0) = 50$. We consider Knock-out option with two different maturities $T = 1/4$ (three months) and $T = 1$ (one year) in Table 5 and Table 6 respectively. For $T = 1/4$, we consider two different number of discretization intervals: $n = 30$ (three-day) and $n = 90$ (one-day) respectively. For $T = 1$, we consider two different number of discretization intervals: $n = 54$ (one-week) and $n = 365$ (one-day) respectively. Again among all the three methods and almost all the simulated examples, our method gives the best performance.

	$n = 30$		
m	1000	5000	10000
X_M	0.725 (43%)	0.824 (35%)	0.925 (28%)
X_{max}	0.948 (26%)	0.995 (22%)	1.025 (20%)
X_E	0.971 (24%)	1.112 (13%)	1.155 (10%)

	$n = 90$		
m	1000	5000	10000
X_M	1.042 (18%)	1.095 (14%)	1.102 (14%)
X_{max}	1.053 (18%)	1.133 (11%)	1.195 (6%)
X_E	1.199 (6%)	1.222 (4%)	1.255 (2%)

The maturity $T = 1/4$ (three months), $\sigma = 0.25, \mu = 0.15, r = 0.1$. The true value ≈ 1.278.

Table 5

	$n = 54$		
m	1000	5000	10000
X_M	8.343 (19%)	8.678 (16%)	9.548 (7%)
X_{max}	9.150 (11%)	9.545 (7%)	9.832 (4%)
X_E	9.535 (7%)	9.875 (4%)	9.953 (3%)

	$n = 360$		
m	1000	5000	10000
X_M	8.816 (14%)	9.066 (12%)	9.795 (5%)
X_{max}	9.972 (3%)	10.071 (2%)	10.082 (2%)
X_E	10.048 (2%)	10.158 (1%)	10.311 (0.3%)

The maturity $T = 1$ (one year), $\sigma = 0.25, \mu = 0.15, r = 0.1$. The true value ≈ 10.282.

Table 6

4. CONCLUDING REMARKS

We apply the Brownian bridge in bias correction of the Monte Carlo simulation of exotic option valuation. We obtain an analytical expression for the expected value of the bias. The simulation results are improved significantly by adding this bias term. Numerical examples are given to demonstrate the efficient performance of our method.

Further results can be done on the valuation of average options, see [4]. In this case, the payoff of the option is given by

$$\frac{1}{T}\int_0^T S(t)dt.$$

We can still apply the Brownian bridge method to each sub-interval $[t_i, t_{i+1}]$ to obtain a better result. One can show easily that

$$B(t) = a(1 - \frac{t}{T}) + b\frac{t}{T} + (W(t) - \frac{t}{T}W(T))$$
$$\text{for } 0 \le t \le T,$$

is a Brownian bridge from a to b on the time interval $[0, T]$, see [5, p. 360]. Using this result, we can get a better estimation of the payoff. Secondly for option like Lookback call, [1] the volatility $\sigma(t, S(t))$ shares similar stochastic process of the stock price. We can still apply our method to get a better estimation of $\sigma(t, S(t))$. It will be interesting to extend our method to the above two cases.

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