Numerical Simulation of Granular Flows Using a Viscous Plasticity Model

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Abstract Many industrial processes involve granular flows. To optimize these processes, it is essential to be able to simulate the flow accurately. However, due to the complexity of granular flows, there is no generally accepted theory at present. This paper presents a finite element formulation of an idealized granular material which is assumed to be viscous-plastic. The flow is modelled within a bin-hopper combination. The governing equations are solved numerically by the finite element method in space and by the finite difference method in time. The resulting system of non-linear equations is solved using the Quasi-Newton method. It is shown that the model captures the overall behaviour of flowing granular materials.

1. INTRODUCTION

The subject of granular flow has significant impact on many industrial processes such as mineral processing. However, unlike other continuum mechanics disciplines, the flow of granular materials is still quite a mystery. This is because granular materials are neither solid nor fluid, but share some characteristics of both, and there is no generally accepted theory for such materials at present.

Over the last few decades, many theoretical and numerical investigations have been carried out to study the flow of granular materials. Most theoretical investigations are focused on developing proper constitutive equations, as these are the equations that distinguish granular flow from other kinds of material flows. Through the joint efforts of mathematicians and other scientists, two types of constitutive models have been developed, namely the plasticity flow model (Spencer, 1982; Collins, 1990; Hill and Wu, 1992) and the non-Newtonian fluid flow model (Shen and Hopkins, 1988; Goldshtein and Shapiro, 1995; Campbell, 1990). A number of attempts have also been made to construct models with both the plasticity feature of solids and the viscous behaviour of fluids (Rombach and Eibl, 1995; Schmidt and Wu, 1989). This type of model is referred to as viscous plasticity model.

In the present work, we study the flow of granular materials using the viscous plasticity model. The stress tensor is assumeed to consist of a rate-dependent part σ_v and a rate independent part σ_s, thus

$$\overset{\circ}{\sigma} = \overset{\circ}{\sigma}_v + \overset{\circ}{\sigma}_s = \mathrm{Hd} + \mathrm{G}\overset{\circ}{\mathrm{d}}, \qquad (1)$$

where H is the so-called elastic-plastic matrix and its form depends on the plasticity model used, G is a viscous matrix, $\overset{\circ}{\sigma}$ and $\overset{\circ}{\mathrm{d}}$ are the co-rotational rates of stress and deformation rate respectively, i.e.

$$\overset{\circ}{\sigma} = \frac{\partial \sigma}{\partial t} + v \cdot \nabla \sigma + \sigma w - w\sigma,$$

$$\overset{\circ}{\mathrm{d}} = \frac{\partial \mathrm{d}}{\partial t} + v \cdot \nabla \mathrm{d} + \mathrm{d}w - w\mathrm{d}, \qquad (2)$$

where d and w are respectively deformation rate and spin tensor defined by

$$\mathrm{d}_{rs} = \tfrac{1}{2}(\mathrm{v}_{r,s} + \mathrm{v}_{s,r}), \; \mathrm{w}_{rs} = \tfrac{1}{2}(\mathrm{v}_{r,s} - \mathrm{v}_{s,r}), \qquad (3)$$

where we have used the so-called index notation with comma representing differentiation and repeated indices indicating summation over the index range and this notation will be used throughout this paper.

The purpose of this paper is to present a finite element formulation within a proper mathematical framework for the flow of an idealized granular material whose constitutive relations can be described by equation (1).

2. VARIATIONAL STATEMENT OF THE PROBLEM

Consider the flow of a granular material which occupies the spatially fixed region Ω with a boundary $\partial\Omega$ consisting of three parts: $\partial\Omega_a$ with rigid constrains, $\partial\Omega_t$ with the prescribed tractions $\overline{f}$, and $\partial\Omega_g$ with rigid constraints in the normal direction and frictional forces along the tangential direction, as shown in figure 1. Under the viscous plasticity assumption, the granular flow is governed by the standard equations of motion, a set of visco-plastic constitutive equations, and a set of boundary conditions. Thus the problem considered can be described by the following boundary value problem.

Problem 2.1: Find v and σ such that

$$\sigma_{ij,i} + X_j - \rho\frac{Dv_j}{Dt} = 0 \quad in\ \Omega, \tag{4}$$

$$\overset{\circ}{\sigma}_{ij} = H_{ijrs}\, d_{rs} + G_{ijrs}\, \overset{\circ}{d}_{rs} \quad in\ \Omega, \tag{5}$$

$$v_i\big|_{t=0} = v_i^0\,,\ \sigma_{ij}\big|_{t=0} = \sigma_{ij}^0 \quad in\ \Omega, \tag{6}$$

$$v_i = 0 \quad on\ \partial\Omega_a\,, \tag{7}$$

$$f_i = \sigma_{ij}n_j = \overline{f}_i \quad on\ \partial\Omega_t\,, \tag{8}$$

$$v_n = v_i n_i = 0,\ f_t = -\mathrm{sgn}(v_t) f_n \mu_w \quad on\ \partial\Omega_f \tag{9}$$

where $\Omega \in R^2, \partial\Omega = \partial\Omega_a \cup \partial\Omega_t \cup \partial\Omega_f$ is the boundary of Ω, n denotes the unit vector in the outward normal direction of $\partial\Omega$, v_i^0 and σ_{ij}^0 are the initial values of velocity and stress.

In what follows, $L^2(\Omega)$ and $H^1(\Omega)$ denote respectively the square integrable function space and the usual sobolev space with norm $\|\cdot\|_2$, namely

$$L^2(\Omega) = \left\{ v: \int_\Omega v_i^2 d\Omega < \infty \right\},$$

$$H^1(\Omega) = \left\{ v \in L^2(\Omega): v\big|_{\partial\Omega_a} = 0\ and\ v_i n_i\big|_{\partial\Omega_f} = 0 \right\}.$$

To solve *Problem 2.1* numerically, we need to derive a corresponding variational boundary value problem. For this purpose we firstly find the weak form of (4) by making the sum of residuals 'orthogonal' to all functions w_j in the test space $H^1(\Omega)$,

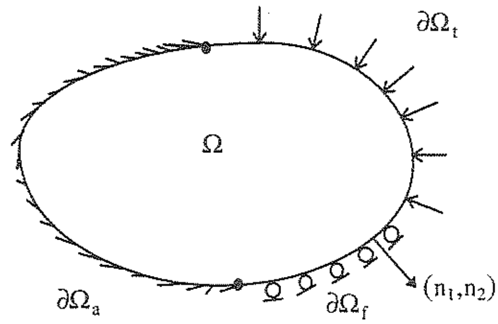

Figure 1: Types of boundary conditions

$$\int_\Omega \left(\sigma_{ij,i} + X_j - \rho\frac{Dv_j}{Dt} \right) w_j d\Omega = 0\ . \tag{10}$$

Integrating (10) by parts and using Green's theorem in the plane, leads to the following weak form

$$\int_\Omega \left(-w_{j,i}\sigma_{ij} + X_j w_j - \rho\frac{Dv_j}{Dt} w_j \right) d\Omega + \int_{\partial\Omega^l} f_j w_j ds = 0, \tag{11}$$

where $\partial\Omega^l$ denotes $\partial\Omega_t \cup \partial\Omega_f$.

Therefore, *Problem 2.1* has now been converted to the following variational boundary value problem.

Problem 2.2: Find $v_j \in H^1(\Omega)$ such that, with relation (5) and condition (6), equations (10) are satisfied for all $w_j \in H^1(\Omega)$.

3. FORMULATION OF THE METHOD

To solve *Problem 2.2* numerically, we pose the problem in the N-dimensional subspace of $H^1(\Omega)$. The variable t is fixed and the space variables are discretized. Thus

$$w_j \approx w_j^h = \sum_{k=1}^N \beta_{jk}\phi_k(x), \tag{12}$$

$$v_j \approx v_j^h = \sum_{k=1}^N \phi_k(x) a_j^k(t), \tag{13}$$

Substituting (12) into (11) yields

$$\sum_{k=1}^N \left[\int_\Omega \left(-\phi_{k,i}\sigma_{ij} - \rho\frac{Dv_j}{Dt}\phi_k + X_j\phi_k \right) d\Omega + \int_{\partial\Omega^l} f_j\phi_k ds \right] \beta_{jk} = 0\ . \tag{14}$$

At interior points and points on $\partial\Omega_t$, we can choose $w_1 = \phi_k$ and $w_2 = 0$ or $w_1 = 0$ and $w_2 = \phi_k$ for all k, namely

$$\beta_{mn} = \begin{cases} 1 & \text{if } n = k \text{ and } m = j \\ 0 & \text{otherwise} \end{cases}. \qquad (15)$$

Thus we have for k=1,..., N-N_b and j=1, 2,

$$\int_\Omega \left(\phi_{k,i}\sigma_{ij} + \rho \frac{Dv_j}{Dt}\phi_k \right) d\Omega = \int_\Omega X_j \phi_k d\Omega + \int_{\partial\Omega^l} f_j \phi_k ds, \qquad (16)$$

where N_b denotes the number of nodes on $\partial\Omega_a$ and $\partial\Omega_f$.

At points on $\partial\Omega_f$, we need to choose w_j properly such that it is in the function space $H^1(\Omega)$. Let $\mathbf{n} = (n_1, n_2)$ be the outward unit normal at a point on $\partial\Omega_f$, then to satisfy the condition that $w_i n_i = 0$, w_j must be chosen as follows: .

$$w_1 = n_2 f_k, \quad w_2 = -n_1 f_k.$$

In other words, the β_{mn} in (14) must be chosen such that

$$\beta_{1n} = \begin{cases} n_2 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases},$$

$$\beta_{2n} = \begin{cases} -n_1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}. \qquad (17)$$

Substituting (17) into (14) yields

$$\int_\Omega \left(\phi_{k,i}\sigma_{ij} + \rho\phi_k \frac{Dv_j}{Dt} - X_j\phi_k \right) n_j d\Omega = \int_{\partial\Omega^l} f_s \phi_k ds,$$

$$(k = N\text{-}N_b+1,...,N). \qquad (18)$$

The next step is to approximate v by v^h. Substituting (13) into (16) yields

$$\sum_{l=1}^{N} \left\{ \left[\int_\Omega \rho\phi_k\phi_l d\Omega \right] \frac{\partial a_j^l}{\partial t} + \left[\int_\Omega \left(\rho v_i \frac{\partial \phi_l}{\partial x_i} \phi_k \right) d\Omega \right] a_j^l \right\}$$

$$+ \int_\Omega \phi_{k,i}\sigma_{ij} - \int_\Omega X_j \phi_k d\Omega - \int_{\partial\Omega^l} f_s \phi_k ds = 0, \qquad (19)$$

which gives,

$$\sum_{l=1}^{N} \left\{ \begin{bmatrix} M_{kl} & 0 \\ 0 & M_{kl} \end{bmatrix} \begin{bmatrix} \frac{da_1}{dt} \\ \frac{da_2}{dt} \end{bmatrix} + \begin{bmatrix} M_{ckl} & 0 \\ 0 & M_{ckl} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} r_{1k} + p_{1k} \\ r_{2k} + p_{2k} \end{bmatrix} \right\} = 0. \qquad (20)$$

where

$$M_{kl} = \int_\Omega \rho\phi_k\phi_l d\Omega,$$

$$M_{ckl} = \int_\Omega \rho v_i \phi_{l,i}\phi_k d\Omega,$$

$$r_{jk} = \int_\Omega \phi_{k,i}\sigma_{ij} d\Omega,$$

$$p_{jk} = \int_\Omega X_j\phi_k d\Omega + \int_{\partial\Omega} f_j\phi_k ds.$$

Similarly, by substituting (13) into (18) , we have

$$\sum_l \left\{ \begin{bmatrix} M_{kl}n_1 & M_{kl}n_2 \end{bmatrix} \begin{bmatrix} \frac{da_1}{dt} \\ \frac{da_2}{dt} \end{bmatrix} + \begin{bmatrix} M_{ckl}n_1 & M_{ckl}n_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right.$$

$$\left[(r_{1k} + p_{1k})n_1 + (r_{2k} + p_{2k})n_2 \right]. \qquad (21)$$

Equations (20) and (21) can now be written in the general form

$$M\frac{da}{dt} + M_c a = r + p. \qquad (22)$$

At a typical instant of time t^n, equation (22) can be approximated by

$$M\frac{a^{n+1} - a^n}{\Delta t} + M_c a^{n+1} + r^{n+1} - p^{n+1} = 0, \qquad (23)$$

where

$$r^{n+1} = \int_\Omega \phi_{k,i}\sigma_{ij}^{n+1}.$$

As from equation (1),

$$\sigma_{ij}^{n+1} = \sigma_{ij}^n + \left(H_{ijrs} d_{rs}^{n+1}\Delta t + G_{ijrs}\left(d_{rs}^{n+1} - d_{rs}^n \right) + \tau_{ij}\Delta t \right), \qquad (24)$$

we have

$$r_k^{n+1} = r_k^n + \sum_l \begin{bmatrix} K_{kl}^{11} & K_{kl}^{12} \\ K_{kl}^{21} & K_{kl}^{22} \end{bmatrix} \begin{bmatrix} a_1^l \\ a_2^l \end{bmatrix} \Delta t +$$

$$\sum_l \begin{bmatrix} G_{kl}^{11} & G_{kl}^{12} \\ G_{kl}^{21} & G_{kl}^{22} \end{bmatrix} \begin{bmatrix} \Delta a_1^l \\ \Delta a_2^l \end{bmatrix}$$

$$= r_k^n + Ka^{n+1}\Delta t + G\Delta a^{n+1} + r_n^n \Delta t + r_{nk}^n, \qquad (25)$$

where

$$K_{kl}^{j1} = \int_{\Omega} \begin{bmatrix} \phi_{k,1} & \phi_{k,2} \end{bmatrix} \begin{bmatrix} H_{1j11} & \frac{1}{2}\left(H_{1j12} + H_{1j21}\right) \\ H_{2j11} & \frac{1}{2}\left(H_{2j12} + H_{2j21}\right) \end{bmatrix} \begin{bmatrix} \phi_{l,1} \\ \phi_{l,2} \end{bmatrix} d\Omega$$

$$K_{kl}^{j2} = \int_{\Omega} \begin{bmatrix} \phi_{k,1} & \phi_{k,2} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\left(H_{1j21} + H_{1j12}\right) & H_{1j22} \\ \frac{1}{2}\left(H_{2j21} + H_{2j12}\right) & H_{2j22} \end{bmatrix} \begin{bmatrix} \phi_{l,1} \\ \phi_{l,2} \end{bmatrix} d\Omega$$

$$r_{nk}^{n} = \begin{bmatrix} r_{nk1}^{n} \\ r_{nk2}^{n} \end{bmatrix}, \quad r_{nkj}^{n} = \Delta t \int \phi_{k,i} \tau_{ij} d\Omega .$$

The formula for C_{kl}^{ji} is similar to that for K_{kl}^{ji}.

Now, substituting (25) into (23) yields

$$M \frac{a^{n+1} - a^{n}}{\Delta t} + M_{c}^{n+1} a^{n+1} + r^{n} + K a^{n+1} \Delta t + C\left(a^{n+1} - a^{n}\right)$$
$$+ r_{n}^{n} - p^{n+1} = 0 , \qquad (26)$$

which can be written as

$$\Psi\left(a^{n+1}\right) = -\left(\frac{M}{\Delta t} + M_{c}^{n+1} + K^{n+1} \Delta t + C\right) a^{n+1}$$
$$+ \left(\frac{M}{\Delta t} + C\right) a^{n} + p^{n+1} + r^{n} - r_{n}^{n} \Delta t . \qquad (27)$$

The Quasi-Newton method is then used to solve system (27), namely

$${}^{i+1}a^{n+1} = {}^{i}a^{n+1} + {}^{i}A^{-1} \Psi\left({}^{i}a^{n+1}\right), \qquad (28)$$

where

$${}^{i}A = -\left.\frac{\partial \Psi}{\partial a^{n+1}}\right|_{a^{n+1} = {}^{i}a^{n+1}} \approx \frac{1}{\Delta t} M + {}^{i}M_{c}^{n+1} + {}^{i}K^{n+1} \Delta t + C .$$

In the present calculation, we choose ${}^{0}a^{n+1} = a^{n}$ and approximate ${}^{i}A$ by $A(a)\big|_{a=a^{n}}$.

4. NUMERICAL RESULTS

In this section, we test the numerical algorithm described in section 3 by simulating the flow of granular materials through hoppers for a specific kind of constitutive model.

In the general constitutive equation (5), the co-rotational rate of stress $\overset{\circ}{\sigma}$ consists of a rate-dependent part $\overset{\circ}{\sigma}_{v}$ and a rate-independent part $\overset{\circ}{\sigma}_{s}$. We assume in the present example that $\overset{\circ}{\sigma}_{v}$ is linearly proportional to the shear rate similar to the deviatoric part of a Newtonian fluid, namely

$$\overset{\circ}{\sigma}_{v} = G_{ijrs} \overset{\circ}{d}_{rs}$$

with

$$G_{ijrs} = 2\mu\left(\delta_{ir}\delta_{js} - \frac{1}{3}\delta_{ij}\delta_{rs}\right) ,$$

where μ is the viscosity of material and δ denotes the delta function. The rate independent part is assumed to be caused by the elastic and plastic deformation of materials. The elastic plastic matrix H in (5) can be derived using two different kinds of theories, namely the plastic flow rule theory and the double-shearing theory. The example presented here is based on the non-associated plastic flow rule theory with a Mohr-Coulomb yield function.

Figure 2 shows the geometry of the hopper and the finite element mesh adopted for the calculation. The granular material has the following properties

Density	ρ=1600kg/m^3
elastic modulus	E=50MPa
Poisson's ratio	ν=0.3
angle of internal friction	ϕ=30°
wall friction coefficient	μ_w=0.4
viscous constant	μ=0.001secMPa/m^2

Figures 3 and 4 show the variation of wall pressures with time during the discharge of material from the hopper. It is noted that the pressure at the outlet area decreases with time and the position of the peak pressure on the hopper wall moves slowly from the outlet to the transition point. This behaviour in general agrees with many experiments reported.

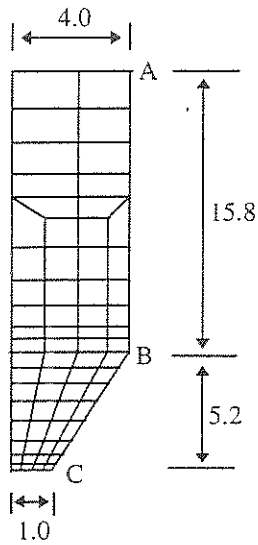

Figure 2 Finite Element Mesh

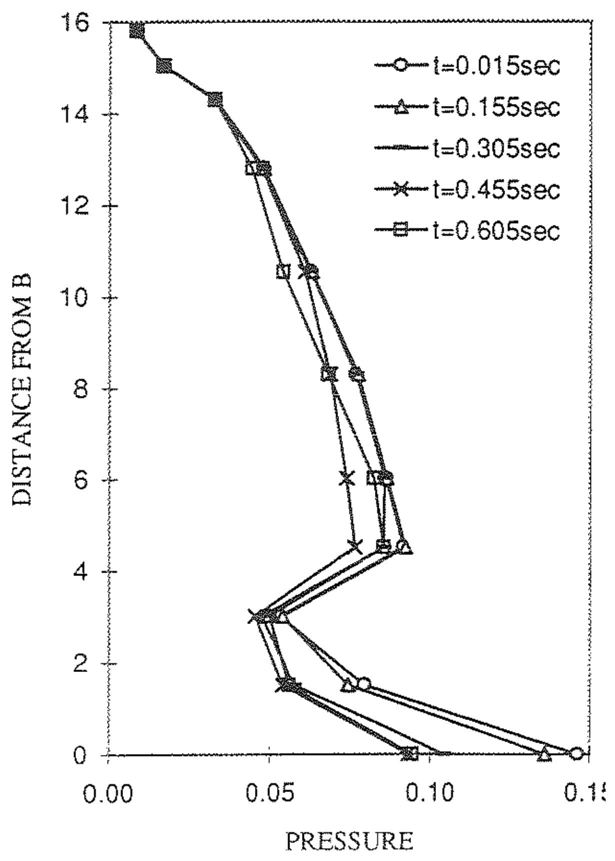

Figure 3. Pressure Distribution on Vertical Wall

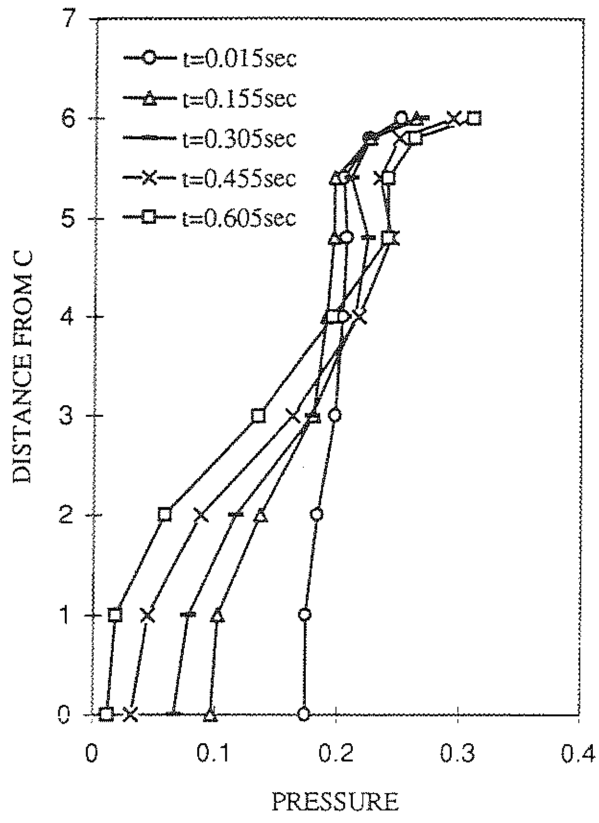

Figure 4. Pressure Distribution on Hopper Wall

5. CONCLUSION

A finite element formulation has been presented for simulating the flow of an idealized viscous-plastic material. A numerical example has shown that the numerical method presented can be used to predict the distribution of pressure on hopper walls.

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REFERENCES

Campbell, C.S., Rapid granular flows, *Annu. Rev. Fluid Mech.*, *22*, 57-92, 1990.

Collins, I.F., The nature of stress and velocity characteristics for critical stress states, *Géotechnique*, *40*, 125-129, 1990.

Goldshtein, A., and M. Shapiro, Mechanics of collisional motion of granular materials, Part 1, general hydrodynamic equations, *J. Fluid Mech.*, *282*, 75-114, 1995.

Harris, D., Constitutive equations for planar deformations of rigid-plastic materials, *J. Mech. Phys. Solids*, *41*, 1515-1531, 1993.

Hill, J.M., and Y.H. Wu, Some axially symmetric flows of Mohr-Coulomb compressible granular materials, *Proc. Royal Soc. Lond. A*, *438*, 67-93, 1992.

Rombach, G., and J. Eibel, Granular flow of materials in silos, *Int. J. Bulk Solids Handling*, 15, 65-70, 1995.

Schmidt, L.C. and Y.H Wu, Prediction of dynamic wall pressures on silos, *Int. J. Bulk Solids Handling*, *9*, 333-338, 1989.

Shen, H., and M. Hopkins, Stress in a rapid, simple shear flow of granular materials with multiple grain sizes, *Particulate Sci. Tech.*, *6*, 1-15, 1988.

Spencer, A.J.M., Deformation of ideal granular materials, in *Mechanics of Solids*, The Rodney Hill 60th Anniversary Volume, edited by H.G. Hopkins and M.J. Sewell, pp. 607-652, Pergamon Press, Oxford, 1982.