Translation of Bounds on Time-domain Behaviour of Dynamical Systems into Parameter Bounds for Discrete-time Rational Transfer-function Models

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Abstract Computation of bounds on the parameters of a linear model of a dynamical system, given observations of the system input and output and bounds on the model-output error, has developed into an interesting alternative to parameter estimation by least-squares, maximum-likelihood or recursive prediction-error methods. It has potential, so far unexploited, for using prior knowledge of bounds on plant behaviour to augment the information in the observations. The paper examines the forms of bounds on the parameters of a discrete-time rational transfer-function model implied by bounds on physically meaningful parameters such as time constants, modal amplitudes, steady-state gains and ringing frequency. Bounds on a single time constant are found to yield parameter bounds which are mainly linear but have a non-linear section, of degree rising rapidly with model order. Simultaneous bounds on two or more time constants give overall parameter bounds ranging from polytopes, easy to handle, to intractably high-degree surfaces, depending on model order and how the original bounds overlap. Bounds on amplitude and steady-state gain of a real mode prove to be linear. Oscillatory modes yield quadratic bounds, ellipsoidal in the numerator- and denominator-parameter subspaces but not overall. Bounds on the initial phase of the ringing bound a bilinear form in the numerator and denominator parameters, at any given value of amplitude. Simultaneous bounds on amplitude and phase look intractable. The ringing frequency of an oscillatory mode is shown to impose parameter bounds of a degree which doubles for each additional pole, but bounds on damping give lower degrees. The practical implications of these results are discussed.

1 INTRODUCTION

In the past 15 years or so, methods have been developed for computation of bounds on the parameters of a linear model of a dynamical system, given observations of the system input and output and bounds on the model-output error (Walter [1990], Norton [1994, 1995], Milanese *et al.*[1996]). In principle, the output-error bounds are merely mapped through the model and observations to the parameters; in practice, approximation is usually necessary. As an alternative to parameter estimation by least-squares, maximum-likelihood or recursive prediction-error methods (Norton [1986], Ljung [1987], Soderstrom and Stoica [1989]), parameter bounding has some advantages: directness and simplicity, lack of assumptions on probabilistic or spectral structure, imposition of realistic limitations on noise amplitude, aptness for worst-case control design or prediction. It also has potential, so far unexploited, for applying prior knowledge of bounds on plant behaviour to augment the information in the observations. The purpose of this paper is to assess the extent of that potential, by examining the forms of the parameter bounds of a discrete-time rational transfer-function model implied by bounds on time-domain quantities such as time constants, modal amplitudes, steady-state gains and ringing frequency, often available from knowledge of the physics of the system.

2 PROBLEM FORMULATION

The system is modelled by

$$Y(z^{-1}) = \frac{B(z^{-1})}{A(z^{-1})}U(z^{-1}) + V(z^{-1}) \qquad (1)$$

where $U(z^{-1})$ and $Y(z^{-1})$ are the z-transformed system observed input and output, $V(z^{-1})$ the transform of the output error, and

$$A(z^{-1}) \equiv 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}$$
$$B(z^{-1}) \equiv b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m} \qquad (2)$$

A specific model may be viewed as a point $\theta \equiv [a_1 \;\; a_2 \ldots a_n \; b_1 \ldots b_m]^T \equiv [\mathbf{a}^T \; \mathbf{b}^T]^T$ in the space of the coefficients.

The modal expansion of (1), assuming for brevity that the poles are distinct, strictly stable and positive if real, is

$$\frac{B(z^{-1})}{A(z^{-1})} = \sum_{k=1}^{n} \frac{g_k}{z - p_k} \qquad (3)$$

with poles $\{p_k\}$ and modal amplitudes $\{g_k\}$. A real pole and its modal amplitude determine respectively the time constant $-T/\ln p_k$ and size of the associated sampled exponential component of the unit-pulse response, where T is the sampling interval. Complex-conjugate poles $\{p_k, \; p_k^*\}$ generate an oscillatory mode; they determine the angular frequency $\angle p_k / T$ and damping time constant $-T/\ln|p_k|$ of the ringing, while their modal amplitudes $\{g_k, \; g_k^*\}$ fix its amplitude $|g_k|$ and initial phase $\angle g_k$.

The problem is to find the bounds imposed on θ when such a quantity is restricted to a known range. In other words, the region of θ space swept out as the specified quantity varies over its range is to be found. Particular interest centres on whether the boundary of this region (the "feasible set") is ellipsoidal or piecewise linear and convex, making it compatible with standard parameter-bounding algorithms (Walter [1990]).

3 BOUNDS ON POLES

3.1 Real poles

Pole $z = p_k$ satisfies

$$A(p_k^{-1}) = 0 \;\; \equiv \;\; \mathbf{p}_k^T \mathbf{a} = -1 \qquad (4)$$

which is a hyperplane in the a-subspace of θ-space, with normal

$$\mathbf{p}_k = \begin{bmatrix} p_k^{-1} & p_k^{-2} & \ldots & p_k^{-n} \end{bmatrix}^T \qquad (5)$$

Upper and lower bounds on p_k yield two such hyperplanes, but it is not immediately clear what part of a-space is swept out by the admissible values $p_k \in [\breve{p}_k, \bar{p}_k]$. It depends on how the signs of $\partial a_i / \partial p_k$ $\;i = 1,2,\ldots,n$ depend on a. If $\partial a_i / \partial p_k$ is positive (negative) throughout, a_i is maximized by $\bar{p}_k\,(\breve{p}_k)$, whereas if it is zero within the range, there is a smooth maximum. At any point a where $A(p_k^{-1}) = 0$,

$$a_i = -p_k^i \left(A(p_k) - a_i p_k^{-i}\right) \;\; i=1, 2, \ldots, n \qquad (6)$$

so

$$\frac{\partial a_i}{\partial p_k} = -i p_k^{i-1} A(p_k) - p_k^i \frac{\partial A(p_k)}{\partial p_k}$$
$$= -p_k^{i-1}\{iA(p_k) - p_k \sum_{l=1}^{n} l a_l p_k^{-l-1}\}$$
$$= \sum_{l=1}^{n} l a_l p_k^{-l} \qquad (7)$$

The derivatives are all zero on the surface

$$\sum_{l=1}^{n} l a_l p_k^{-l} = 0 \qquad (8)$$

where, from the algebra leading to (7), both $A(p_k^{-1})$ and $\partial A(p_k^{-1}) / \partial p_k$ are zero. This surface is the envelope of (4), along which the tangent point moves as p_k varies. [It is also the locus of a for a repeated pole]. The feasible-a side of each p_k bound changes as a moves past the point of tangency along the bound.

Example: second-order model.

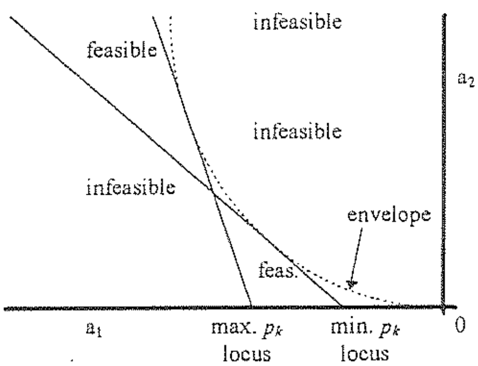

Figure 1 Feasible set defined by bounds on one pole, 2nd-order model

In Fig. 1, all a to the left of the $\breve{p}_k$ bound (and right of the $\bar{p}_k$ bound) are feasible for a_2 up to the lower point of tangency; above it, a is feasible to the right of both bounds, so the near-triangular segment between the two tangency points is feasible. Above the upper tangency, the feasible region is right of the $\breve{p}_k$ bound and left of the $\bar{p}_k$ bound. Overall, the feasible set consists of threee sections: two triangles bounded by the straight-line individual pole bounds and by $-\hat{p}_1 - \hat{p}_2 < a_1 < -\breve{p}_1 - \breve{p}_2$, $\breve{p}_1\breve{p}_2 < a_2 < \hat{p}_1\hat{p}_2$, and the near-triangle formed by the linear pole bounds and the envelope described parametrically by (8), easily shown to be

$$a_1^2 = 4a_2 \tag{9}$$

Approximation of the envelope section by a straight line may be acceptable, allowing marginally underdamped models, if the pole bounds are tight enough. □

The feasible set defined by independent bounds on all poles of an nth-order model is simply the intersection of the n sets for the n poles. Its boundary therefore generally comprises sections of hyperplanes (4) and non-linear sections (8).

Example: second-order model again.
Disjoint ranges for the two poles give a quadrilateral intersection, which can be handled directly by standard recursive parameter-bounding algorithms (Walter [1990]). The other two cases, (i) overlapping ranges and (ii) one range containing the other, are more complicated. In case (i), the near-triangular sections overlap, so part of the non-linear bound is active. Moreover, the linearly bounded part consists of the union of two triangles and a quadrilateral. Linear-bounding algorithms would have to be modified to deal with it, and more seriously, its non-convexity prevents efficient approximation by the convex sets, ellipsoids or parallelotopes (Vicino and Zappa [1993]), often employed. Case (ii) excludes the quadrilateral but is otherwise similar. □

More generally, for a model of any order (8) can be turned into an equation $f(a_1, \ldots, a_n) = 0$ by repeated use of (4). For a 3rd-order model, the result is

$$\begin{aligned} &4a_1^3 a_3 - a_1^2 a_2^2 - 18a_1 a_2 a_3 \\ &\quad + 4a_2^3 + 27a_3^2 = 0 \end{aligned} \tag{10}$$

For $n=4$ the equation contains 36 terms and is of degree 10 overall and 8 in a_1. To compound the difficulty, the number of configurations of the ranges for the n poles rises rapidly with n: for $n=3$, 15 cases must be considered, and for $n=4$, 105 cases arise. However, the case of disjoint ranges remains straightforward, with entirely linear bounds.

3.2 Complex-conjugate poles

Now bounds on angle as well as size of the poles must be considered. The pole pair

$$\begin{aligned} p_k &= ce^{j\phi} = \alpha + j\beta, \\ p_k^* &= ce^{-j\phi} = \alpha - j\beta \end{aligned} \tag{11}$$

has amplitude and angle bounds

$$\begin{aligned} &\breve{c}^2 \le p_k p_k^* = \alpha^2 + \beta^2 \le \hat{c}^2, \\ &\breve{\phi} \le \cos^{-1}\left(\frac{0.5(p_k + p_k^*)}{\sqrt{p_k p_k^*}} \right) \le \hat{\phi} \end{aligned} \tag{12}$$

and influences a through

$$\begin{aligned} A(z^{-1}) \equiv\ &(1 - 2\alpha z^{-1} + c^2 z^{-2}). \\ &(1 + a_1' z^{-1} + \ldots + a_{n-2}' z^{-n+2}) \end{aligned} \tag{13}$$

3.2.1 Bounds on pole amplitude

For $n=2$, bounds on c give directly

$$-\hat{c}^2 \le a_2 \le -\breve{c}^2 \tag{14}$$

Less obviously, as $\alpha = c\cos\phi > 0$ for any sampling rate above four per cycle of ringing,

$$-2\hat{c} \le a_1 \le 0 \tag{15}$$

These box bounds are easily handled, but (15) is loose.

For $n \ge 3$, equating powers in (13) gives

$$\begin{aligned} a_1 &= a_1' - 2\alpha \\ a_2 &= a_2' - 2\alpha a_1' + c^2 \\ &\ldots \\ a_{n-1} &= -2\alpha a_{n-2}' + c^2 a_{n-3}' \\ a_n &= c a_{n-2}' \end{aligned} \tag{16}$$

from which α and a_1' to a_{n-2}' must be eliminated to find an equation of the form

$$f(a_1, a_2, \ldots, a_n, c) = 0 \tag{17}$$

For $n=3$, (14) yields bounds of the form

$$c^2 a_1 a_3 - c^4 a_2 - a_3^2 + c^6 = 0 \tag{18}$$

for $c^2 = \breve{c}^2, \hat{c}^2$. The quadratic in a_1, a_3 is sign-indefinite, so (18) does not fit ellipsoidal parameter-bounding. Its linearity in a_1 and a_2 suggests computing the parameter bounds separately at a number of values of a_3.

For $n=4$, (16) gives

$$c^6 a_1^2 a_4 - c^8 a_1 a_3 - c^4 a_1 a_3 a_4 + c^{10} a_2 - 2c^6 a_2 a_4 + c^2 a_2 a_4^2 + c^6 a_3^2 - a_4^3 + c^4 a_4^2 + c^8 a_4 - c^{12} = 0 \quad (19)$$

which would require a general non-linear bounding technique, *e.g.* Jaulin and Walter [1993].

3.2.2 Bounds on angle of complex poles

In (16), α is $c\cos\phi$, and the aim is to eliminate c and derive an equation

$$g(a_1, a_2, \ldots, a_n, \phi) = 0 \quad (20)$$

Denoting $\cos\phi$ by γ, we obtain for $n=2$

$$a_1 + 2\gamma\sqrt{a_2} = 0 \quad (21)$$

so bounds on ϕ in the range 0 to $\pi/2$ (covering all adequate sampling rates) imply

$$\cos^2\hat{\phi} \le \frac{a_1^2}{4a_2} \le \cos^2\check{\phi} \quad (22)$$

with a_1 negative. These parabolic bounds give a non-convex feasible set which can, however, be approximated arbitrarily closely by the region between (sections of) two concentric ellipsoids. For $n=3$, we get

$$a_1^3 a_3 - a_1^2 a_2^2 - (\eta^2 + 3\eta)a_1 a_2 a_3 + \eta a_1 a_3 + (\eta+1)a_2^3 + \eta^3 a_3^2 = 0 \quad (23)$$

where $\eta \equiv 2\cos 2\phi - 1$. A bound of this form is unsuited to easy computation, even in a 2-dimensional subspace.

For $n=4$, a 30-term equation of degree 8 results.

4 BOUNDS ON MODAL AMPLITUDE

4.1 Real poles

In the modal expansion (3),

$$g_k = \frac{p_k B(p_k^{-1})}{A_k(p_k^{-1})} \quad (24)$$

where

$$(1 - p_k z^{-1})A_k(z^{-1}) \equiv A(z^{-1}) \quad (25)$$

so

$$g_k A_k(p_k^{-1}) - p_k B(p_k^{-1}) = 0 \quad (26)$$

From (25), the coefficients in A_k are linear in **a**, so (26) is a linear relation between **a** and **b** for given p_k. By differentiating (25) with respect to z^{-1} and setting $z=p_k$,

$$A_k(p_k^{-1}) = -p_k^{-1}\left(\frac{\partial A(z^{-1})}{\partial z^{-1}}\right)_{z=p_k} \quad (27)$$

giving from (26)

$$g_k\left[p_k^{-1} \;\; 2p_k^{-2} \;\; \ldots \;\; np_k^{-n}\right]\mathbf{a} + \left[1 \;\; p_k^{-1} \;\; \ldots \;\; p_k^{-m+1}\right]\mathbf{b} \equiv g_k \mathbf{p}_k'^T \mathbf{a} + \mathbf{p}_k^{(m)T}\mathbf{b} = 0 \quad (28)$$

in obvious notation. For a given pole value, bounds on g_k thus confine $[\mathbf{a}^T \;\; \mathbf{b}^T]^T$ between two hyperplanes, which have parallel intersections with any constant-**b** subspace, with $\mathbf{p}_k'$ as common normal.

The dependence of these parameter bounds on p_k can be removed by eliminating it between (28) and (4). For $n=2$, the result is

$$g a_1^2 (b_1 - g) - a_1 b_1 b_2 + a_2(b_1 - 2g)^2 + b_2 = 0 \quad (29)$$

of degree 3 overall. For constant **b**, (29) is a parabola but for constant **a** (such that the poles are real) it is hyperbolic. Higher-order models have higher-degree loci, even less well adapted to the standard bound-updating algorithms.

4.2 Complex poles

The size and angle of the complex-conjugate pair $\{g_k, g_k^*\}$ must be considered.

4.2.1 Bounds on size

The derivation of (28) did not rely on the pole being real, so the complex modal amplitude is

$$g_k = -\frac{\mathbf{p}_k^{(m)T}\mathbf{b}}{\mathbf{p}_k'^T \mathbf{a}} \quad (30)$$

and its size is given by

$$|g_k|^2 = g_k g_k^* = \frac{\mathbf{b}^T \mathbf{p}_k^{(m)} \mathbf{p}_k^{(m)*T} \mathbf{b}^*}{\mathbf{a}^T \mathbf{p}_k' \mathbf{p}_k'^{*T} \mathbf{a}^*} = \frac{\mathbf{b}^T \mathbf{P}_k^{(m)} \mathbf{b}}{\mathbf{a}^T \mathbf{P}_k' \mathbf{a}} \quad (31)$$

where

$$\left[\mathbf{P}^{(m)}\right]_{hi} = \frac{p_k^{-h+1} p_k^{*-i+1} + p_k^{-i+1} p_k^{*-h+1}}{2}$$

$$= c^{-(h+i)+2} \cos(h-i)\phi \qquad (32)$$

$$\left[\mathbf{P}'_k\right]_{hi} = \frac{hp_k^{-h} ip_k^{*-i} + ip_k^{-i} hp_k^{*-h}}{2}$$

$$= hic^{-(h+i)} \cos(h-i)\phi \qquad (33)$$

Hence a bound on $|g_k|$ bounds the ratio between quadratic forms, in **a** and **b**. It can be shown that both are positive-semidefinite. The proof for $\mathbf{P}_k^{(m)}$ will be sketched; the other differs only in detail. First, write

$$\mathbf{P}_k^{(m)} = \mathbf{D}\,\mathrm{Re}\{\mathbf{F}_m\}\mathbf{D}$$

$$\text{where } \mathbf{D} = diag(1, c^{-1}, \ldots, c^{-m+1}) \qquad (34)$$

$$\text{and } \left[\mathbf{F}_m\right]_{hi} = e^{j(h-i)\phi} = \xi^{h-i}$$

then define

$$q_m = \mathbf{x}^T \mathbf{F}_m \mathbf{x}, \quad s_h = \sum_{l=1}^{h} \xi^{h-l} x_l \qquad (35)$$

for any real, non-zero **x**. From

$$s_h = \xi s_{h-1} + x_h$$

$$q_h = q_{h-1} - (\xi s_{h-1})^2 + s_h^2 \qquad (36)$$

with $q_1 = x_1^2 = s_1^2$

it follows by induction that

$$\mathrm{Re}(q_m) = \mathrm{Re}\{s_m\}^2 + \mathrm{Im}\{s_m\}^2 \geq 0 \qquad (37)$$

and hence, with **D** real, $\mathbf{P}_k^{(m)} \geq 0$. Bounds

$$\left|g_k\right|^2_{\min} \leq \left|g_k\right|^2 \leq \left|g_k\right|^2_{\max} \qquad (38)$$

therefore give, generically,

$$\frac{\mathbf{b}^T \mathbf{P}_k^{(m)} \mathbf{b}}{\left|g_k\right|^2_{\max}} \leq \mathbf{a}^T \mathbf{P}'_k \mathbf{a} \leq \frac{\mathbf{b}^T \mathbf{P}_k^{(m)} \mathbf{b}}{\left|g_k\right|^2_{\min}} \qquad (39)$$

at a specified value of **a**, and

$$\left|g_k\right|^2_{\min} \mathbf{a}^T \mathbf{P}'_k \mathbf{a} \leq \mathbf{b}^T \mathbf{P}_k^{(m)} \mathbf{b} \leq \left|g_k\right|^2_{\max} \mathbf{a}^T \mathbf{P}'_k \mathbf{a} \qquad (40)$$

at given **b**; each pair of bounds confines **a** or **b** between two concentric, similar ellipsoids.

Example: an underdamped second-order model.

The model

$$\frac{B(z^{-1})}{A(z^{-1})} = \frac{z^{-1} - 0.7z^{-2}}{1 - 1.7z^{-1} + 0.845z^{-2}}$$

has poles z=0.85±0.35j and gives

$$\mathbf{P}_k^{(m)} = \begin{bmatrix} 1 & 1.0059 \\ 1.0059 & 1.1834 \end{bmatrix},$$

$$\mathbf{P}'_k = \begin{bmatrix} 1.1834 & 2.3809 \\ 2.3809 & 5.6020 \end{bmatrix}$$

for k=1, 2. At the correct value of **b**, the bounds on **a** are ellipses with axis directions $[0.9166\ -0.4]^T$, $[0.4\ 0.9166]^T$ (the eigenvectors of $\mathbf{P}'_k$) and semi-axis lengths 1.722, 0.254 (the reciprocals of the square roots of the eigenvalues). The bound on g and the value of **b** change only the size (not the shape or orientation) of the ellipse. □

4.2.2 Bounds on angle

To obtain a relation between the model parameters and the angle θ *of* g_k, put $g_k = |g_k| e^{j\theta} \equiv |g_k|(\cos\theta + j\sin\theta)$ in

$$g_k \mathbf{p}'^T_k \mathbf{a} + \mathbf{p}_k^{(m)T} \mathbf{b} = 0 \qquad (28)$$

separate the real and imaginary parts and eliminate $|g_k|$ between them. The result is

$$(\alpha_1\alpha_2 + \beta_1\beta_2)\sin\theta = (\alpha_1\beta_2 - \beta_1\alpha_2)\cos\theta \qquad (41)$$

where

$$\mathbf{p}'^T_k \mathbf{a} = \alpha_1 + j\beta_1, \quad \mathbf{p}_k^{(m)T} \mathbf{b} = \alpha_2 + j\beta_2 \qquad (42)$$

This can be reduced to

$$\mathbf{a}^T \mathbf{S} \mathbf{b} = 0 \qquad (43)$$

where

$$s_{hi} = hc^{-h-i+1} \sin(\theta + (i-h-1)\phi) \qquad (44)$$

with c, ϕ the size and angle of the complex pole, as before.

A bound on θ thus imposes a hyperplane (through the origin) bound on **a** at fixed **b**, and *vice versa*.

4.3 Bounds on contribution of mode to steady-state value of unit-step response

Following a unit-step input, the output settles according to

$$y_t \rightarrow \frac{B(1)}{A(1)} = \sum_{k=1}^{n} \frac{g_k}{1 - p_k} \quad \text{as } t \rightarrow \infty \qquad (45)$$

so for a real mode, bounds

$$\breve{h}_k \leq \frac{g_k}{1-p_k} \leq \hat{h}_k \tag{46}$$

on its final contribution to the step response give hyperplane bounds on **a** and **b**, as did (28), the sole difference being the rescaling effect of $1 - p_k$ in either the **a** or the **b** directions.

A complex-conjugate mode pair has

$$|h_k|^2 = \frac{g_k g_k^*}{(1-p_k)(1-p_k^*)} = \frac{|g_k|^2}{1-2\alpha+c^2} \tag{47}$$

in the notation of (13), so again only a rescaling is needed at any particular pole value, this time in (40).

The overall steady-state gain of the model gives notably simple parameter bounds:

$$\breve{h}_o \leq \frac{B(1)}{A(1)} \leq \hat{h}_o \Rightarrow \begin{cases} \breve{h}_o \mathbf{1}_n^T \mathbf{a} - \mathbf{1}_m^T \mathbf{b} \leq -\breve{h}_o \\ \hat{h}_o \mathbf{1}_n^T \mathbf{a} - \mathbf{1}_m^T \mathbf{b} \geq -\hat{h}_o \end{cases} \tag{48}$$

where $\mathbf{1}_r$ denotes an r-vector of 1's. The simplicity of these bounds was pointed out long ago (Norton [1976]) in connection with least-squares parameter estimation.

5 CONCLUSIONS

Of the items with known prior bounds considered above, those yielding high-degree parameter bounds prevent use of the standard parameter-bounding algorithms. As general non-linear bounding is intrinsically expensive in computation, they have little promise for on-recursive use, for instance in adaptive control or in tracking time-varying behaviour. On the other hand, the items giving linear or positive-semidefinite quadratic bounds are compatible with existing algorithms, and can be treated as extra observations with only minor modifications. A noteworthy feature is that the implied bounds are often much simpler in a subspace (for instance of **a** or **b**) than overall. In the practically important case of a second-order model, the parameter bounds are all simple in functional form, but non-convexity of the overall bounds may be a problem, as seen when two poles are known to fall in specific ranges.

The analyses presented here leave largely unexamined the overall bounds imposed jointly by simultaneous bounds on two or more items, *e.g.* a pole and its modal amplitude. As that sort of prior information will often be available, its effects should be investigated.

5.1 Acknowledgement

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5.2 References

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