# Fuzzy Modelling and Tracking Control of Nonlinear Systems

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**Abstract** This paper presents a fuzzy modelling and tracking control methodology for complex systems by combining the merits of fuzzy logic and conventional linear control theory. Here, fuzzy logic is used to formulate a system model by aggregating a set of linearized local subsystems which identify the nonlinear system approximately, and a fuzzy feedback controller is designed to guarantee that the output tracking error of the controlled system with respect to a desired trajectory converges to zero.

## 1. INTRODUCTION

In the design of modern and classical control systems, the first step is to establish a suitable mathematical model to describe the behaviour of the controlled plant. However, in practical situations, such a requirement is not feasible because the controlled system have high nonlinearities and uncertain dynamics, and simple linear or nonlinear differential equations cannot sufficiently represent the corresponding practical systems, and therefore, the designed controller based on such a model cannot guarantee the good performance such as stability and robustness. During the last few years, fuzzy logic control has been suggested as an alternative to conventional control techniques for complex nonlinear systems due to the fact that fuzzy logic combines human heuristic reasoning and expert experience to approximate a certain desired behaviour function (Takagi et al., 1985).

This paper presents a fuzzy modelling and tracking control methodology for complex systems by combining the merits of fuzzy logic and conventional linear control theory. Here, fuzzy logic is used to formulate a system model by aggregating a set of linearized local subsystems which identify the nonlinear system approximately, and a fuzzy feedback controller is designed to guarantee that the output tracking error of the controlled system with respect to a desired trajectory converges to zero.

The organisation of the rest of the paper is as follows. In Section 2, a brief review of fuzzy logic and fuzzy systems is given. In Section 3, fuzzy modelling and tracking controller design for nonlinear systems is presented. In Section 4, a simulation example using one-link rigid robotic manipulator is given in support of the proposed control scheme. Section 5 gives concluding remarks.

## 2. BRIEF REVIEW ON FUZZY LOGIC AND FUZZY SYSTEMS

Unlike Boolean logic in which the state (value) of any variable/statement assumes either 0 or 1, fuzzy logic allows states (membership values) between them. The *grade of membership* of a fuzzy variable can be regarded as its *capability of belonging* to the described linguistic term (Zadeh, 1973). For example, let x and y be fuzzy variables and "good" the described linguistic term. Then, we can tell how "good" x and y are by their grades of membership. More precisely, we use the following definitions:

*Definition 1*: A fuzzy set $A$ of a universe of discourse $U$ is represented by a collection of ordered pairs of a generic element $x \in U$ and its grade of membership function $\mu_A(x)$, i.e.,

$$A = \sum_{i=1}^{N} \mu_A(x_i)/x_i = \{(\mu_A(x_1)/x_1), (\mu_A(x_2)/x_2), \ldots, (\mu_A(x_N)/x_N)\}$$

where $N$ is the number of elements in $U$.

Note that the symbol $\Sigma$ here denotes collection of discrete elements. The corresponding notation for a continuous universe of discourse $U$ is

$$A = \int_U \mu_A(x)/x$$

*Definition 2*: Fuzzy set A is said to be a fuzzy singleton if it consists of only one element $\mu_A(x_s)/x_s$. In particular, if the value of $\mu_A(x_s)$ is 1, then A becomes a nonfuzzy singleton and $A = 1/x_s$.

*Definition 3*: The union of two fuzzy sets $A$ and $B$ in the universe of discourse $U$ is defined by

$$A \cup B = \int_U \max\{\mu_A(x), \mu_B(x)\}/x \quad \text{for } x \in U$$

or

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

*Definition 4*: The intersection of two fuzzy sets $A$ and $B$ in the universe of discourse $U$ is defined by

$$A \cap B = \int_U \min\{\mu_A(x), \mu_B(x)\}/x \quad \text{for } x \in U$$

or

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

In this paper, we consider a fuzzy system whose basic configuration is shown in Fig. 1.

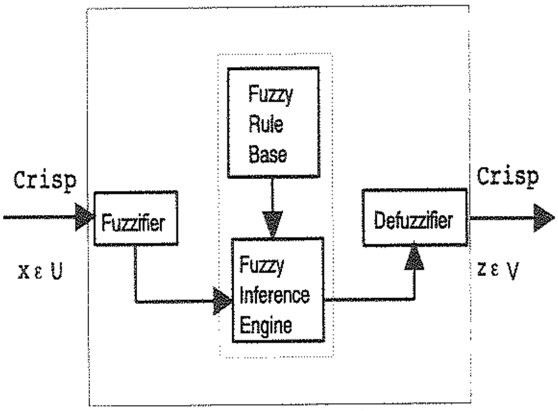


**Fig.1** Basic configuration of fuzzy systems

There are four principal elements in such a fuzzy system: fuzzifier, fuzzy rule base, fuzzy inference engine, and defuzzifier.

The *fuzzifier* performs a mapping from the observed crisp input space $U \subset R^n$ to the fuzzy sets defined in $U$. The most commonly used fuzzifier is the singleton fuzzifier, which maps $x \in U$ into fuzzy set $A_x$ in $U$ with $\mu_{A_x}(x) = 1$ and $\mu_{A_x}(x') = 0$ for all $x' \in U$ with $x' \neq x$.

The *fuzzy rule base* consists of a set of linguistic rules in the form of "IF a set of conditions are satisfied, THEN a set of consequences are inferred." Let's consider the case where the fuzzy rule base consists of $M$ rules in the following form:

$R^j$ : IF $x_1$ is $A_1^j$ and $x_2$ is $A_2^j$ $\cdots$ and $x_n$ is $A_n^j$,

THEN $z$ is $B^j$,

where $j = 1,2,\ldots,M$, $x_i (i = 1,2,\ldots,n)$ are the input variables to the fuzzy systems, $z \in V$ is the output variable of the fuzzy system, and $A_i^j$ and $B^j$ are linguistic terms characterised by fuzzy membership functions $\mu_{A_i^j}(x_i)$ and $\mu_{B^j}(z)$, respectively. Each rule $R^j$ can be viewed as a fuzzy implication: $A_1^j \times \cdots \times A_n^j \rightarrow B^j$, which is a fuzzy set in $U \times V$ with

$$\begin{aligned}&\mu_{A_1^j \times \cdots \times A_n^j \rightarrow B^j}(x_1', \ldots, x_n', z) \\ &= \mu_{A_1^j}(x_1') \otimes \cdots \otimes \mu_{A_n^j}(x_n') \otimes \mu_{B^j}(z)\end{aligned} \quad (1)$$

and the most commonly used operations for "⊗" are "product" and "min".

The *fuzzy inference engine* is decision making logic which employs fuzzy rules from the fuzzy rule base to determine a mapping from the fuzzy sets in the input space $U$ to the fuzzy sets in the output space $V$. Let $A_x$ be an arbitrary fuzzy set in $U$; then each rule $R^j$ determines a fuzzy set $A_x \circ R^j$ in $V$ based on the following *compositional rule*:

$$\begin{aligned}\mu_{A_x \circ R^j}(z) &= \max_{x' \in U}[\mu_{A_x}(x') \otimes \mu_{A_1^j \times \cdots \times A_n^j \rightarrow B^j}(x', z)] \\ &= \max_{x' \in U}[\mu_{A_x}(x') \otimes \mu_{A_1^j}(x_1') \otimes \cdots \otimes \mu_{A_n^j}(x_n') \otimes \mu_{B^j}(z)]\end{aligned} \quad (2)$$

The *defuzzifier* performs a mapping from the fuzzy sets in $V$ to a crisp value in $V$. The following *centriod (centre-average) defuzzifier*, which performs a mapping from the fuzzy sets $A_x \circ R^j$ in $V$ to a crisp value $z \in V$, is the most commonly used method:

$$z = \frac{\sum_{j=1}^{M} \bar{z}^j \mu_{A_x \circ R^j}(\bar{z}^j)}{\sum_{j=1}^{M} \mu_{A_x \circ R^j}(\bar{z}^j)} \quad (3)$$

where $\bar{z}^j$ is the point in $V$ at which $\mu_{B^j}(z)$ achieves its maximum value (usually, we assume that $\mu_{B^j}(z) = 1$).

## 3. FUZZY MODELLING AND FUZZY TRACKING CONTROLLER DESIGN OF NONLINEAR SYSTEMS

### 3.1 Linearization of the system

Consider the following nonlinear dynamic system

$$\dot{x}(t) = f(x, u) \quad (4)$$

where $x = [x_1, x_2, \ldots, x_n]^T$ is the state vector, also $f(x,u)$ is a nonlinear function, and $u = [u_1, u_2, \ldots, u_m]^T$ is the control input. The control objective is to force the plant state vector $x$ to follow a specified desired trajectory, $x_d$. Assuming $f(x,u)$ is differentiable with respect to $x$ and $u$ respactively, then equation (4) can be linearized at some point $(x_i, u_i)$ by Taylor's expansion up to the first order such that

$$\dot{x} = \dot{x}_i + \left.\frac{\partial f}{\partial x}\right|_i \tilde{x} + \left.\frac{\partial f}{\partial x}\right|_i \tilde{u} \qquad (5)$$
$$\dot{x}_i = f(x_i, u_i)$$

where

$$\tilde{x} = x - x_i,\ \tilde{u} = u - u_i$$

and $u_i$ can be obtained from the following equilibrium condition (Palm et al., 1997)

$$\dot{x}_i = 0$$

From expression (5), we have the following local linearized error dynamic equation

$$\dot{\tilde{x}} = A_i \tilde{x} + B_i \tilde{u} \qquad (6)$$

where

$$A_i = \left.\frac{\partial f}{\partial x}\right|_i \in R^{n\times n},\ B_i = \left.\frac{\partial f}{\partial u}\right|_i \in R^{n\times m} \qquad (7)$$

*Remark 1*: If equation (6) is in a controllable form, the feedback control law

$$\tilde{u} = -K_i \tilde{x} \qquad (8)$$

can be designed by using conventional linear system theory (Ogata, 1990) so that the eigenvalues of $(A_i - B_i K_i)$ are the specified ones. The feedback gain $K_i$ can be obtained by using the Ackerman's formula in the case of single-input system as follows

$$K_i = [0, \ldots, 0, 1] Q_i^{-1} \alpha(A_i) \qquad (9)$$

where

$$\alpha(s) = s^n + \alpha_n s^{n-1} + \ldots \alpha_2 s + \alpha_1$$

is a desired stable polynomial, and

$$Q_i = [B_i\ A_i B_i\ A_i^2 B_i \ldots A_i^{n-1} B_i]$$

### 3.2 Fuzzy modelling and controller design

Many Physical systems are very complex in practice so that it is very difficult to obtain their rigorous mathematical models. In recent years, fuzzy logic has been applied to the field of system modelling and control engineering (Feng et al., 1997) by means of combining human heuristic reasoning and expert experience. In this paper, the fuzzy model is established by the following fuzzy inference rules which include local linearized subsystems and feedback controllers.

$R^i$: IF $x_1$ is $F_1^i$ AND... $x_n$ is $F_n^i$
THEN

$$\dot{\tilde{x}} = A_i \tilde{x} + B_i \tilde{u}$$
$$\tilde{u} = -K_i \tilde{x} \qquad (10)$$
$$i = 1, 2, \ldots, l$$

where $R^i$ denotes the i-th fuzzy inference rule, $l$ the number of inference rules, $F_j^i\ (j = 1, 2, \ldots, n)$ are fuzzy sets, $\tilde{x} = x - x_d$ is the tracking error of the system with desired trajectory $x_d$ and $\tilde{u} = u - u_d$.

Let $\mu_i(x)$ be the normalized membership function of the inferred fuzzy set $F^i$ where

$$F^i = \bigcap_{j=1}^{n} F_j^i \qquad (11)$$

and

$$\sum_{i=1}^{l} \mu_i(x) = 1 \qquad (12)$$

By using a standard fuzzy inference method, that is, using a singleton fuzzifier, product fuzzy inference and centre-average defuzzifier, the following global tracking error fuzzy nominal model for the controlled nonlinear system can be obtained,

$$\dot{\tilde{x}} = A_0 \tilde{x} + B_0 \tilde{u} \qquad (13)$$
$$\tilde{u} = -K \tilde{x} \qquad (14)$$

where

$$A_0 = \sum_{i=1}^{l} \mu_i A_i \qquad B_0 = \sum_{i=1}^{l} \mu_i B_i$$
$$K = \sum_{i=1}^{l} \mu_i K_i \qquad (15)$$

*Remark 2*: Here, we assume the fuzzy model is globally controllable, that is, $(A_0, B_0)$ is a controllable pair.

## 4. A SIMULATION EXAMPLE

To illustrate the proposed robust tracking control scheme in this paper, a simulation example is carried out for a one-link robotic manipulator. The dynamic equation of the one-link robotic manipulator is given by

$$ml^2\ddot{\theta} + d\dot{\theta} + mgl\cos(\theta) = u \qquad (16)$$

with

$m = 1\text{kg}$ - payload,

$l = 1\text{m}$ - length of the link,

$g = 9.81\text{m/s}^2$ - gravitational constant,

$d = 1\text{kgm}^2/\text{s}$ - damping factor,

$u$ - control variable $(\text{kgm}^2/\text{s}^2)$.

Assuming we are interested in the dynamics of the system in the range of $[-90°, 90°]$, then the fuzzy nominal model can be obtained by linearizing the nonlinear equation (16) over a number of points, such as $0°, \pm 45°, \pm 90°$. The following fuzzy nominal model has been obtained.

$R^1$ : IF $x_1$ is about $0°$

THEN $\dot{\tilde{x}} = A_1\tilde{x} + B_1\tilde{u}$

$R^2$ : IF $x_1$ is about $-45°$

THEN $\dot{\tilde{x}} = A_2\tilde{x} + B_2\tilde{u}$

$R^3$ : IF $x_1$ is about $+45°$

THEN $\dot{\tilde{x}} = A_3\tilde{x} + B_3\tilde{u}$

$R^4$ : IF $x_1$ is about $-90°$

THEN $\dot{\tilde{x}} = A_4\tilde{x} + B_4\tilde{u}$

$R^5$ : IF $x_1$ is about $+90°$

THEN $\dot{\tilde{x}} = A_5\tilde{x} + B_5\tilde{u}$

where

$$x_1 = \theta,\ x_2 = \dot{\theta},\ \tilde{x} = [\tilde{x}_1, \tilde{x}_2]^T,\ u_i = mgl\cos(\theta_i)$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -6.94 & -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 6.94 & -1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 \\ -9.81 & -1 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 \\ 9.81 & -1 \end{bmatrix}, \qquad B_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The fuzzy sets for $x_1$ are chosen as in Fig.2.

The desired closed loop poles for each local model are chosen as [-4, -3]. Thus the feedback control gains are found by using of pole placement method as follows

$K_1 = [12 \quad 6]$,

$K_2 = [5.1 \quad 6]$,

$K_3 = [18.9 \quad 6]$,

$K_4 = [2.2 \quad 6]$,

$K_5 = [21.8 \quad 6]$.

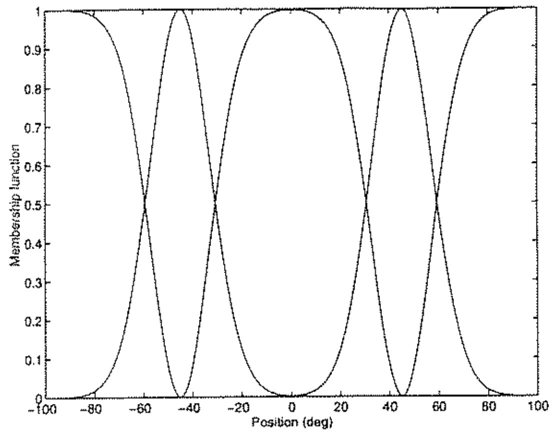


**Fig.2** Fuzzy sets of $x_1$

The control objective in this simulation is to force the one-link robotic manipulator to follow a desired trajectory which is generated by the following reference model

$$\begin{bmatrix} \dot{x}_d \\ \ddot{x}_d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix} \begin{bmatrix} x_d \\ \dot{x}_d \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_s \qquad (17)$$

where $u_s$ is chosen as in Fig.3.

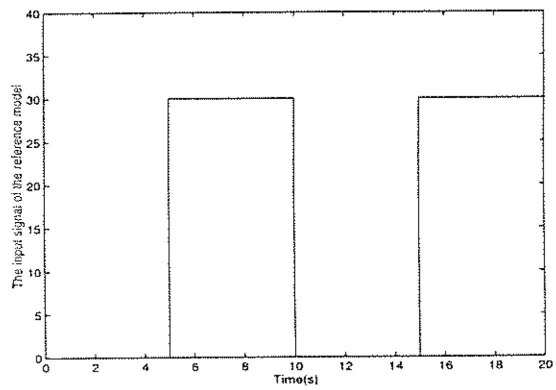


**Fig.3** The control input of the reference model

In this example, the initial values of x and $x_d$ are selected as

$$[x_1(0), \dot{x}_1(0)] = [17.2°, 0], [x_d(0), \dot{x}_d(0)] = [0, 0].$$

Fig. 4 shows the output tracking using the fuzzy feedback controller. It can be seen that the output tracking error is asymptotically converges to zero. Fig. 5 shows the control input.

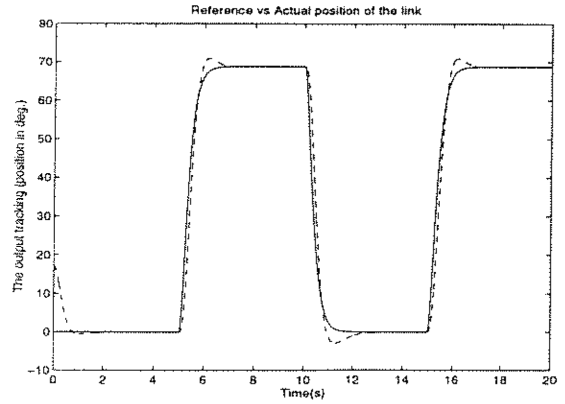

**Fig.4** The output tracking of the link

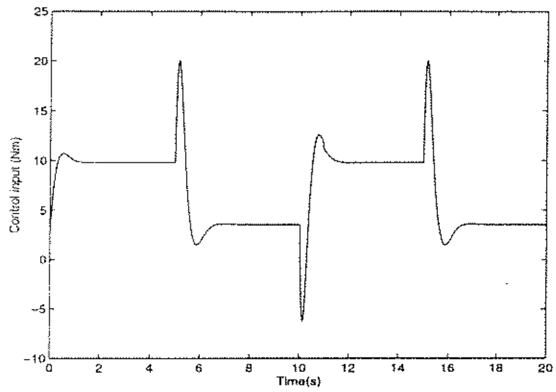

**Fig.5** Fuzzy tracking control input

## 5. CONCLUSIONS

A fuzzy tracking control scheme is proposed for nonlinear systems in this paper. The main contribution of this scheme is that a system model for a nonlinear system is established by fuzzy synthesis of a set of linearized local subsystems, where the conventional linear feedback control technique is used to design a feedback controller for the fuzzy system. A simulation example has been given to support the proposed control scheme.

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