Chaotic Dynamics in Tank Level Fluctuations

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Abstract This paper describes how chaotic level fluctuations have been observed in the closed loop feedback level control of a single tank system. The tank system is being studied because of its application to the process industries. The approach makes use of differential equations in the simulations used and compares this to models that use difference equations. A difference equation model indicated the presence of period doubling bifurcations. The differential equation approach with a simple gain element as the valve model showed chaotic behaviour at high gains.

1. INTRODUCTION

Chaos has emerged as a new phenomena wherein a system that is described by completely deterministic equations generates output that appears to be stochastic, i.e. bounded yet with the appearance of randomness, Baker and Gollub [1990]. An example of this is the seemingly random changes in the water level in a filling tank.

The pioneering work in chaos theory was that of Lorenz [1963] and was based on a set of set of simplified non-linear differential equations, which modelled weather on a computer. Lorenz discovered the solutions to his equations led to apparently random trajectories bounded in a fixed region of state space; the structure would be named the Lorenz attractor. Many reports of chaotic phenomena have appeared since.

This paper investigates the appearance of chaotic dynamics in the level control of a single water tank. Chang and Lee [1994] have taken a difference equation approach and we repeat part of that modelling. Like Lorenz, the paper then looks at how the control system might be modelled with differential equations and whether or not chaotic dynamics can be predicted through simulation and through testing.

The long term aim of this study is not necessarily to design a better tank level control system, but rather to be able to generalise results for other applications, particularly in the process industries e.g. exothermic reactors, Elnnashaie *et al.*, [1993].

2 EXPERIMENTAL

The tank system under investigation consisted of a vertically mounted tank equipped with a 4-20 mA differential pressure level transmitter, a centrifugal pump, rotameter, an I/P converter, a modulating valve, return tank and a PID controller. The general layout is shown in Figure 1, and is described in more detail in Fuller [1993].

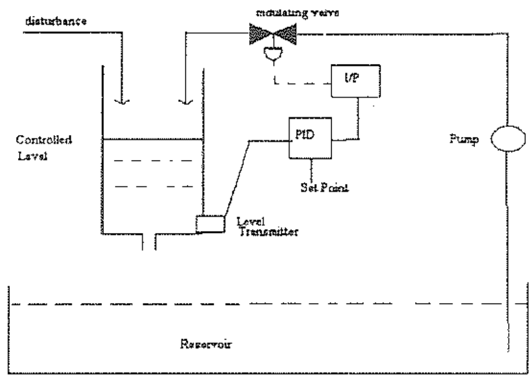

Figure 1. General Arrangement

At certain values of the PID controller parameters, small fluctuations in the level were observed which did not seem to follow any recognisable pattern. Figure 2 shows the overall level magnitudes and Figure 3 the same changes in more detail. Examination of this experimentally determined time series seems to indicate the existence of chaos.

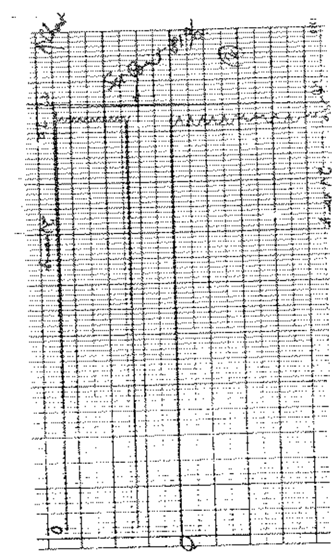

Figure 2. Noise-like Level Changes (Chart recording)

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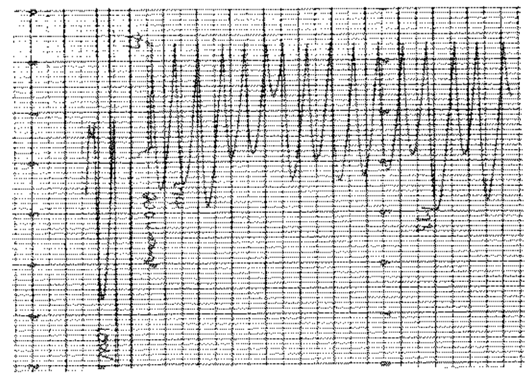

Figure 3. Close up view of Level Changes.

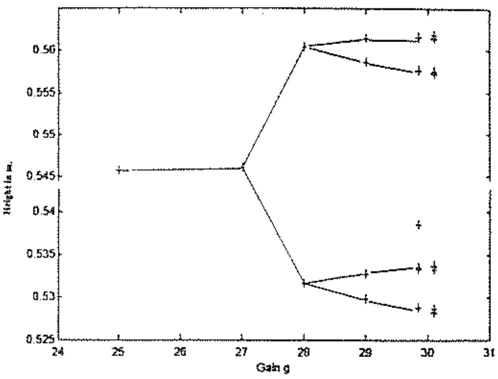

Figure 4. Bifurcation Diagram

3. DIFFERENCE EQUATIONS

From chaos theory it is known that if the system model can be represented by a mapping $x_{n+1}=f(x_n)$, then it can display chaos in one dimension, May [1976].

The equations that describe the dynamics of the level may be derived from a mass balance as described in Ogata [1990]:

$$\rho A \frac{dh}{dt} = \rho d + \rho Q - \rho Q_{out} \quad (1)$$

where ρ is the fluid density, A is the cross sectional area of the tank, h is the level in the tank, t is time, d is the disturbance flow, Q is the input flow rate, and Q_{out} is the output flow rate.

The normalised equation becomes, as shown by Chang and Lee [1994]:

$$\frac{du}{d\Theta} = d' + (1 + q(t)) - k\sqrt{u} \quad (2)$$

where u is the normalised tank level, h/h_o; Θ is dimension-less time, $\frac{tQo}{Aho}$; d' is the normalised disturbance, d/Q_o; $q(t)$ is the normalised flow difference from the steady state operating point; and k is the dimensionless control valve characteristic or dimensionless Cv.

Under the action of a proportional controller, the flow would depend on the error between the actual level and the desired setpoint, sp. Making use of first differences, equation (2) may be converted into the difference equation:

$$u_{n+1} = u_n + \Delta\Theta\, (d + (1 + g(u_{sp} - u_n) - k\sqrt{u}\,) \quad (3)$$

This may be coded into a MATLAB *m*-file, Etter [1993], and solutions found, with the dimensionless gain, g, as the varying parameter. The result for our tank system is the bifurcation diagram of Figure 4, which is in general agreement with Chang and Lee [1994].

4. DIFFERENTIAL EQUATIONS

The success of the difference equation approach to modelling this chaotic type behaviour raised the question of whether it is possible to derive a similar result using ordinary differential equations. If so, it might then be possible to identify a three dimensional structure in an attempt to find a "strange attractor". A strange attractor is essentially defined as the asymptotic limit set to which trajectories tend as time increases, Gulick [1992].

In particular for a continuous model, a dissipative system must consist of at least three autonomous ordinary differential equations in order to exhibit chaos, Sparrow [1981] and Brockett, [1982].

4.1 Valve Modelling Effects

In particular the way the modulating valve is modelled may be particularly pertinent to the possible solutions that result. The valve may be modelled as a constant gain, a first order lag or a second order system, Weber [1992], depending on the circumstances. Our tank system is shown in Figure 5 with a P only controller and a second order valve model. Note that the non-linearity comes from the delimited relationship between the valve input current to valve lift or stem position (which is not shown in the Figure).

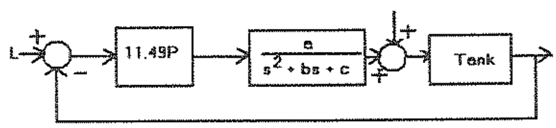

Figure 5. System with a Second Order Valve Model

We have so far modelled the valve as a gain only element. This approach resulted in the following equations describing the tank system:

$$\frac{dx}{dt} = -0.033x + 165I$$

$$\frac{dh}{dt} = -0.0211\sqrt{h} + 44.44(Q + d) \quad (4)$$

$$Q = 7.456\times10^{-5}\exp(0.0181x)$$

where x is the valve lift (stem position) and I is the current signal to the valve I/P converter/positioner.

This is a set of two differential equations and one algebraic equation. Yet, even this simplification of the valve dynamics showed some promise for identifying the onset of chaos. Signs of an approach to chaotic behaviour were seen in the simulated system response, however the response settled out to a stable fixed point for typical (small) values of controller gain. Figure 6 shows these simulated responses, and Figure 7 the point attractor.

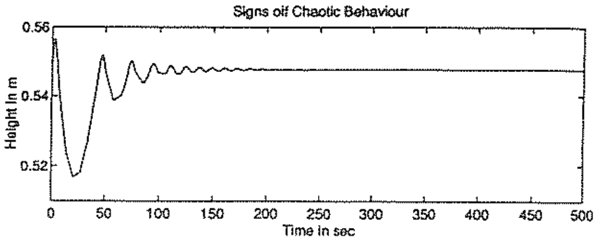

Figure 6. Signs of Chaotic Behaviour

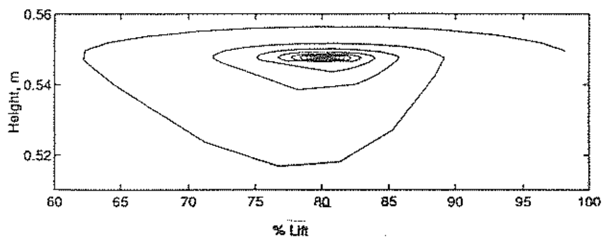

Figure 7. Fixed Point Attractor

It was interesting to note that for atypical (very high) controller gain values the response appeared to exhibit chaotic behaviour above the saturation limit of the I/P converter. Figure 8 shows these "chaotic"-like level fluctuations.

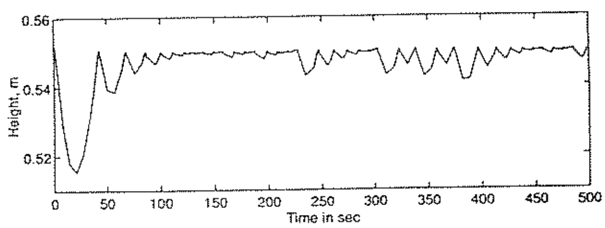

Figure 8. "Chaotic"-like Level Fluctuations

It is then possible to go further and construct a three dimensional plot of level, flow, and valve lift which results in a diagram that looks very much like a strange attractor. However, even though it is not our aim to look at the quality of the control, the task of constructing a strange attractor seems rather fruitless given the atypical high values of the controller gains.

Indeed, what this result may indicate is that the chaotic type fluctuations observed in the real tank arise from as yet unmodelled dynamics, and a higher order valve model should be investigated. Other evidence for higher order models being more applicable is an additional non-linearity consisting of an observable hysteresis in the dynamic behaviour of the valve.

5. CONCLUSIONS AND RECOMMENDATIONS

This paper investigated the appearance of chaotic dynamics in the level control of a single water tank.

The difference equation approach taken by Chang and Lee [1994] was applied to the system to successfully model these fluctuations.

The use of the differential equations approach similar to that of Lorenz [1963] did not yield the same result with the control element modelled by a classical gain element, unless the gain values were atypically very high.

However, the saturation behaviour of the model indicated that a further investigation into the use of higher order valve models, e.g. second order, is warranted.

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