A New Linearized Equation of Random Wave Forces

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Abstract Based on the linear wave theory and Morison's equation, the probability density of the random wave force acting on an isolated pile is derived and a new linearized Morison equation is proposed. In contrast to the linearized model proposed by Borgman, the new model predicts the random wave force with a larger value. A subsequent analysis shows that the proposed new linearized equation is more accurate in calculating random wave forces corresponding to small exceedences acting on an isolated pile.

1. INTRODUCTION

Determination of random wave forces acting on marine structures is essential in ocean engineering design. Over the last few decades, many attempts have been made to develop formulae for the calculation of random wave forces. For structures comprised of slender piles, the generally accepted formula is due to Morison and his colleagues (see, Sarpkaya and Isaacson, 1981). In Morison's model, the horizontal random wave force f acting on per unit length of a pile is assumed to consist of two parts, a fluid drag force and an inertia force, namely

$$f = f_D + f_I \,, \tag{1}$$

where f_D and f_I are respectively the drag force and the inertia force acting on per unit length of the pile and are respectively related to the horizontal velocity u and the horizontal acceleration a of the fluid particle by,

$$f_D = K_d u|u| \,, \tag{2}$$

$$f_I = K_i a \,, \tag{3}$$

where

$$K_d = \frac{1}{2}\rho D C_d, \quad K_i = \frac{1}{4}\pi\rho D^2 C_m, \tag{4}$$

in which ρ is the density of water, D represents the diameter of the pile section, C_d and C_m denote respectively the drag coefficient and the inertia coefficient.

Although Morison's model provides a good estimate of the random wave force acting on slender piles, the model is non-linear in velocity and thus it is difficult to develop efficient response analysis methods for dynamically sensitive structures subjected to random drag forces. Today, the only practical way to carry out a stochastic dynamic analysis of a drag-dominated structure without linearizing the drag force is to apply the technique of time domain Monte Carlo simulation. However, the computational burden involved in application of this procedure for the estimation of , for example, long-term fatigue is almost prohibitive at present for a detailed analysis. Therefore, it is practically important to establish simplified models for engineering design analysis. The importance of developing a simplified model is also in that with such a model one can derive analytical solutions for the dynamic behaviour of certain structures and accordingly provides not only a good understanding of the dynamic behaviour of the structures but also a means for validating numerical methods. The most extensively used representation of drag forces on structures is obtained by linearization of the Morison equation. Borgman (1967), based on the analysis of wave force spectrum, proposed a linearized Morison equation and the equation has been widely used in calculating random wave forces and the response analysis of structures. However, it has been reported (Tickell 1977) that the distribution of the drag force, derived by Borgman's equation and the linear wave theory, significantly underestimates the drag force corresponding to small exceedences. Hence, further analysis is needed to modify Borgman's model.

In this paper, we derive a new linearized Morison model based on the linear wave theory and the Morison formula. Firstly, we assume that the linearized Morison model takes the form of

$$\overline{f} = Au + K_i a \,,$$

with A independent of u. Then, the coefficient A is such chosen that the variance of the probability distribution of $\overline{f}$ is the same as that of f. Hence, this paper is organized as follows. In section two, we present the form of u and a and then derive the joint probability distribution of u and a and consequently the distribution of f. In section there, we derive the variance of f and $\overline{f}$ and then determine the coefficient A by matching these two variances.

2. PROBABILITY DENSITY OF RANDOM WAVE FORCES

Let η denote the displacement of a random wave surface at a spatially fixed point as shown in Fig.1. From the linear random wave theory, η can be expressed as (see, e.g., Longuet-Higgins 1957)

$$\eta = \sum_{n=0}^{\infty} a_n \cos\psi_n , \tag{6}$$

where $\psi_n = \omega_n t + \varepsilon_n$, t = time, a_n = amplitude of the nth wave component, ω_n = frequency, and ε_n denotes independent random phase uniformly distributed over the interval $(0, 2\pi)$. From (6) one can determine, by solving the water wave equations (Borgman 1972; Sarpkaya and Isaacson 1981), the horizontal velocity u(z) and acceleration a(z) of a water particle, namely

$$u = u(z,t) = \sum_{n=1}^{\infty} \frac{k_n g}{\omega_n} \cdot \frac{\cosh(k_n z)}{\sinh(k_n d)} a_n \cos\psi_n , \tag{7}$$

$$a = a(z,t) = -\sum_{n=1}^{\infty} k_n g \cdot \frac{\cosh(k_n z)}{\sinh(k_n d)} a_n \sin\psi_n , \tag{8}$$

where d is the still water depth; g = gravity constant and $\omega_n = k_n g \cdot \tanh(k_n d)$. It can be shown from the Lyapunov theorem that both u and a are random variables with Gaussian distribution, namely

$$P_u(u) = \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left(-\frac{u^2}{2\sigma_u^2}\right), \tag{9}$$

$$P_a(a) = \frac{1}{\sqrt{2\pi}\sigma_a} \exp\left(-\frac{a^2}{2\sigma_a^2}\right), \tag{10}$$

where P_u and P_a are respectively the probability distributions of u and a, σ_u^2 and σ_a^2 denote respectively the variances of random variables u and a and are given by,

$$\sigma_u^2 = \sigma_u^2(z) = \int_0^{\infty} \omega^2 \frac{\cosh^2(kz)}{\sinh^2(kd)} S_{\eta\eta}(\omega) d\omega , \tag{11}$$

$$\sigma_a^2 = \sigma_a^2(z) = \int_0^{\infty} \omega^4 \frac{\cosh^2(kz)}{\sinh^2(kd)} S_{\eta\eta}(\omega) d\omega , \tag{12}$$

in which $S_{\eta\eta}(\omega)$ represents the wave surface spectrum.

From (7) and (8), it is further noted that the correlation coefficient of u and a is zero. Thus u and a are independent random variables and their joint probability density is given by

$$P_{u,a}(u,a) = \frac{1}{2\pi\sigma_u\sigma_a} \exp\left(-\frac{u^2}{2\sigma_u^2} - \frac{a^2}{2\sigma_a^2}\right). \tag{13}$$

The probability densities of the drag force, the inertia force and the total random wave force can then be determined from above equations, which is presented as follows.

2.1 Probability Density of the Drag Force

As f_D is directly related to the horizontal velocity u by (2), the probability density of the drag force f_D can thus be determined from the probability distribution of u given by (9), thus

$$P_D(f_D) = \frac{1}{2\sqrt{K_d |f_D|}} f_u\left(\sqrt{\frac{|f_D|}{K_d}} \cdot \mathrm{sgn}(f_D)\right) = \frac{1}{2\sigma_u \sqrt{2\pi K_d |f_D|}} \exp\left(-\frac{|f_D|}{2K_d\sigma_u^2}\right), \tag{14}$$

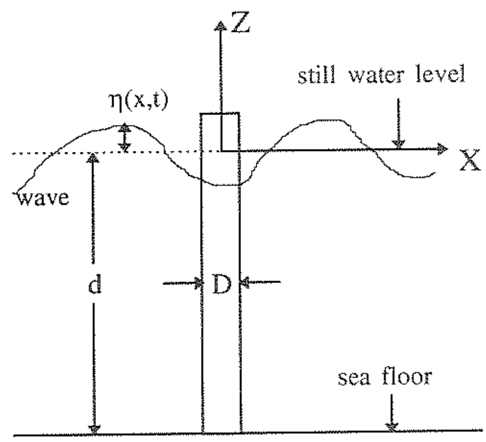

Figure 1. Definition Sketch

where sgn(x) is the sign function given by

$$\operatorname{sgn}(x)=\begin{cases}1 & (x>0)\\ 0 & (x=0)\\ -1 & (x<0)\end{cases}\ . \tag{15}$$

Hence, the drag force obeys a symmetric distribution with zero mean and the variance given by

$$\sigma_{f_D}^2=\int_{-\infty}^{+\infty} f_D^2 P_D(f_D)df_D=3\sigma_u^4 K_d^2 \tag{16}$$

2.2 Probability Density of the Inertia Force

As the inertia force is directly related to the horizontal acceleration via (3), the probability density of f_I can thus be determined from the distribution of the horizontal acceleration (10), namely

$$P(f_I)=\frac{1}{K_i} f_a\left(\frac{f_I}{K_i}\right)$$
$$=\frac{1}{K_i\sigma_a\sqrt{2\pi}}\exp\left(-\frac{f_I^2}{2K_i^2\sigma_a^2}\right), \tag{17}$$

which is a Gaussian distribution with zero mean and a variance given by

$$\sigma_{f_I}^2=\int_{-\infty}^{+\infty} f_I^2 P_I(f_I)df_I=K_i^2\sigma_a^2 . \tag{18}$$

2.3 Joint Probability Density of the Drag Force and Inertia Force

As u and a are independent random variables, and the drag force f_D and the inertia force f_I are respectively functions of u and a, f_D and f_I are also independent and thus their joint probability density $P(f_D, f_I)$ can be determined by

$$P(f_D, f_I)=P_D(f_D)P_I(f_I). \tag{19}$$

2.4 Probability Density of the Total Random Wave Force

As the total random force f is the sum of f_D and f_I, the probability density of the total random wave force per unit length is given by

$$P(f)=\int_{-\infty}^{+\infty} P_D(\mu)P_I(f-\mu)d\mu$$
$$=C\int_0^{+\infty}\frac{1}{\sqrt{\mu}}\exp(-d_1\mu)\left[\exp\left(-d_2(f-\mu)^2\right)\right.$$
$$\left.+\exp\left(-d_2(f+\mu)^2\right)\right]d\mu , \tag{20}$$

where

$$C=\frac{1}{4\pi\sigma_u\sigma_a K_i\sqrt{K_d}},$$
$$d_1=\frac{1}{2K_d\sigma_u^2},\quad d_2=\frac{1}{2K_i^2\sigma_a^2}. \tag{21}$$

Since

$$\exp\left(-d_2(f\pm\mu)^2\right)$$
$$=\exp\left(-d_2f^2\right)\sum_{n=0}^{\infty}\frac{H_n\left(\sqrt{d_2}f\right)}{n!}\cdot\left(\mp\sqrt{d_2}\mu\right)^n , \tag{22}$$

we have

$$P(f)=\frac{2C}{\sqrt{d_1}}\exp\left(-d_2f^2\right)\cdot$$
$$\cdot\sum_{k=0}^{\infty}\frac{H_{2k}\left(\sqrt{d_2}f\right)}{(2k)!}\left(\frac{d_2}{d_1^2}\right)^k\Gamma\left(2k+\frac{1}{2}\right), \tag{23}$$

where $\Gamma(x)$ is the so-called $\Gamma-$function, $H_n(x)$ is the nth degree Hermite polynomial. The probability distribution (23) of the total wave force is non-Gaussian, although it is symmetric with zero mean.

3. A LINEARIZED MORISON EQUATION

From (20), (16) and (18), we can determine the variance of the total random wave force corresponding to the Morison equation (1), namely

$$\sigma_f^2=\int_{-\infty}^{+\infty} f^2 P(f)df$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f^2 P_D(\mu)P_I(f-\mu)d\mu df$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\mu+y)^2 P_D(\mu)P_I(y)d\mu dy$$
$$=3K_d^2\sigma_u^4+K_i^2\sigma_a^2 . \tag{24}$$

On the other hand, the total random wave force given by the linearized model (5) obeys the Gaussian distribution with variance

$$\sigma_{\bar f}^2=A^2\sigma_u^2+K_i^2\sigma_a^2 . \tag{25}$$

By requiring the variance unchanged in the linearization, we have

$$A = \sqrt{3} K_d \sigma_u . \quad (26)$$

Thus, we obtain a new linearized Morison equation which has the form of

$$f = \sqrt{3} K_d \sigma_u u + K_i a . \quad (27)$$

In contrast to Borgman's (Borgman 1967) equation,

$$f = \sqrt{\frac{8}{\pi}} K_d \sigma_u u + K_i a , \quad (28)$$

our model predicts the drag force with a higher value. As it has been reported that Borgman's model underestimates the drag force, our model can thus be expected to provide a more accurate approximation. This argument is further supported by noting that the probability densities of the drag force per unit length corresponding to the proposed model and Borgman's are respectively given by

$$P_1(f_D) = \frac{1}{\sqrt{6\pi} K_d \sigma_u^2} \exp\left(-\frac{f_D^2}{6K_d^2 \sigma_u^4} \right), \quad (29)$$

$$P_2(f_D) = \frac{1}{4K_d \sigma_u^2} \exp\left(-\frac{\pi f_D^2}{16K_d^2 \sigma_u^4} \right). \quad (30)$$

In contrast to the probability density of the drag force corresponding to the Morison equation as given by (14), the probability densities given by (29) and (30) approach zero much faster as $f_D \to \infty$. Thus, as reported by Tickell (1977), the Gaussian distributions derived by linearized approximations significantly underestimate the drag force corresponding to small given exceedences. The drag force calculated from our model (27) is about 1.08 times that calculated from Borgman's model (28), which further demonstrates that the proposed linearized Morison equation (27) is more accurate than (28) in calculating the random wave forces corresponding to small given exceedences on an isolated pile.

4. CONCLUSIONS

By analysing the probability distribution and variances of random wave force, a new linearized Morison equation has been derived based on the linear wave theory. In contrast to Borgman's linearized model, our model predict the random wave forces acting on piles with a higher value, which has been shown to be a more reliable estimation.

REFERENCES

Borgman, L. E., Random hydrodynamic forces on objects, *Annals of Mathematical Statistics*, 38, 37-51, 1967.

Borgman, L. E., Statistical models for ocean waves and wave forces, *Advances in Hydroscience*, 8, 132-181, 1972.

Longuet-Higgins, M. S., The statistical analysis of a random moving surface, *Phil. Trans. Roy. Soc.* A249(966), 321-387, 1957.

Sarpkaya T., and Isaacson M., Mechanics of wave forces on offshore structure, *Van Nostrand Reinhold, New York*, 1981.

Tickell, R. G., Continuous random wave loading on structural members, *The Structural Engineer*, 55(6), 209-222, 1977.