Long-range dependence and singularity of two-dimensional turbulence

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Abstract This paper proposes a class of models to represent, in the same framework, the long-range dependence and singularity/intermittency of 2-D turbulence.A method is given to estimate and separate these two effects. Application to a vorticity field indicates that the resulting energy spectrum follows the power law, in the low-frequency as well as the high-frequency inertial range, predicted by Kraichnan's theory of 2-D turbulence.

1 Introduction

Kraichnan's theory of two-dimensional turbulence predicts an inverse energy (L^2-norm of velocity) cascade which gives rise to the power law $E(\lambda) \sim |\lambda|^{-5/3}$ for the energy spectrum in the low-frequency inertial range (Kraichnan, 1967). This theory also predicts a direct enstrophy (L^2-norm of vorticity) cascade which leads to the power law $E(\lambda) \sim |\lambda|^{-3}$ for the high-frequency inertial range. It is known that the energy spectrum $E(\lambda)$ of the velocity field is related to the enstrophy spectrum $Z(\lambda)$ of the vorticity field via the formula $Z(\lambda) = |\lambda|^2 E(\lambda),\ \lambda \in \mathbb{R}^2$ (Do-Khac *et al.*, 1994). Consequently, the above scaling laws can be written for $Z(\lambda)$ as

$$Z(\lambda) \sim \begin{cases} |\lambda|^{1/3}, & |\lambda| \to 0, \\ |\lambda|^{-1}, & |\lambda| \to \infty. \end{cases}$$

Self-similarity theories, in particular that of fractional Brownian motion (fBm), have been commonly used to model the $|\lambda|^{-3}$ scaling of the velocity field. The scaling $|\lambda|^{1/3}$ in the low-frequency range means that the vorticity field does not display long-range dependence (LRD), and, due to lack of an appropriate method, its validation does not seem to have received much attention in laboratory as well as numerical experiments (Farge *et al.*, 1996). Also, to our knowledge, there has been no previous work reporting on the estimation of both scaling laws in the same setting.

In this paper, we propose a method to estimate both scaling behaviours (at low frequencies as well as in the high-frequency range) for the vorticity field. As a result, we advocate that the scaling $|\lambda|^{-3}$ in the energy spectrum is the contribution of both singularity / intermittency and LRD of the velocity random field. It is then essential to be able to separate these two effects. This separation is based on a class of models which represent simultaneously the LRD and singularity / intermittency of 2-D turbulent flows. In the simplest setting, the increment random fields of this class have spectral density of the form

$$f(\lambda) = \frac{c}{|\lambda|^{2\gamma}\left(1+|\lambda|^2\right)^{\alpha}}, \quad c > 0, 0 < \alpha \leq 1, \quad (1)$$
$$-1/2 < \gamma < 1/2, \quad \alpha+\gamma > 1/2, \quad \lambda \in \mathbb{R}^2.$$

The imposed conditions on γ and α mean that the spectral density (1) is properly defined and the resulting increment random field is stationary. In other words, the random fields are not assumed stationary, but have stationary increments with spectral density (1). It is noted that $f(\lambda) \sim |\lambda|^{-2\gamma}$ as $|\lambda| \to 0$ and $f(\lambda) \sim |\lambda|^{-2(\gamma+\alpha)}$ as $|\lambda| \to \infty$. Hence, the LRD is represented by the exponent γ, while the singularity / intermittency is indicated by the exponent α .Fractional Brownian motion is a special case of (1) when $\alpha = 1$. The component $|\lambda|^{-2\gamma}$ is the Fourier transform of the Riesz kernel, while $\left(1+|\lambda|^2\right)^{-\alpha}$ is the Fourier transform of the Bessel kernel (Anh *et al.*, 1997b). The existence of random fields with spectrum of the form

(1) (the fractional Riesz-Bessel motion) is established in Anh *et al.* (1997a).

This paper gives a discrete approximation to (1) and a method for parameter estimation of the resulting discrete models. Here, we rely on the wavelet theory to be able to separate the LRD effect and the heavy-tail effect in the data. Numerical results will be reported for a vorticity field generated from classical 2-D turbulence equations. The results agree with the scaling behaviours predicted by Kraichnan's theory.

2 Wavelet transform of stationary increment random fields

Let $\{X(t), t \in \mathbb{R}^2\}$ be a continuous (in mean square) random field with mean 0. Denote the increments of $X(t)$ by $\Delta(\tau) X(t) = X(t+\tau) - X(t), t, \tau \in \mathbb{R}^2$. The random field $X(t)$ is said to have second-order stationary increments if the expectations

$$D(t;\tau_1,\tau_2) = E\left[\left(\Delta(\tau_1) X(t+s)\right)\left(\Delta(\tau_2) X(s)\right)\right]$$

are independent of s for all $s, t, \tau_1, \tau_2 \in \mathbb{R}^2$. The function $D(t;\tau_1,\tau_2)$ then has the spectral representation

$$\begin{aligned} D(t;\tau_1,\tau_2) &= \int_{\mathbb{R}^2_-} \left(1-e^{-i\tau_1\cdot\lambda}\right)\left(1-e^{i\tau_2\cdot\lambda}\right) \\ &\quad \times \frac{1+|\lambda|^2}{|\lambda|^2} F(d\lambda) + A\tau_1\cdot\tau_2, \end{aligned} \quad (2)$$

where $\mathbb{R}^2_- = \mathbb{R}^2 - \{0\}$, $F(d\lambda)$ is a nonnegative measure on $\mathbb{R}^2_-$ such that $\int_{\mathbb{R}^2_-} F(d\lambda) < \infty$, A is a constant Hermitian positive definite matrix, and $\tau_1 \cdot \tau_2$ is the scalar product of two vectors $\tau_1, \tau_2 \in \mathbb{R}^2$ (Yaglom, 1957).

We shall assume that F is absolutely continuous and that there exist positive numbers γ, α, c such that its Radon-Nikodym derivative $f(\lambda)$ has the form (1). We shall consider only random fields for which $A = 0$ in the representation (2). An important example is the class of locally stationary and locally isotropic random fields with a power law structure function (Yaglom, 1957, pp. 311, 316 & 317), which is known to play an important role in the statistical theory of turbulence (Obukhov, 1954). Fractional Brownian motion and those characterised by a spectral density of the form (1) belong to this class.

Since the estimation method of Section 4 relies on the wavelet theory, we derive here some needed results on the wavelet transforms of stationary increment random fields.

Let $\psi \in L_2(\mathbb{R}^2)$ with its Fourier transform $\widehat{\psi}$ satisfying the admissibility condition:

$$\widehat{\psi}(0) = 0, \quad C_\psi = \int_0^\infty \left|\widehat{\psi}(a\lambda)\right|^2 \frac{da}{a} < \infty, \quad \lambda \in \mathbb{R}^2 \text{ a.e.} \quad (3)$$

The function ψ is then a wavelet function. The continuous wavelet transform of $X(t)$ is defined as

$$\begin{aligned} W_a(t) &= a^{-1}\int_{\mathbb{R}^2} X(s)\,\psi\left(\frac{s-t}{a}\right) ds, a>0, t \in \mathbb{R}^2 \\ &= a\int_{\mathbb{R}^2} X(au+t)\,\psi(u)\,du. \end{aligned} \quad (4)$$

Now, using the spectral representation (2) with $A = 0$, the definition (4) and Fubini's theorem, the covariance function of $W_a(t)$ at a fixed scale a is given by

$$\begin{aligned} E\left[W_a(t+\tau) W_a(t)\right] &= a^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} E(X(au+t+\tau) \\ &\qquad X(av+t))\psi(u)\,\psi(v)\,dudv \\ &= a^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2_-}\left(1-e^{-i(au+\tau)\cdot\lambda}\right)\left(1-e^{iav\cdot\lambda}\right)\right. \\ &\qquad \left.\times \frac{1+|\lambda|^2}{|\lambda|^2} f(\lambda)\,d\lambda\right]\psi(u)\,\psi(v)\,dudv \\ &= a^2 \int_{\mathbb{R}^2_-}\left[\int_{\mathbb{R}^2}\left(1-e^{-i(au+\tau)\cdot\lambda}\right)\psi(u)\,du\right. \\ &\qquad \left.\times \int_{\mathbb{R}^2}\left(1-e^{iav\cdot\lambda}\right)\psi(v)\,dv\right] \frac{1+|\lambda|^2}{|\lambda|^2} f(\lambda)\,d\lambda \\ &= a^2 \int_{\mathbb{R}^2_-} e^{-i\tau\cdot\lambda}\left|\widehat{\psi}(a\lambda)\right|^2 \frac{1+|\lambda|^2}{|\lambda|^2} f(\lambda)\,d\lambda \quad \text{using (3)}, \end{aligned} \quad (5)$$

which is independent of t. Consequently, the random field $W_a(t)$ is stationary with spectral density

$$f_{W_a}(\lambda) = a^2\left|\widehat{\psi}(a\lambda)\right|^2 \frac{1+|\lambda|^2}{|\lambda|^2} f(\lambda). \quad (6)$$

Writing $\psi_{jk} = 2^{-j/2}\psi\left(2^{-j}t-k\right), j \in \mathbb{Z}, t, k \in \mathbb{R}^2$, then, in view of (5), the variance of W_{2^j} is given by

$$R(0;j) = 2^{2j}\int_{\mathbb{R}^2_-}\left|\widehat{\psi}\left(2^j\lambda\right)\right|^2 \frac{1+|\lambda|^2}{|\lambda|^2} f(\lambda)\,d\lambda. \quad (7)$$

3 Parameter estimation

In this section, we outline a method to estimate the spectral density of a discrete-parameter random field whose covariance function approximates that of the random field generated by (1) as $|\lambda| \to 0$ and $|\lambda| \to \infty$. We need the following results.

Lemma 1 *Let* $f(x,y) \in L_1(\mathbb{R}^2)$ *and* $g(x,y) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f(x+2\pi j, y+2\pi k)$. *Then* $g(x,y) \in L_1\left([-\pi,\pi]^2\right)$ *and* $c_{jk} = c(j,k)$, *where*

$$\begin{aligned} c(j,k) &= \int_{\mathbb{R}^2} f(x,y)\, e^{-i(jx+ky)}dxdy, \\ c_{jk} &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x,y)\, e^{-i(jx+ky)}dxdy. \end{aligned}$$

Proof. Extension to $\mathbb{R}^2$ of Lemma 1 of Meyer, 1992, p.8. ■

Lemma 2 *For* $\alpha > 0$, *there exists a finite measure* μ_α *on* $\mathbb{R}^2$ *so that its Fourier transform* $\widehat{\mu}_\alpha$ *is given by*

$$\widehat{\mu}_\alpha(\lambda) = \frac{|\lambda|^{2\alpha}}{\left(1+|\lambda|^2\right)^{\alpha}}.$$

Proof. See Stein (1970), p. 133. ■

In view of Lemma 2, we can write (1) in the form

$$f(\lambda) = \frac{c\widehat{\mu}_\alpha(\lambda)}{|\lambda|^{2(\gamma+\alpha)}}. \tag{8}$$

Lemma 1 then implies that a discrete approximation of the component $1/|\lambda|^{2(\gamma+\alpha)}$ of (8) is

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \frac{1}{\left((\lambda_1+2\pi j)^2+(\lambda_2+2\pi k)^2\right)^{\gamma+\alpha}}, \quad \lambda_1,\lambda_2 \in (0,\pi]$$

$$\sim \frac{1}{|\lambda|^{2(\gamma+\alpha)}} \quad \text{as} \quad |\lambda| \to 0. \tag{9}$$

The result (9) suggests that the exponent $\gamma+\alpha$ can be estimated by linear regression on the periodogram $I(\lambda)$ of $X(t)$:

$$\ln I(\lambda_j,\lambda_k) = c_1 - (\gamma+\alpha)\ln\left[\left(\frac{2\pi j}{n}\right)^2+\left(\frac{2\pi k}{n}\right)^2\right] + u_{jk}, \quad j,k = 1,\ldots,[n/2]-1, \tag{10}$$

n is the sample size and u_{jk} is white noise (Geweke and Porter-Hudak, 1983). On the other hand, the component $\widehat{\mu}_\alpha(\lambda)$ of (8) can be estimated by the maximum entropy spectral method of Chiang (1984). This latter method, which is an extension of the Burg maximum entropy method to two dimensions, can be implemented using the multivariate Levinson recursion.

Estimation of model (8) gives the exponent $\gamma+\alpha$ of the combined effects of LRD and singularity / intermittency. In order to separate these two effects, we need to estimate γ, the low-frequency exponent, alone. Since the scaling $|\lambda|^{1/3}$ in a neighbourhood of the origin means short-range dependence for the vorticity field, it is not appropriate to use a Geweke-Porter-Hudak type procedure. Our method starts with Eq. (7). It is plausible to estimate γ by a 1-D method; in other words, the vorticity field is vectorised for this estimation. Eq. (7) then becomes

$$R(0;j) = 2^j \int_{\mathbb{R}} \left|\widehat{\psi}\left(2^j\lambda\right)\right|^2 \frac{1+\lambda^2}{\lambda^2} f(\lambda)\, d\lambda, \tag{11}$$

where $f(\lambda)$ is also of the form (1). For the wavelet function ψ, we shall use the Haar wavelet:

$$\psi(t) = \begin{cases} 1, & 0 \le t < 1/2, \\ -1, & 1/2 \le t < 1, \\ 0, & \text{elsewhere}, \end{cases}$$

$$\widehat{\psi}(\lambda) = \frac{i\lambda}{4}e^{-i\lambda/2}\left(\frac{\sin\lambda/4}{\lambda/4}\right)^2.$$

Then, with the scale parameter $a = 2^j$,

$$\begin{aligned} R(0;j) &= 2^{2j(1+\gamma)}\int_{\mathbb{R}}\left|\widehat{\psi}(\lambda)\right|^2 \frac{\left(1+2^{-2j}\lambda^2\right)^{1-\alpha}}{|\lambda|^{2(1+\gamma)}}d\lambda \\ &= 2^{2j(1+\gamma)}\int_{\mathbb{R}} h(\lambda;j)\, d\lambda, \end{aligned}$$

where $h(\lambda;j) = \left(\frac{\sin\lambda/4}{\lambda/4}\right)^4 \frac{\left(1+2^{-2j}\lambda^2\right)^{1-\alpha}}{|\lambda|^{2\gamma}}$. Noting that $\frac{\sin\lambda}{\lambda} \to 1$ as $\lambda \to 0$ and $\frac{\sin\lambda}{\lambda} \sim \frac{1}{\lambda}$ as $\lambda \to \infty$, we have $\int_{\mathbb{R}} h(\lambda;j)\,d\lambda < \infty$ as $j \to \infty$ and for $1+\alpha+\gamma > 0$. Hence, as $j \to \infty$,

$$R(0;j) \sim 2^{2j(1+\gamma)}M, \quad 0 < M < \infty,$$

which suggests the linear regression:

$$\ln R(0;j) = c_2 + 2(1+\gamma)\ln 2^j + v_j, \tag{12}$$

v_j being white noise.

4 Experimental results

The above method is applied to a decay vorticity field generated from the Navier-Stokes equations (in the

Table 1: Parameter estimates for the vorticity field of Figure 1

$\widehat{\gamma}+\widehat{\alpha}$		$2\widehat{\gamma}$	
1.006	(15)	−0.314	(100)
1.090	(20)	−0.292	(200)
1.150	(25)	−0.296	(300)

velocity-vorticity form). Farge *et al.* (1992) observed that the nonlinear dynamics is preserved if only the strongest wavelet coefficients are used. This leads to a numerical scheme to integrate Navier-Stokes equations in an adaptive wavelet basis, and a forcing scheme which injects enstrophy only into the strongest wavelet coefficients in order to excite the vortices without affecting the background flow (Schneider and Farge, 1997). A typical 512×512 image of the vorticity field at time step $t = 10$ is shown in Figure 1. Its 2-D periodogram is shown in Figure 2.

We derived the numerical results (reported below) on this image. Since the estimation of the exponent γ is a 1-D method, we performed this estimation on the horizontal lines 100, 200 and 300 of the image. The line 100 is displayed in Figure 3, while its periodogram is shown in Figure 4. It is clear that the vorticity field exhibits short-range dependence (with a positive slope near frequency 0 in the periodogram).

In using the Geweke-Porter-Hudak method for estimating $\gamma+\alpha$, it is necessary to compute the regression (10) in a small neighbourhood of the origin. Geweke and Porter-Hudak (1983) suggested to use $j,k = 1,...,\sqrt{n}$ (i.e. $\sqrt{512}$ in our case). We report the estimates of $\gamma+\alpha$ for 15, 20 and 25 Fourier frequencies in Table 1. The estimates of 2γ are reported for the lines 100, 200 and 300 of the image. It should be noted that the above estimates are obtained for the spectral density $f(\lambda_1,\lambda_2)$ of the vorticity field. In applied works, such as that of Kraichnan (1967), the spectral density of a 2-D isotropic random field is meant to be the quantity

$$\varphi(|\lambda|) = \Phi'(|\lambda|), \quad \lambda \in \mathbb{R}^2,$$

where $\Phi(k) = \int\int_{|\lambda|<k} f(\lambda_1,\lambda_2)\, d\lambda_1 d\lambda_2$ (Eq. (4.98) of Yaglom, 1986). The relationship between $\varphi(|\lambda|)$ and $f(|\lambda|)$ is

$$\varphi(|\lambda|) = \frac{2\pi}{\Gamma(1)} |\lambda| f(|\lambda|) \qquad (13)$$

(Eq. (4.119) of Yaglom, 1986). Hence, in the form (13), the scaling behaviours of the vorticity field of Figure 1 is

$$Z(|\lambda|) \sim \begin{cases} |\lambda|^{0.314}, & |\lambda| \to 0, \\ |\lambda|^{-1.012}, & |\lambda| \to \infty \end{cases}$$

(using $j,k = 1,...,15$ and line 100) as predicted by Kraichnan's theory.

5 Conclusions

This paper introduces a class of models which are capable of representing two key features of 2-D turbulence: LRD and singularity/intermittency. Their spectral form allows this representation to be examined simultaneously. The range of the parameters means that the random field may display long-range ($0 < \gamma < 1/2$) or short-range dependence ($-1/2 < \gamma < 0$). For severe singularity or intemittency, the parameter must be within the range $0 < \alpha < 1/2$. With the Haar wavelet playing a key role, the method of this paper allows these two effects to be estimated and separated. Additional features of the data can also be incorporated in the model (such as short-term features represented by an *ARMA*-type component in one dimension). However, in two dimensions,this incorporation may require the introduction of artificial causal directions in order to define a "past" and a "future" of a pixel in the image. For the purpose of studying scaling laws, this spatial *ARMA*-type modelling is not essential.. Chiang's method of representing short-term features in a 2-D stationary spectrum is sufficient for an iterative estimation of the LRD and singularity/intermittency exponents. (It should be noted that a specification of causal directions in space is not needed for the spectral approach of this paper). The estimated results indicate that the generated vorticity field does not display much singularity (with $\widehat{\alpha} \approx 1.163$) at time step t=10. Estimates on images of later time steps show that the resulting vorticity field scales as $|\lambda|^{-2}$ at t=200 and as $|\lambda|^{-3}$ at t=300, for example. These estimates indicate a decay turbulence and that the scaling exponent may vary in the range $1 < 2(\gamma+\alpha) < 3$.

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References

[1] Anh, V.V., Angulo, J.M. and Ruiz-Medina, M.D., Possible long-range dependence of fractional random fields (1997a) (submitted).

[2] Anh, V.V., Kloeden, P.E., Angulo, J.M. and Ruiz-Medina, M.D., Modified fractional Brownian motion, Stochastics & Stochastic Reports, 1997b (revised version with the Editor).

[3] Chiang, Tse-pei, Entropy of Gaussian random field and spectral analysis of random field data, China-Japan Symposium on Statistics, pp. 42-44, Peking University Press, Beijing, 1984.

[4] Do-Khac, M., Basdevant, C., Perrier, V. and Dang-Tran, K., Wavelet analysis of 2-D trubulent flows, Physica D 76(1994) 252-277.

[5] Geweke, J. and Porter-Hudak, S., The estimation and application of long-memory time series models, J. Time Series Anal. 4(1983) 221-238.

[6] Farge, M., Goirand, E., Meyer, Y., Pascal, F. and Wickerhauser, M.V., Improved predictability of two-dimensional turbulent flows using wavelet packet compression, Fluid Dynamics Research, 10(1992) 229-250.

[7] Farge, M., Kevlahan, N., Perrier, V. and Goirand, E., Wavelets and turbulence, Proc. IEEE 84(1995) 639-669.

[8] Kraichnan, R.H., Inertial ranges in two-dimensional turbulence, Phys. Fluid 10(1967) 1417-1423.

[9] Kraichnan, R.H., Models of intermittency in hydrodynamic turbulence, Physical Review Letters 65(1990) 575-578.

[10] Obukhov, A.M., The statistical description of continuous fields, Trudu Geofiz. Inst. Akad. Nauk. SSSR, 151(1954) 3-42 (in Russian).

[11] Schneider, K. and Farge, M., Wavelet forcing for numerical simulation of two-dimensional turbulence, C.R. Acad. Sci. Paris, Série IIb (Fluid Mechanics) (in press).

[12] Stein, E.M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, New Jersey, 1970.

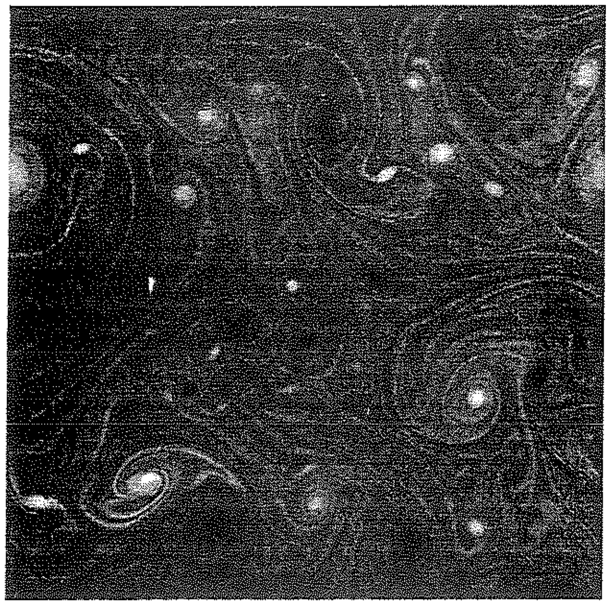

Figure 1: A vorticity field at time step t=10.

[13] Yaglom, A.M., Some classes of random fields in n -dimensional space, related to stationary random processes, Theory of Probability and its Applications, 2(1957) 273-320.

[14] Yaglom, A.M., The Correlation Theory of Stationary and Related Random Processes, Vol.1, Springer-Verlag, New York, 1986.

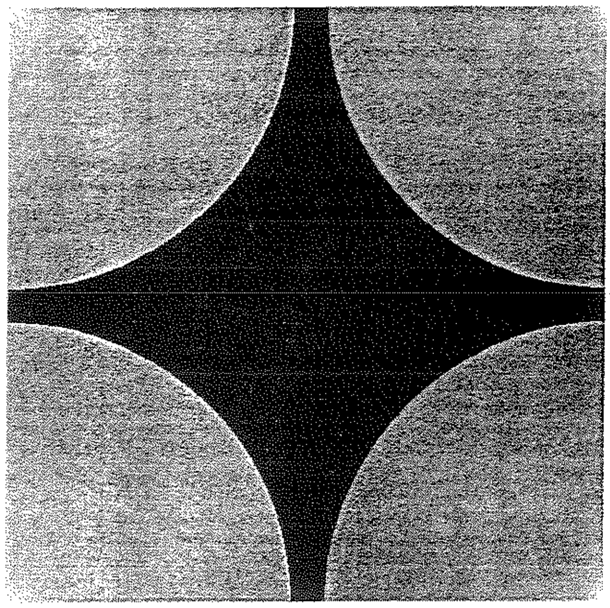

Figure 2: Periodogram of the vorticity field of Figure 1.

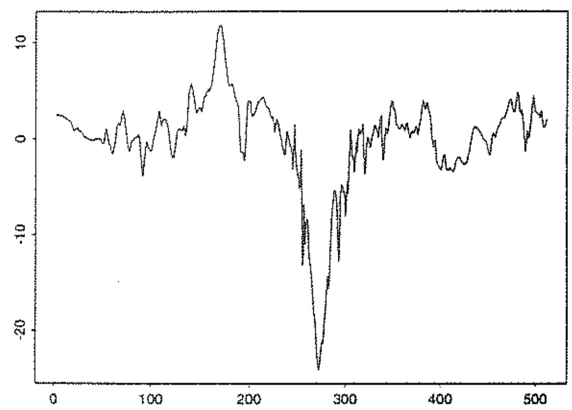

Figure 3: Line 100 of the vorticity field of Figure 1.

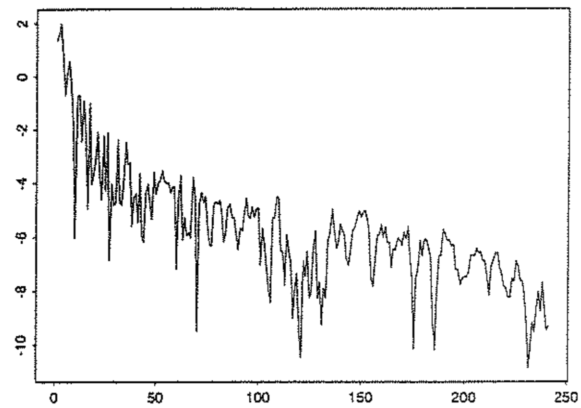

Figure 4: Periodogram of the time series of Figure 3.