Stochastic Models of Interest Rates in Economics, Finance and Actuarial Science

Y.K. Tse
Department of Economics and Statistics
National University of Singapore

Abstract In this paper we review some recent developments in stochastic interest rate models in the economics, finance and actuarial literature. We survey the empirical evidence with respect to various one-factor and two-factor stochastic models of interest rates. The econometric issues in testing the models are discussed. We also discuss three approaches in determining the term structure of interest rates and the pricing of interest-rate contingent claims, namely, the arbitrage-free method, the general equilibrium method and the perfect replication method. Some of the recent findings in the actuarial literature on the stochastic behaviour of interest rates are summarized.

1 INTRODUCTION

Interest rate is an important determinant of the values of assets as well as liabilities. In the economics and finance literature, voluminous research has been done on the linkage between interest rate and the pricing of various assets, in particular, fixed-income securities and interest-rate contingent claims. Until recently, valuation of the liabilities of an insurance company had been done mainly in the fixed-interest rate environment. Thus, interest rate is assumed to be constant during the period of assessment and is exogenously determined. Since the early 1980s, interest rate has become more volatile. As a result, the actuarial profession has increasingly recognized the importance of treating interest rate as a stochastic variable. In this paper, we review some recent developments in stochastic interest rate models in the economics, finance and actuarial literature.

Traditional economic research has focused mainly on the linkage between monetary policy and real economic activities. The term structure has been shown to have valuable information on future economic activities. Research has also suggested that, by shifting the structure of risks, monetary policy may have important consequences for real long-term rates. Theories have been proposed to explain the term structure by various hypotheses concerning the expectations of economic agents as well as the market structure. In the last decade new research directions have been initiated by considering interest rate as a stochastic variable. The developments in the finance literature on the continuous–time processes of security prices have greatly influenced and enhanced our understanding of interst rate movements. Various approaches have been proposed to model the prices of interest–rate contingent claims, including pure discount bonds. Term structure models are then derived from these asset prices.

Cash flow analysis is an important tool for actuaries in the valuation of insurance products. Among the various assumptions in the setup of a cash flow analysis, interest rate models and assumptions play an important role. Actuaries have been researching on the selection of interest rate models for their analysis. Results have been drawn from the finance and statistics literature in identifying proper strategies in the construction of assumptions for cash flow analysis. It appears that interest in this area is growing rapidly and it is a purpose of this paper to summarize some of the recent findings.

This paper is organized as follows. In Section 2 we review the stochastic models of interest rates in the continuous–time. Our focus is on the empirical evidence with respect to various one–factor and two–factor models. Some of the econometric issues in testing the models are discussed. Section 3 reviews three approaches in determining the term structure of interest rates and the pricing of interest–rate contingent claims. These are the arbitrage–free method, the general equilibrium method and the perfect replication method. In Section 4 we summarize some of the recent findings in the actuarial literature on the stochastic behaviour of interest rates. Section 5 concludes the paper.

2 STOCHASTIC INTEREST RATE MODELS

2.1 Terminologies and Notations

Consider a pure discount (zero–coupon) bond that pays one dollar at time T. Let $P(t, T)$, $t \leq T$, denote the price of the bond at time t. Thus, $P(T, T) = 1$. The yield–to–maturity of the bond, denoted by $R(t, T)$, is defined as $-\log P(t, T)/(T - t)$. A plot of the yield–to–maturity against the time–to–maturity, $\tau = T - t$, is called the term structure of interest rates. As we shall only consider default–free bonds such as Treasury securities, $R(t, T)$ is the risk-free rate of return at time t for bonds with time–to–maturity τ. We define $r_t = \lim_{T\to t} R(t, T)$ as the instantaneous risk-free rate of interest and introduce the notation $r_{t,\tau} = R(t, T)$. As such, r_t is unobservable and should be regarded as a state (latent) variable. Many term structure models postulate that r_t is the only determinant of the term structure. These models are called one–factor models. They have the characteristic that interest rates of all maturities are perfectly correlated. Alternatively, the number of state variables driving the term structure may be extended to two or more. These models are called multi–factor models. One possible extension is to assume that, in addition to r_t, the term structure is also determined by the long–term rate of interest defined as $\ell_t = \lim_{T\to\infty} R(t, T)$. One–factor models in the literature typically assume that r_t follows a continuous–time diffusion process. In the next subsection, we shall review some of these diffusion processes and discuss the empirical findings related to these processes.

2.2 One–Factor Models

We assume r_t follows a diffusion process driven by the following stochastic differential equation:

$$dr = \kappa\,(\theta - r)\,dt + \sigma\, r^{\gamma}\, dZ, \tag{1}$$

where dZ is a standard Wiener process. This model assumes r_t follows mean–reversion, with θ being the long–run instantaneous riskfree rate of interest and κ being the speed–of–adjustment coefficient. If we denote the instantaneous variance of r_t by $V(r_t)$, the elasticity of variance, defined by $(\partial V(r_t)/\partial r_t)/(r_t/V(r_t))$ is given by 2γ. Thus, the model is characterized by constant elasticity of variance and we call γ the elasticity parameter. Many models in the literature are special cases of equation (1). For example, Merton (1973) considered the special case $\theta = \gamma = 0$, and Dothan (1978) assumed $\kappa = 0$ and $\gamma = 1$. When $\theta = 0$ and $\gamma = 1$, r_t follows a geometric Brownian motion. Perhaps, the most important special cases of equation (1) are the following: (i) the Ornstein–Uhlenbeck (OU hereafter) model (see Vasicek (1977)) with $\gamma = 0$, (ii) the Cox, Ingersoll and Ross (CIR hereafter) (1985b) model with $\gamma = 0.5$,[1] and (iii) the Brennan and Schwartz (BS hereafter) (1979) model with $\gamma = 1$.

[1] The CIR model is sometimes called the square–root process.

One drawback of the OU model is that r_t may become negative. On the other hand, while r_t may become zero in both the CIR and BS models, it can never become negative.

As we shall see in Section 3, stochastic interest rate models have important implications for the pricing of interest–rate contingent claims such as bonds and bond options. Depending on the assumed process for r_t, some pricing models admit analytic closed form solutions while others may require numerical evaluations. On theoretical bases, models that permit closed form solutions may be desirable since they provide greater analytical insights. On the other hand, models that represent reality inacurrately, or rely on unrealistic assumptions, may incur model risk. A balance between the two issues is required for a successful evaluation of interest–rate contingent claims. In the rest of this subsection we review the empirical evidence in the literature with respect to the applicability of the one–factor models.

The following points are important in assessing the empirical results. First, interest rate data are only available at discrete time points. Except for some special cases (such as the geometric Brownian motion, the Vasicek model and the CIR model), the conditional density of r_t is unknown. Some authors use discrete models that approximate the continuous–time process. But as argued by Lo (1988), the discretized maximum likelihood estimator (MLE) is in general inconsistent. Second, r_t is an unobservable state variable. Direct estimation of equation (1) or its discretized approximation is impossible. Some authors get around this problem by using yields of short maturities, such as one-month yields. Thus, data on $r_{t,\tau}$ with τ being one month are used. Obviously, this procedure introduces measurement errors into the estimation. Alternatively, one may use the steady-state distribution of the interest rate process and calculate the unconditional moments. The parameters of the models may be estimated using Hansen's (1982) generalized method of moments (GMM). The models can then be tested by examining the overidentifying restrictions on the objective function.

Of the models nested in equation (1), the CIR model has attracted most research interests. One of the reasons for its importance is that closed form solutions of the prices of pure discount bonds and bond options are available. This enables researchers to use bond price data and avoid using short-term interest rate data, which may induce measurement errors. Unfortunately, the bond price formula depends on r_t as well as the parameters of the CIR process.[2] Using cross section data, however, it is possible to treat r_t as a parameter to be estimated. Brown and Dybvig (1986) examined the CIR model using Treasury securities. Ignoring the tax effect, they considered coupon bonds as a portfolio of pure discount bonds. They examined the deviation of the theoretical bond prices, as predicted by the CIR model, versus the observed bond prices over a cross section of bond data. Using nonlinear least squares methods, they were able to estimate some functions of the parameters of the model as well as the riskfree rate of interest.[3] Examining the cross section estimates over time, they concluded that the model systematically overestimates the short-term interest rates. In addition, the model appears to fit Treasury Bills better than other Treasury issues. This suggests that the tax effect has not been adequately accounted for.

Unlike the CIR process, the BS model admits no analytic solution for the bond prices. Also, the conditional density of r_t is unknown so that an exact MLE is impossible. Ogden (1987) estimated the BS model using the discretized MLE. He used yield data on Treasury bills with ninety days to mature, and the results showed that the short-term interest rate is expected to revert halfway to the long-run level in about one year.[4]

Making use of the results on the steady-state properties of the bond prices, Oldfield and Rogalski (1987) provided some tests for the OU and CIR processes. They argued that the differenced logarithmic bond prices should be serially correlated for both models. For the OU process, the variance of the differenced logarithmic prices is expected to be equal among subperiods for a fixed differencing interval and maturity. In contrast, the variance for the CIR process should be unstable for observations over short time intervals, although this instability should disappear over longer differencing inter-

[2]In addition to r_t and the CIR model parameters, the bond price formula also depends on the market risk parameter, denoted by λ. See Section 3 below for further discussions.

[3]In the CIR bond pricing formula, not all parameters are identified. In particular, only $\kappa + \lambda$, $\kappa\theta$ and σ are estimable from the cross section data.

[4]Ogden used nonlinear optimization methods to obtain the MLE. Analytic solution of the discretized MLE for the BS model can be found in Tse (1995a). See also Tse (1992).

vals. This characteristic provides an indirect test for the OU model versus the CIR model. Although formal significance tests for the varying degree of stability of the variance cannot be obtained, this approach circumvents the problem of unobservable state variables and the derivation of the exact likelihood function. Oldfield and Rogalski reported that the instability in the variance appears to decrease with increasing differencing interval. They argued that this is inconsistent with the OU process and the term structure is better represented by the CIR model.

A serious limitation of one–factor models is that the correlations between yields of different maturities are unity. However, casual observations show that nominal yields of different maturities are not perfectly correlated. Thus, it seems more relevant to examine the applicability of the one–factor models to real interest rates. Following CIR's assumption that the changes in the price level have no effect on the real variables in the economy (money–neutrality assumption), Gibbons and Ramaswamy (1993) examined the applicability of the CIR model to the real interest rates. They derived the first moment of real returns on nominal bonds. The covariances of nonoverlapping real returns with different maturities were also obtained. These unconditional moments are based on the steady–state distributions of the real yields and do not depend on the unobservable state variable. Using Treasury securities with maturity up to 12 months Gibbons and Ramaswamy estimated the parameters of the CIR model by the GMM. They failed to reject the overidentifying restrictions implied by the CIR model. The estimated parameter values, however, preclude a humped curve for any value of r_t. Thus, the model only allows for upward– and downward–sloping real term structures.

To calculate the real return of Treasury bills, Gibbons and Ramaswamy adjusted the nominal yields using the consumer price index (CPI). The CPI is based on prices of consumption goods sampled at various times during a month. Thus, to take CPI as the measured price level at a single time point would introduce autocorrelations in the real return series. This problem appears to be difficult to overcome when price data are required. Pearson and Sun (1994) proposed a method that does not require price data. They assumed a price process suggested by CIR, in which the price level p_t depends on a state variable y_t that is independent of r_t. In this model the price of a nominal bond admits a closed form solution that depends on the state variables r_t and y_t, as well as the parameters of the stochastic processes driving r_t and p_t. The price formula, however, does not depend on p_t. Under the assumption of independence between r_t and y_t, the joint density of these variables can be obtained straightforwardly. Inverting the state variables as functions of bond prices with different maturities, Pearson and Sun obtained the conditional density of the bond prices. They estimated the model using MLE. The results for the likelihood ratio test rejected the CIR model. It should be noted, however, that the Pearson and Sun approach provides a joint test for the CIR model and the specified price process. Thus, they effectively tested a two–factor model for the nominal term structure.

An important implication of the CIR

model is that the long–term rate of interest ℓ_t is independent of r_t. In particular,

$$\ell_t = \frac{2\kappa\theta}{\kappa+\lambda+\gamma}. \quad (2)$$

Thus, if the parameters of the CIR model are stable, ℓ_t is expected to be a constant. The volatility observed in long–term nominal yields easily rejects this condition, suggesting that the CIR model is inapplicable to nominal yields. Brown and Schaefer (1994) constructed a series of real long–term yields using British government index–linked bonds. They observed a time series with a high degree of stability. Using the Brown–Dybvig approach they estimated the parameters of the CIR model from cross sections of bonds with different maturities. The model appears to provide a good fit for cross sections. However, the hypothesis that the parameter values are stable over time is firmly rejected.

As a model derived from the general equilibrium setup with closed form solution for the prices of bonds and bond options, the CIR model assumes a special role in the literature. This explains the overwhelming interest in testing the model. However, within the general diffusion process given in equation (1) there is no *apriori* reason for setting the constraint $\gamma = 0.5$. Furthermore, analytic tractability can also be found in the OU process, although the process has a drawback of permitting the interest rate to take negative values. From the empirical point of view it may be useful to start from the general unrestricted model and proceed to examine if the data support a particular model. This approach was adopted by Chan, Karolyi, Longstaff and Sanders (CKLS hereafter) (1992a). Using monthly data on one–month U.S. nominal yield they estimated the parameters of the unrestricted model using GMM.[5] The estimated value of γ is found to be 1.4999, which is about two standard deviations above 1. Thus, the OU, CIR and BS models are all rejected against the unrestricted model. The results suggest that the volatility is highly dependent on the level of interest rates.

Several empirical issues should be noted in the CKLS approach. First, nominal one–month yields are used. Due to the existence of term premium, measurement errors are introduced. As pointed out by Pearson and Sun, the term premium may be very significant, depending on the values of the riskfree rate of interest and the market risk parameter. Second, the approach provides no information on the market risk parameter. Third, the moment functions used by CKLS are not based on the steady–state distribution of interest rates.[6] They are derived from the discretized approximation to equation (1). Thus, the approach is subject to errors due to discretization. Broze, Scaillet and Zakoian (1993) proposed an indirect estimation method that can eliminate the bias due to the discretization of the model. The method is simulation–based and involves the construction of an approximate likelihood function. Asymptotic properties of the indirect estimates were analysed in Broze, Scaillet and Zakoian (1995). These authors, however,

[5] Some empirical results on the Japanese data using the MLE method were given by CKLS (1992b).

[6] Except for some special cases such as the OU and CIR models, the steady–state distribution of r_t in the unrestricted model is unknown.

assumed that r_t is observable or a good proxy for it is available. The problem of measurement errors is not considered in their analysis.

Following the CKLS approach Tse (1995b) examined the stochastic behaviour of short-term interest rates in eleven countries. He considered eight stochastic models of interest rates nested within equation (1). Three-month money market rates were used in the study. The results showed that no single model can satisfactorily describe the structure of interest rates for all countries. For France, Holland and U.S., the elasticity of variance is above 1.5. Canada, Italy, Switzerland and U.K. are the countries where the elasticity of variance is low. The OU model may be preferred for these countries. For Australia, Belgium, Germany and Japan, the elasticity of variance is moderate. For these countries, there is no clear-cut statistical evidence in choosing between the CIR and BS models.

The CIR model permits three shapes of yield curves: downward-sloping, humped and upward-sloping. As inverted-humped yield curves have been observed in practice, the CIR model is incomplete in describing the data. Constantinides (1992) proposed an alternative approach to model the nominal interest rates that can overcome this difficulty. His approach assumes that there exists a pricing kernel $M(t)$ such that

$$P(t, T) = \mathrm{E}_t[M(T)]/M(t), \qquad (3)$$

where $\mathrm{E}_t[\,\cdot\,]$ denotes the expectation conditional on the information at time t. The existence of $M(t)$ may be justified by assuming an economy in which the consumers have Von Neumann–Morgenstern preferences. However, instead of exploring along this line, Constantinides examined directly the time series process of $M(t)$. Assuming that $M(t)$ depends on some state variables which are either a Wiener process or an OU process, he derived the closed form solutions for bond prices and bond option prices. Analytic formulae for the unconditional moments of the interest rates were also given. Thus, calibration of the model parameters using the GMM approach can be obtained. Constantinides reported some empirical results on a one-factor model. The model permits all four shapes of yield curves, depending on the value of the state variable. Another interesting result is that the half life of the state variable is approximately equal to the length of the business cycle.

Commenting on the inability of the CIR model to take account of humped term premiums as noted above, Longstaff (1989) proposed an alternative model within the general equilibrium framework constructed by CIR. Designated the double square root (DSR) model, r_t is given by the following stochastic differential equation:

$$dr = \kappa\,(\mu - \sqrt{r})\,dt + \sigma\,\sqrt{r}\,dZ, \qquad (4)$$

where $\mu = \sigma^2/(4\kappa)$. Thus, unlike the CIR model the DSR model has only two independent parameters. An important implication of the DSR model is that interest rate reverts to the long-run level μ^2 more slowly from above than from below. Longstaff calibrated the DSR model using the GMM approach and argued that the DSR model provided better description of the data than the CIR model.

The inability of the one-factor CIR model to explain the empirical behaviour of interest rates has led some researchers to explore multi-factor models. Stambaugh (1988) examined the issue of one-factor versus multi-factor models within the general equilibrium term structure model of CIR. He argued that the CIR model implies that the expected excess returns are linear functions of the forward premiums. The number of latent variables determining the forward premiums equals the number of state variables in the model. His GMM analysis of the U.S. Treasury securities with different maturities rejects a one-factor CIR model, while a two- or three-factor model may adequately describe the variations in excess returns. This study focused on the number of state variables determining the term structure. The identities of the term structure, however, are not investigated. In the next subsection we shall review some of the multi-factor models of the term structure.

2.3 Multi-Factor Models

As noted above, one-factor models impose a constraint that the yields of bonds with different maturities are perfectly correlated. The shapes of the yield curves of one-factor models may also be restricted. To overcome these drawbacks and constraints, some authors proposed multi-factor models for the term structure. Richard (1978) suggested a two-factor model in which bond prices are determined by the instantaneous rate of interest r_t and the anticipated instantaneous rate of inflation π_t. Using arbitrage-free arguments,[7] he derived a solution for the bond price, which is determined by a partial differential equation. For the special case that r_t and π_t are uncorrelated square-root processes and that the prices of risks of r_t and π_t are functions of the square root of these variables, Richard derived closed form solutions for the bond prices. Similar to the one-factor CIR model, however, Richard's model implies that the yield of a consol bond is constant.

Brennan and Schwartz (1979) proposed an alternative two-factor model. They assumed that bond prices are determined by the instantaneous rate of interest r_t and the long term rate of interest ℓ_t. These state variables are assumed to follow a joint diffusion process. Correlation between the two variables is permitted. Using an arbitrage-free argument similar to that of Richard, they derived the partial differential equation which determines the prices of bonds of any maturity. Apart from the state variables r_t and ℓ_t, the bond prices also depend on the market prices of risks of r_t and ℓ_t. However, making use of the fact that ℓ_t is a function of the price of a traded asset, i.e., the consol bond, BS showed that the price of risk of ℓ_t can be expressed as a function of r_t, ℓ_t and the price of risk of r_t. This reduces the number of determinants for bond prices to three. The differential equation for determining the bond prices was derived. No analytic solution, however, is obtainable. An important implication of the model is that the price of a bond is independent of the expected rate of return of the consol bond. BS likened this result to the finding of the Black-Scholes (1973) model for pricing stock options that the option price is independent of the expected return of the stock.

To derive a model that is empirically tractable, BS made some simplifying as-

[7] See Section 3.3 below for an outline of the arbitrage-free approach.

sumptions. First, they assumed that the market price of risk is constant. Second, they specified that the standard deviation of the unanticipated instantaneous changes in interest rate is proportional to the current level of interst rate, both for r_t and ℓ_t.[8] Third, they required the logarithm of r_t to regress to a level that is dependent on both r_t and ℓ_t. The interest rate processes were then linearized and discretized to obtain a simultaneous equation system. To proxy the state variables, BS used yields on 30-day Canadian Banker's Acceptances as r_t and the average yields on Government of Canada bonds with maturities in excess of 10 years as ℓ_t. They obtained some estimates of the two-equation system. The results showed that half of the adjustments in r_t occurs within 10 months. The estimated model was used to predict yields-to-maturity. The relationship between the actual and predicted values, however, was found to be erratic.

Chen and Scott (1992) developed a two-factor model within the framework of general equilibrium pricing constructed by CIR. They assumed that r_t is determined by two state variables y_{1t} and y_{2t} such that

$$r_t = y_{1t} + y_{2t}. \tag{5}$$

These state variables are assumed to follow two independent square-root process. To capture the observed variability of interest rates, Chen and Scott suggested that the parameters of the model should be set such that y_{1t} has a strong mean reversion and y_{2t} has extremely low mean reversion. Although Chen and Scott expected the variation in the short-term interest rate to be captured by y_{1t} and the variation in the long-term interest rate to be captured by y_{2t}, these variables were not given clearcut economic interpretations. As a consequence of the CIR model, the price of pure discount bonds can be calculated as the expected values of future interest rates in a risk-adjusted world. Closed form solution was provided. Solutions were also developed for a variety of interest rate contingent claims such as coupon bonds, coupon bond futures, European options on pure discount bonds, Eurodollar futures and floating rate caps. An advantage of this model is that the solutions for bond options can be evaluated as univariate integrals. However, due to the lack of intepretations for the state variables, it appears that the model can only be assessed on its ability to predict the prices of interest rate contingent claims. No empirical calibration of the model was provided.

An alternative model developed within the general equilibrium framework was proposed by Longstaff and Schwartz (1992). They assumed that returns on investments are determined by two state variables. One of these variables influences the expected return component only and is unrelated to the production uncertainty, while the other variable represents the expected return as well as the variance component. Making use of the assumption of a logarithmic utility function, Longstaff and Schwartz transformed the determining variables to the instantaneous riskfree rate r_t and its variance V_t. In this model both r_t and V_t have a long-run stable distribution. Using the fundamental partial differential equation given by CIR, Longstaff and Schwartz de-

[8] This means that the elasticity parameter is set to 1.

rived closed form solutions for the prices of pure discount bonds and discount bond options. These results imply that the changes in yields are known functions of r_t and V_t. The cross–section constraints imposed on the term structure can be written as:

$$\triangle r_{t,\tau} = b_\tau \triangle r_t + c_\tau \triangle V_t, \qquad (6)$$

where b_τ and c_τ are maturity–specific functions that depend on the parameters of the stochastic processes determining r_t and V_t.

Making use of the constraints imposed in equation (6), the two–factor model can be tested against the general CIR model using the GMM approach. As r_t and V_t are unobservable, proxies have to be used. Longstaff and Schwartz used yield on 1–month U.S. Treasury bills for r_t. For V_t they estimated the values by fitting a generalized autoregressive conditional heteroscedasticity (GARCH) model to r_t. Using securities with eight different maturities they tested the overidentifying restrictions on the data. The results showed that the restrictions cannot be rejected. This conclusion applied to yields with maturity of up to five years. Further tests on the one–factor model, however, rejected the overidentifying restrictions.

Research in multi–factor models have been motivated by the need to avoid perfect correlation between yields of different maturities. Limited empirical findings on multi–factor stochastic processes of interest rates are available. Furthermore, these empirical results are dependent on the use of proxies for the unobservable state variables and are thus subject to measurements errors.

As we have seen, stochastic interest rate models have important implications for the pricing of interest rate contingent claims. The results in pricing these assets in turn provide methods for calibrating the interest rate processes. There are several different approaches to the modelling and determination of the prices of the interest rate contingent claims. In the next section, we shall review the developments of these approaches.

3 TERM STRUCTURE MODELS AND PRICING OF INTEREST RATE CONTINGENT CLAIMS

3.1 Overview

The literature on models of term structure is voluminous. Traditional theories focus on the relationship between the forward rates and the future spot rates. The expectation hypothesis, the liquidity premium hypothesis, the preferred habitat hypothesis and the market segmentation theory belong to this category.[9] These theories offer different explanations to the variations in the term premiums. As pointed out by CIR (1985b), however, these theories are only hypotheses that do not go beyond saying whether the forward rates should or need not equal the expected future spot rates. They are limited in their predictions and causal explanations. Making use of continuous–time analysis, modern theories in term structure are able to provide richer models with specific implications and predictions suitable for empirical testing.

In the continuous–time setting there are three modern approaches to the modelling of the term structure of interest

[9]See McEnally and Jordan (1991) for a review of these hypotheses.

rates. The arbitrage-free approach, pioneered by Vasicek (1977) made use of an argument similar to the Black–Scholes (1973) model for the pricing of option on stocks. This approach assumes that there are one or more state variables that determine the whole term structure. The stochastic models driving these state variables are specified. The prices of interest rate contingent claims (including the pure discount bonds) are then derived by imposing the condition that there are no arbitrage opportunities in the market. The general equilibrium approach proposed by CIR (1985a, 1985b) begins with a framework of the underlying economy. Assumptions about the state variables determining the economy are made. The preferences of a representative investor are also specified. In this approach both the term structure and the prices of interest rate contingent claims are derived endogenously as solutions satisfying the equilibrium conditions. The perfect replication method was developed as a variation to the arbitrage-free approach. In this approach the evolution of future interest rates are constrained to follow a structure such that given the initial yield curve there are no arbitrage opportunities in the market. This approach was first developed by Ho and Lee (1986) in the discrete time setting. It was subsequently extended by Heath, Jarrow and Morton (HJM hereafter) (1992) and Ritchken and Sankarasubramanian (1995), among others. In this section we review these approaches of term structure modelling.

3.2 Arbitrage–Free Method

The arbitrage-free approach was pioneered by Vasicek (1977) for the one-factor model. Vasicek assumed that the instantaneous rate of interest follows a diffusion process and that the price of a pure discount bond depends on the instantaneous interest rate only. The market is assumed to be efficient so that there are no riskfree arbitrage opportunities. Thus, r_t is driven by the following stochastic differential equation:

$$dr = \mu(r,\, t)\, dt + \sigma(r,\, t)\, dZ, \tag{7}$$

where dZ is a standard Wiener process, and $\mu(r,\, t)$ and $\sigma(r,\, t)$ are, respectively the drift rate and the instantaneous standard deviation of r_t.[10] Using Ito's lemma, with the simplifying notation $P = P(t,\, T)$, $\mu_r = \mu(r,\, t)$ and $\sigma_r = \sigma(r,\, t)$, the price of a pure discount bond satisfies the following equation:

$$dP = P\mu_P\, dt + P\sigma_P\, dZ, \tag{8}$$

where

$$\mu_P P = P_t + \mu_r P_r + \frac{1}{2}\sigma_r^2 P_{rr} \tag{9}$$

and

$$\sigma_P P = -\sigma_r P_r. \tag{10}$$

In the above formulae, P_t, P_r and P_{rr} denote partial derivatives. Suppose an investor diversifies between two bonds with maturities τ_1 and τ_2. If we denote the drift rates and the instantaneous standard deviations by $\mu_{Pi}P$ and $\sigma_{Pi}P$, for $i = 1, 2$, respectively, then the absence of arbitrage opportunities implies that

$$\frac{\mu_{P1} - r_t}{\sigma_{P1}} = \frac{\mu_{P2} - r_t}{\sigma_{P2}}. \tag{11}$$

[10] Note that we have used dZ to denote generically a standard Wiener process. The stochastic parts of equations (1) and (7) (also (8) and (13) below) need not be related.

Since this result is true for arbitrary values of τ_1 and τ_2, the common ratio in equation (11) must be independent of τ. Denoting the common ratio by $q = q(r, t)$, the arbitrage–free condition implies:

$$P_t + (\mu_r + \sigma_r q) P_r + \frac{1}{2}\sigma_r^2 P_{rr} - rP = 0. \qquad (12)$$

This is the fundamental equation that determines the price of a pure discount bond.

The quantity q is called the market price of risk, as it gives the excess return over the instantaneous rate of interest per unit risk as measured by the instantaneous standard deviation. To derive specific solutions for the term structure, Vasicek assumed that the price of risk is a constant independent of t and r_t, and that r_t follows an Ornstein–Uhlenbeck process given by the following equation:

$$dr = \kappa(\theta - r)\,dt + dZ. \qquad (13)$$

Vasicek derived the closed form solution for the bond price. Exact solution for a European option on pure discount bond is given by Jamshidian (1989).

Brennan and Schwartz (1979) extended the Vasicek approach to a two–factor model. Bond prices are assumed to depend on r_t and ℓ_t. In this model, the arbitrag–free condition implies that there are two restrictions on the drift rates and standard deviations of the bond prices. That is, the term structure depends on two market prices of risk. However, making use of the fact that ℓ_t is a function of a traded asset, that is, the consol bond, one of the market price parameters can be substituted out.

Although Vasicek's approach implies equation (11), it does not say anything about the functional form of the market price of risk q. Many authors make arbitrary assumptions about the functional form of q, such as the simple assumption that q is a constant. This approach, however, was criticized by CIR (1985b). They argued that not all choices of q will lead to bond prices which do not admit arbitrage opportunities. Thus, closing the model by assuming a specific form for q may lead to internal inconsistency. CIR's general equilibrium model is developed to overcome these difficulties. It assures that the term structure model is completely specified without losing internal consistency.

3.3 General Equilibrium Method

CIR (1985a, 1985b) considered a continuous time competitive economy with a single good. Production opportunities are represented by n linear activities. The vector of expected rates of return is α and the covariance matrix of the rates of return is GG'. Both α and G are functions of k state variables represented by vector Y. The vector of expected changes in Y is μ and its covariance matrix is SS'. The elements of Y are deemed to determine the technological changes and production opportunities open to the economy. CIR assumed that the economy consists of individuals with identical Von Neumann–Morgenstern utility functions. They seek to maximize the expected value of the aggregate discounted utilities and subsequently choose the optimal level of consumptions as well as investments in the production activities and contingent claims. In equilibrium, the interest rate and the expected rates of return in the contingent claims would adjust until all wealth is invested in the production processes. The market clearing condition leads to two fundamental equations that determine the equilibrium instantaneous rate of interest and the value of any contingent claim. CIR further specified the preference structure to follow a logarithmic utility function. Denoting $F = F(Y, t)$ as the equilibrium value of a contingent claim, CIR showed that the following differential equation must be satisfied:

$$\frac{1}{2}\,\mathrm{tr}\,(SS'F_{YY}) + [\mu' - a^{*\prime}GS']\,F_Y + F_t + \delta - rF = 0, \tag{14}$$

where δ is the payout flow received by the contingent claim, a^* is the vector of optimal proportions of wealth to be invested in production activities, and F_{YY}, F_Y and F_t denote partial derivatives.

To derive a specific term structure model CIR considered a single-factor model with $k = 1$ such that Y is a scalar. They assumed Y is driven by the following stochastic differential equation:

$$dY = (\xi Y + \zeta)\,dt + v\sqrt{Y}\,dZ. \tag{15}$$

It is then straightforward to show that r_t follows the continuous-time mean-reversion process defined in equation (1) for some constants κ, θ and σ^2. In particular, r is a constant multiple of Y and the distribution of r_t conditional on r_s, $s < t$, follows a noncentral chi-square. As t approaches infinity we obtain the steady state distribtuion of r_t, which follows a gamma variable. Making use of equation (14), it can be shown that the price of a pure discount bond that matures at T, $P = P(t, T)$, satisfies the following partial differential equation:

$$\frac{1}{2}\,\sigma^2 r P_{rr} + \kappa\,(\theta - r)\,P_r + P_t - \lambda r P_r - rP = 0, \tag{16}$$

where P_{rr}, P_r and P_t denote partial derivatives, and $\lambda = a^{*\prime}GS'$ is the market risk parameter. It is noted that λ is a constant independent of the time to maturity $\tau = T - t$.[11] CIR gave closed form solution for P, which depends on the parameters $\kappa\theta$, $\kappa + \lambda$ and σ^2. Thus, the yield to maturity $r_{t,\tau}$ can be easily calculated from the price of the pure discount bonds.[12]

CIR's approach has been applied to multi-factor models by several authors. For example, Longstaff and Schwartz (1992) and Chen and Scott (1992) developed two-factor models in the CIR framework. As noted above, the fundamental equation (11) depends on the assumption of logarithmic utility. Furthermore, few authors have developed alternative models beyond the square-root process, although the fundamental evaluation equation applies to a general class of diffusion processes.[13] It is not sure how robust the CIR model is with respect to the assumption of the utility function. Allowing the structure of Y to take alternative processes will enhance the generality of the model, although analytic solutions may not be easy to obtain.

3.4 Perfect Replication Method

[11] Note that λr is the covariance of changes in r_t with the percentage changes in the optimally invested wealth. CIR called λ the "market" risk parameter. It should be distinguished from the market price of risk q as defined in equation (11).

[12] CIR also provided close form solution for the European option on pure discount bond.

[13] The model suggested by Longstaff (1989) is perhaps an exception.

Making use of the arbitrage–free condition Ho and Lee (1986) proposed a model for the determination of future interest rate movements and the pricing of contingent claims. Their model is constructed to replicate the current term structure perfectly. Thus, they take the current yield curve as given and derive the feasible subsequent term structure movements. The movements are constrained to be consistent with an efficient market so that there are no arbitrage opportunities. Ho and Lee considered interest rate movements in discrete time. The future term structures are assumed to depend on a set of discount functions that form a binomial lattice. The term structure may evolve from one vertex to another by different paths, but the path will not affect the value of the discount function at the vertex. Thus, the discount function is path independent. Prices of contingent claims can be evaluated by backward substitution, in a way similar to the Cox, Ross and Rubinstein (1979) method for the pricing of stock options.

As a consequence of a one–factor model, the bond prices in the Ho–Lee model are perfectly correlated. HJM (1992) generalized the Ho–Lee model to a continuous economy with multiple factors. They take the state vector as the entire forward rate curve. Movements in the forward rates are assumed to depend on a finite number of standard Wiener processes. Making use of the results in Harrison and Pliska (1981), HJM characterize the forward rate process so that it is consistent with an arbitrage–free economy.

A difficulty with the HJM model is that the interest rate process is generally path dependent. Thus, the future evolution of interest rates depends on the entire history of the interest rates and not just the current yield curve. Simulation of interest rates and the valuation of the price contingent claims may be very computer intensive. Ritchken and Sankarasubramanian (1995) proposed an alternative approach in which the path–dependence is summarized in a single sufficient statistic. Their model is two–factor Markovian. An algorithm is suggested, with the implementation of a control variate to improve the efficiency.

In the Ho–Lee approach the risk–neutral probabilities at each node are established to ensure that all pure discount bonds are priced correctly according to the current term structure. Alternatively, we may use the arbitrage–free condition to establish the possible states of the world, given assumed risk–neutral probabilities. Tilley (1992) outlined an algorithm for this approach. Equal–probability paths of interest rates are generated by random sampling. The interest rates are then adjusted, epoch by epoch, to ensure arbitrage–free conditions. This is achieved by imposing the constraint that the expected present value of a payment at the next epoch is equal to the given initial spot price.

Unlike the arbitrage–free method and the general equilibrium method, the perfect replication approach does not depend explicitly on the market risk parameter. Indeed, the market risk is reflected implicitly in the current term structure. The perfect replication method, however, depends critically on the current term structure. Pricing errors may incur if the current term structure cannot be measured accurately. Furthermore, by assuming the current term structure to be arbitrage-free, the perfect replication method does not enable one to identify arbitrage opportunities. qwp

4 INTEREST RATE MODELS IN ACTUARIAL SCIENCE

Apart from being an important determinant of the value of many assets, interest rate may also be an important factor in determining the value of liabilities. The net value of an insurance company may be very sensitive to the movements of interest rates. Actuaries call the risks associated with interest rate fluctuations the C–3 risks. An insurance company may be selected against when interest rate moves, causing a block of business which was originally profitable to become unprofitable. Take the case of a Single Premium Deferred Annuity (SPDA) product for example. When interest rate increases the credited rate is less than the new money rate. Disintermediation may occur due to the surrender of policies. The lapses may be kept down by crediting old policies with new money rates. This strategy, however, may incur losses for existing business. On the other hand, when interest rate drops the duration of the SPDA lengthens due to fewer lapses. In this case, the policy holders bear the reinvestment risks.

To properly assess the C–3 risk actuaries use a technique called cash flow analysis.[14] In cash flow analysis liability and asset cash flows are projected into the future under various interest rate scenarios. Depending on the product being analyzed, other assumptions such as lapse rate, prepayment rate, expenses, reinvestment rate, market rate and credited rate strategies have to be specified. Interest rate assumption, however, is perhaps the most important factor for a cash flow testing.

A scenario set is defined as a sequence of interest rate scenarios. The scenario set may be constructed in several ways. One approach is to arbitrarily select a set of scenarios that seem to cover the major possibilities.[15] There are some drawbacks to this approach. First, the number of scenarios is unlimited and no single scenario is likely to occur in practice. Second, probability statement about the outcome of the test is impossible.

Apart from arbitrary scenarios, two other methods are widely used in practice: the transitional probability method and the lognormal method. In the transitional probability method a universe of standard yield curves are defined. These yield curves may be constructed to represent historical observations. A matrix of transitional probabilities is then created. The matrix consists of probabilities p_{ij} of a yield curve C_i being followed

[14]Cash flow analysis is a well-established technique used by actuaries. The U.S. Actuarial Standard Board gave some guidelines as to how cash flow analysis should be conducted. The Actuarial Standard of Practice No. 7 prescribes appropriate procedures for performing cash flow testing. It instructs actuaries performing such testing to describe their procedures and document their assumptions. The Actuarial Standard of Practice No. 14 provides guidelines to actuaries in determining whether or not to perform cash flow testing as part of forming a professional opinion. The Standard provides guidance to actuaries in determining the type and depth of such testing.

[15]The New York Regulation 126 arbitrarily defines a set of seven scenarios. See Jetton (1990) for further discussions.

by C_j. A yield curve C_0 that resembles most closely to the existing rates is determined. Subsequent interest rate scenarios are then generated using Monte Carlo methods. Depending on a set of "reasonable conditions" Christiansen (1992) proposed several interest rate generators that vary by the intended purposes. In a subsequent paper Christiansen (1994) refined the development of the generator, called the Markov Chain Generator (MCG), to include 11 shapes of yield curves. A matrix of transitional probabilities was determined. An algorithm for determining the initial shape code of the yield curve was also suggested.

The lognormal method assumes that successive ratios of interest rates are jointly lognormally distributed. Thus, if we denote R^S as the short-term (3-month, say) interest rate and R^L as the long-term (10-year, say) interest rate, then $\log(R^S_{t+1}/R^S_t)$ and $\log(R^L_{t+1}/R^L_t)$ follow jointly a bivariate normal distribution.[16] Random numbers are generated to obtain a sequence of interest rates (R^S_t, R^L_t).[17] Intermediate interest rates can be obtained by interpolation.

As the reliability of cash flow testing depends critically on the underlying interest rate assumptions, the importance of generating realistic interest rate scenarios cannot be overemphasized. Becker (1991) examined the applicability of the lognormal assumption of interest rates. He studied the U.S. Treasury securities with six different maturities. Under the lognormal assumption, the following separate hypotheses are implied: (i) x_t are serially independent, (ii) x_t are normally distributed, and (iii) x_t have mean zero and constant variance. Becker examined these hypotheses separately. He concluded that except for the hypothesis that x_t have zero mean, all other hypotheses were rejected.

Becker's findings raise the question whether the lognormal assumption should be used in cash flow testing. Although Becker's results convincingly reject the hypothesis that interest rates follow a lognormal distribution, it does not follow automatically that cash flow analysis using the lognormal assumption would lead to unreliable results. Two questions need to be answered: (i) Are there other models that describe the distribution of interest rates better than the lognormal model? and (ii) How robust are the results of cash flow testing to the assumptions of interest rates? Klein's (1993) work attempted to answer these questions.

As an alternative to the lognormal model Klein (1993) investigated the applicability of the stable Paretian model to interest rate data. Mandelbrot (1963) proposed the stable Paretian distribution as a distribution for describing data of security prices. Fama and Roll (1968, 1971) provided the foundation for the statistical analysis of this distribution. The full stable Paretian distribution that allows for asymmetry is characterized by four parameters. Following the definition proposed by Zolotarev (1957) and adopted by

[16] We denote R^S and R^L generically as R. We also define x_t as $\log(R_{t+1}/R_t)$.

[17] Two advantages of the lognormal assumption are that it cannot result in negative interest rates and that it is multiplicative over time, i.e., over n time periods, we have $R_{t+n} = R_t \exp(\sigma \sum_{i=1}^{n} Z_i)$, where Z_i are independently and identically distributed standard normal variables and σ is the standard deviation of the logarithm of successive ratios of R_t.

McCulloch (1986), the logarithmic characteristic function of the stable Paretian distribution is given by:

$$\Psi(u) = \log \mathrm{E}(e^{iuX}) \tag{17}$$

$$= \begin{cases} iu\delta - |cu|^{\alpha}\,[1 - i\beta\,\mathrm{sign}(u) \\ \qquad \times \tan(\frac{\pi\alpha}{2})] \quad \alpha \neq 1 \\ \\ iu\delta - |cu|\,[1 + i\beta\,\frac{2}{\pi}\,\mathrm{sign}(u) \\ \qquad \times \log|u|] \quad \alpha = 1, \end{cases}$$

where X is a stable Paretian variable, u is the parameter of the characteristic function, $i^2 = -1$ and α, β, δ and c are, respectively, the characteristic exponent, the skewness parameter, the location parameter and the scale parameter. The normal distribution is a special member of the family with $\alpha = 2$, and is the only stable Paretian distribution for which the variance exists. When $\alpha < 2$, absolute moments of order less than α exist, while those of order greater than or equal to α do not. With the exception of the normal distribution the stable Paretian model does not admit closed form solution for its density function in general. Thus, estimation using the MLE method is intractable. Fama and Roll (1968, 1971) suggested a fractile method based on the ordered statistics. This method was improved by McCulloch, who took account of asymmetry, corrected the bias in the Fama–Roll results and considered a broader range of α.

Klein applied the stable Paretian model to fit 30–year U.S. Treasury yield data. He examined the hypothesis that the logarithm of successive yield ratios are independently and identically distributed as stable Paretian variables and argued that the evidence supports this hypothesis. However, using the Fama–Roll results Klein's analysis was restricted to symmetric stable Paretian distributions. This limitation was relaxed by Cardinal in his discussion on Klein's paper. Cardinal used McCulloch's tables to estimate the parameter β. Using Monte Carlo methods, Cardinal found that the hypothesis β is zero cannot be rejected. An important property of the stable Paretian distribution is that it is invariant under addition. That is, a sum of independently and identically distributed stable Paretian variables with characteristic exponent α is again stable Paretian with the same exponent. Cardinal's analysis showed that there is no evidence of instability in the estimates of α with respect to varying sum sizes. This finding, however, is limited due to the effects of a small data set.

Klein examined the sensitivity of cash flow testing to the lognormal and stable Paretian distributions. He compared the surplus values of a SPDA product under the two interest rate assumptions. In his study he incorporated detailed actuarial assumptions such as lapse rates, credited rates, taxes and expenses. His results show that the final surplus is very sensitive to the assumptions of the interest rate processes. In particular, the probability of a negative surplus for the stable Paretian model exceeds that of the lognormal model substantially. However, due to the lack of actual surplus data for comparison, Klein's study cannot indicate which model is preferable in practice.

Recently, Tse (1995c) examined the following models of short–term interest rates: the lognormal model, the stable Paretian model and the (discretized) mean–reversion model as given in equation (1). To isolate the effects of the choice of interest rate models to cash flow testing he examined the generated interest rate scenarios directly. The 95 percent intervals of the samples of simulated paths, as well as the distributions of the accumulated values and the average yields over several durations were considered. The results showed that the lognormal model is likely to generate unrealistic scenarios even for horizon of five years. While the stable Paretian model gives results similar to the mean–reversion model for five-year horizon in terms of the 95 percent interval, it may generate very extreme values in the upper tail of the distribution. The generated scenarios for the mean reversion model is sensitive to the estimate of the long–run interest rate level. Overall he favoured the use of the mean–reversion model as the effect of mean–reversion prevents the interest rates from reaching unreasonably high levels.

In the insurance literature, the theory of life contingencies was traditionally developed in a deterministic way. Thus, mortality is assumed to follow a known mortality table or function, and interest rate is assumed to be constant. In the early 1980s interest rate became very volatile and actuaries began to realize that the assumption of constant interest rate is unrealistic. Boyle (1976) explored the stochastic structure of the force of interest as normally distributed random variables. Evaluation of the moments of the present value functions became a topic of major importance. Dhaene (1989) developed a practical method for computing the moments of insurance functions when interest rates are assumed to follow an autoregressive integrated moving average process. The results can be applied to many insurance products. Papachristou and Waters (1991) considered a simple lognormal model and focused on evaluating the present value of the profit of a long–term sickness policy. For annuity business, Beekman and Fuelling (1991) suggested a method for determining contingent the reserves when interest rates are stochastic. Further results can be found in Vanneste, Goovaetrs and Labie (1994). As actuaries have recognized the importance of incorporating stochastic interest rates in the valuation and pricing of insurance products, further research on this topic in the insurance literature is expected.[18]

4 Conclusions

We have reviewed some recent developments in stochastic interest rate models in economics, finance and actuarial science. While there has been much progress in the theories of the determination of the term structure and the pricing of interest–rate contingent claims, there are still many unanswered questions with respect to the empirical models that best describe the historical interest rate data. Empirical evidence for multi–factor models are especially lacking. In the actuarial profession, the search for a stochastic model for use in cash flow testing is expected to continue for some time. Unlike the problem of model selection for pricing financial as-

[18]The recent survey by Vetzal (1994) has demonstrated the research interest shown in the insurance literature. The fact that the second edition of the well–known text book by Kellison (1991) has a new chapter on "stochastic approaches to interest" is also a testimony to the growing importance of the topic.

sets, the selection of an interest rate model for the cash flow tesing of a block of insurance products is made difficult because of the lack of an appropriate benchmark for assessment. Nonetheless, much progress has been made in understanding the limitations of the current methods that are being practised.

References

[1] Becker, D.N. (1991), "Statistical Tests of the Lognormal Distribution as a Basis for Interest Rate Changes", *Transactions of the Society of Actuaries*, **43**, 7 - 72.

[2] Beekman, J.A. and C.P. Fuelling (1991), "Extra Randomness in Certain Annuity Models", *Insurance: Mathematics and Economics*, **10**, 275 - 287.

[3] Black, F. and M.J. Scholes (1973), "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, **81**, 637 - 659.

[4] Boyle, P.P. (1976), "Rates of Return as Random Variables", *Journal of Risk and Insurance*, **43**, 693 - 713.

[5] Brennan, M.J. and E.S. Schwartz (1979), "A Continuous Time Approach to the Pricing of Bonds", *Journal of Banking and Finance*, **3**, 133 - 155.

[6] Brown, R.H. and S.M. Schaefer (1994), "The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model", *Journal of Financial Economics*, **35**, 3 - 42.

[7] Brown, S.J. and P.H. Dybvig (1986), "The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates", *Journal of Finance*, **41**, 617 - 632.

[8] Broze, L., O. Scaillet and J.M. Zakoian (1993), "Testing for Continuous–Time Models of the Short–Term Interest Rate", *CORE Discussion Paper* 9331.

[9] Broze, L., O. Scaillet and J.M. Zakoian (1995), "Quasi Indirect Inference for Diffusion Processes", *CORE Discussion Paper* 9505.

[10] Chan, K.C., G.A. Karolyi, F.A. Longstaff and A.B. Sanders (1992a), "An Empirical Comparison of Alternative Models of the Short–Term Interest Rate", *Journal of Finance*, **47**, 1209 - 1227.

[11] Chan, K.C., G.A. Karolyi, F.A. Longstaff and A.B. Sanders (1992b), "The Volatility of Japanese Interest Rates: A Comparison of Alternative Term Structure Models", *Pacific–Basin Capital Markets Research*, Vol III, 119 - 136.

[12] Chen, R.R. and L. Scott (1992), "Pricing Interest Rate Options in a Two–Factor Cox–Ingersoll–Ross Model of Term Structure", *The Review of Financial Studies*, **5**, 613 - 636.

[13] Christiansen, S.L. (1992), "A practical Guide to Interest Rate Generators for C–3 Risk Analysis", *Transactions of the Society of Actuaries*, **44**, 101 - 134.

[14] Christiansen, S.L. (1994), "The Markov Chain Interest Rate Scenario Generator Revisited", *Journal of Actuarial Practice*, **2**, 101 – 124.

[15] Constantinides, G.M. (1992), "A Theory of the Nominal Term Structure of Interest Rates", *The Review of Financial Studies*, **5**, 531 – 552.

[16] Cox, J.C., J.E. Ingersoll and S.A. Ross (1985a), "An Intertemporal General Equilibrium Model of Asset Prices", *Econometrica*, **53**, 363 – 384.

[17] Cox, J.C., J.E. Ingersoll and S.A. Ross (1985b), "A Theory of the Term Structure of Interest Rates", *Econometrica*, **53**, 385 – 407.

[18] Cox, J.C., S.A. Ross and M. Rubinstein (1979), "Option Pricing: A Simplified Approach", *Journal of Financial Economics*, **7**, 229 – 263.

[19] Dhaene, J. (1989), "Stochastic Interest Rates and Autoregressive Integrated Moving Average Processes", *ASTIN Bulletin*, **19**, 131 – 138.

[20] Dothan, U.L. (1978), "On the Term Structure of Interest Rates", *Journal of Financial Economics*, **6**, 59 – 69.

[21] Fama, E.F. and R. Roll (1968), "Some Properties of Symmetric Stable Distributions", *Journal of the American Statistical Association*, **63**, 817 – 836.

[22] Fama, E.F. and R. Roll (1971), "Parameter Estimates for Symmetric Stable Distributions", *Journal of the American Statistical Association*, **66**, 331 – 338.

[23] Gibbons, M.R. and K. Ramaswamy (1993), "A Test of the Cox, Ingersoll, and Ross Model of the Term Structure", *Review of Financial Studies*, **6**, 619 – 658.

[24] Hansen, L.P. (1982), "Large Sample Properties of Generalized Method of Moments Estimators", *Econometrica*, **50**, 1029 – 1054.

[25] Harrison, J.M. and S. Pliska (1981), "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and Their Applications*, **11**, 215 – 260.

[26] Heath, D., R. Jarrow and A. Morton (1992), "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", *Econometrica*, **60**, 77 – 105.

[27] Ho, T.S. and S. Lee (1986), "Term Structure Movements and Pricing Interest Rate Contingent Claims", *Journal of Finance*, **41**, 1011 – 1028.

[28] Jamshidian, F. (1989), "An Exact Bond Option Formula", *Journal of Finance*, **44**, 205 – 209.

[29] Jetton, M.F. (1988), "Interest Rate Scenarios", *Transactions of the Society of Actuaries*, **40**, 423 – 476.

[30] Kellison, S.G. (1991), *The Theory of Interest*, second edition, Homewood, IL: Irwin.

[31] Klein, G.E. (1993), "The Sensitivity of Cash-Flow Analysis to the Choice of Statistical Model for Interest Rate Changes", *Transactions of the Society of Actuaries*, **45**, 79 – 124.

[32] Lo, A.W. (1988), "Maximum Likelihood Estimation of Generalized Ito Processes with Discretely Sampled Data", *Econometric Theory*, **4**, 231 – 247.

[33] Longstaff, F.A. (1989), "A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates", *Journal of Financial Economics*, **23**, 195 – 224.

[34] Longstaff, F.A. and E.S. Schwartz (1992), "Interest Rate Volatility and the Term Structure: A Two–Factor General Equilibrium Model", *Journal of Finance*, **47**, 1259 – 1282.

[35] Mandelbrot, B.B. (1963), "The Variation of Certain Speculative Prices", *Journal of Business*, **36**, 394 – 419.

[36] McCulloch, J.H. (1986), "Simple Consistent Estimators of Stable Distribution Parameters", *Communications in Statistics: Simulation and Computation*, **15**, 1109 – 1136.

[37] McEnally, R.W. and J.V. Jordan (1991), "The Term Structure of Interest Rates", chapter 56 in F.J. Fabozzi, T.D. Fabozzi and I.M. Pollack, eds., *The Handbook of Fixed Income Securities*, third edition, Homewood, IL: Business One Irwin.

[38] Merton, R.C. (1973), "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, **4**, 141 – 183.

[39] Ogden, J.P. (1987), "An Analysis of Yield Curve Notes", *Journal of Finance*, **42**, 99 – 110.

[40] Oldfield, G.S. and R.J. Rogalski (1987), "The Stochastic Properties of Term Structure Movements", *Journal of Monetary Economics*, **19**, 229 – 254.

[41] Papachristou, D.J. and H.R. Waters (1991), "Some Remarks Concerning Stochastic Interest Rates in Relation to Long Term Insurance Policies", *Scandinavian Actuarial Journal*, **2**, 103 – 117.

[42] Pearson, N. and T.S. Sun (1994), "Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model", *Journal of Finance*, **49**, 1279 – 1302.

[43] Richard, S.F. (1978), "An Arbitrage Model of the Term Structure of Interest Rates", *Journal of Financial Economics*, **6**, 33 – 57.

[44] Ritchken, P. and L. Sankarasubramanian (1995), "Volatility Structures of Forward Rates and the Dynamics of the Term Structure", *Mathematical Finance*, **5**, 55 – 72.

[45] Stambaugh, R.F. (1988), "The Information in Forward Rates: Implications for Models of the Term Structure", *Journal of Financial Economics*, **21**, 41 – 70.

[46] Tilley, J.A. (1992), "An Actuarial Layman's Guide to Building Stochastic Interest Rate Generators", *Transactions of the Society of Actuaries*, **44**, 509 – 538.

[47] Tse, Y.K. (1992), "MLE of Some Continuous Time Financial Models:

Some Monte Carlo Results", *Mathematics and Computers in Simulation*, **33**, 575 – 580.

[48] Tse, Y.K. (1995a), "Maximum Likelihood Estimation of Cox–Ingersoll–Ross and Brennan–Schwartz Models of Interest Rate Movements", *Hong Kong Economic Papers*, forthcoming.

[49] Tse, Y.K. (1995b), "Some International Evidence on the Stochastic Behaviour of Interest Rates", *Journal of International Money and Finance*, forthcoming.

[50] Tse, Y.K. (1995c), "Short–Term Interest Rate Models and Generation of Interest Rate Scenarios", *mimeo.*

[51] Vanneste, M., M.J. Goovaerts and E. Labie (1994), "The Distributions of Annuities", *Insurance: Mathematics and Economics*, **15**, 37 – 48.

[52] Vasicek, O. (1977), "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, **5**, 177 – 188.

[53] Vetzal, K.R. (1994), "A Survey of Stochastic Continuous Time Models of the Term Structure of Interest Rates", *Insurance: Mathematics and Economics*, **14**, 139 – 161.

[54] Zolotarev, V.M. (1957), "Mellin–Stieltjes Transforms in Probability Theory", *Theory of Probability and Its Applications*, **2**, 433 – 460.