Estimating the Parameters of Stochastic Differential Equations by Monte Carlo Methods

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Abstract We propose a method for the simultaneous estimation of the drift and diffusion coefficients of stochastic differential equations (SDE) from panel data. The method involves matching the distribution of the experimental/field data with a panel of simulated data generated by a Monte Carlo experiment. The fit between the two distributions is assessed by means of the chi-square goodness-of-fit statistic leading to a confidence function computed from an incomplete gamma function. A numerical optimisation algorithm then optimises the choice of parameters to maximise this function. Preliminary evidence is presented which suggests that it is possible to estimate the coefficients of the generating SDE very accurately.

Keywords Stochastic differential equations, Wiener process, Ito's stochastic integral, Chi-squared Goodness-of-Fit Statistic, Gamma Function

1. INTRODUCTION

Many natural phenomena can be modelled by a system of ordinary differential equations which are perturbed by random disturbances. Indeed, stochastic differential equations (SDE) now find applications in many disciplines including *inter alia* engineering, economics and finance, environmetrics, physics, population dynamics and medicine. The problem of estimating the parameters of the deterministic part of the equation, or drift parameters, has been studied extensively for both continuously observed data [1], [2] and when the process is observed at discrete instants only [3]. This research focuses on providing a numerical method, based on the maximum likelihood principle, for the simultaneous estimation of both the drift and diffusion parameters of SDE from discrete panel data. This is potentially important in practice since in many applications, such as modelling the behaviour of stock prices, the deterministic component in the dynamics is only of secondary importance [2].

2. STOCHASTIC DIFFERENTIAL EQUATIONS

Central to the formulation of the behaviour of systems by SDE is the Wiener process. A standard Wiener process, $W(t)$, is a continuous Gaussian process with independent increments such that

$$W(0)=0 \quad E[W(t)]=0 \quad \mathrm{Var}[W(t)-W(s)]=t-s \quad 0 \le s \le t. \tag{1}$$

Based on this definition, we shall be concerned with SDE of type

$$dX = a(t, X(t))dt + b(t, X(t))dW \tag{2}$$

where dW is the increment of the Wiener process and $X(t)$ is the solution to be determined. In integral notation, this solution satisfies

$$X(t) = X(s) + \int_s^t a(r, X(r))dr + \int_s^t b(r, X(r))dW(r). \tag{3}$$

bins as opposed to 2 reduces (quite dramatically in some cases) the standard deviations of the coefficients. Whilst the effect is less pronounced when a larger number of trials is used, it remains true that both the drift and diffusion coefficients are more accurately resolved by increasing the number of bins.

(b) For a given sample size, increasing the number of trials, m, improves both the parameter estimates and their standard errors. However, most of this improvement occurs when m is increased from 20 to 50.

(c) Increasing the uniform sampling frequency when the data refers to the same time interval produces little or no improvement in the estimation results. It is clear that 500 points are barely better than 50 points over the same interval but require 10 times as much computing effort. This, we believe is attributable to confidence levels calculated from data in similar time regimens not being independent.

Table 1 Parameter estimates in simulation exercise with 1000 replicates

Number of data points	Number of simulations	10 data points in experimental bins	5 data points in experimental bins
50	20	1.078±0.179 0.546±0.225	1.053±0.071 0.481±0.081
	50	1.011±0.060 0.536±0.055	0.992±0.057 0.515±0.051
	100	0.997±0.042 0.514±0.035	0.984±0.043 0.504±0.035
200	20	1.176±0.240 0.680±0.296	1.059±0.073 0.490±0.078
	50	1.019±0.057 0.539±0.052	1.004±0.056 0.521±0.047
	100	1.005±0.039 0.517±0.033	0.997±0.040 0.512±0.033
500	20	1.288±0.300 0.814±0.330	1.062±0.072 0.493±0.078
	50	1.021±0.055 0.540±0.051	1.010±0.054 0.525±0.046
	100	1.008±0.036 0.519±0.031	1.003±0.039 0.516±0.032

5. CONCLUSION

We believe we have presented a viable method of estimating the parameters of SDE's from discrete panel data. The method should be particularly useful in the first instance in medical and environmental applications where trials of the same process will yield the required panel data. Economic and financial applications will be limited in the first instance to situations where such data is available, particularly in microeconometrics. Clearly much work remains to be done. The estimation method also appears to work well for sample paths of SDE's obtained by stochastic numerical integration, but adapting these numerical techniques to yield parameter estimates for a single data series believed to be generated by SDE is a priority.

REFERENCES

Basawa, I.V., and Prakasa Rao, B.L.S., Statistical Inference for Stochastic Processes, *Academic Press*, London, 1980.

Black, F., and Scholes, M., The pricing of options and corporate liabilities, *Journal of Political Economy, 81*, 637–659, 1973.

Dacunha-Castelle, D., and Florens-Zmirou, D., Estimators of the coefficients of a difusion from discrete observations, *Stochastics, 19*, 263–284, 1986.

Doob, J.L., Stochastic processes, John Wiley, New York, 1953.

Harvey, A., Forecasting, Structural Time Series Models and the Kalman Filter CUP, Cambridge, 1989.

Ito, K., Stochastic differential equations, *Memoirs, Amer. Math. Soc., 4*, 1951.

Ito, K., On a formula concerning stochastic differentials, *Nagoya Mathematical Journal, 3*, 55–65, 1951.

Jazwinski, A.H., Stochastic Processes and Filtering Theory, *Academic Press,* San Diego, 1970.

Kalman, R.E., A new approach to linear filtering and prediction, *ASME Transactions Part D (Journal of Basic Engineering), 83*, 95–108, 1961.

Kloeden, P.E., Platen, E., and Schurz, H., Numerical Solution of SDE Through Computer Experiment, *Springer-Verlag*, Berlin, 1994.

Kutoyants, Y.A., Parameter Estimation for Stochastic Processes, *Heldermann Verlag,* Berlin, 1984.

Stratonovich, R.L., A new representation for stochastic integrals and equations, *SIAM Journal on Control, 4*, 362–371, 1966.