The Distributions of Change Points in Long Memory Processes

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EXTENDED ABSTRACT

Long memory processes are an important aspect of current time series modelling. They are needed in cases where the series lies between non-stationary ARIMA processes and nearly non-stationary ARMA processes. In other words, integration is allowed to be fractional. Long memory processes show that a time series value at any time has significant dependency on the entire past.

On the other hand, processes with structural breaks, or more precisely stochastic regime switching, implies changes in the object identity. Structural changes could occur when some parts of series have changing behaviours.

It is known that series with structural breaks could be identified as a spurious long memory process time series. The reverse, that a pure long memory process could appear to have structural changes, is also possible. It is therefore important to be able to differentiate between a true long memory process and a series with structural breaks. . However, the distinction between them is difficult as they share many common analytical features. Our work is in developing tests to enable the distinction between pure long memory processes and time series with structural breaks to be made.

In term of the theory of time series, a long memory process comes from a stable data generating mechanism over the whole length of the series, whereas series with structural changes does not.

Several techniques have been developed for detecting structural breaks and have shown various strengths and weaknesses. We have used Atheoretical Regression Tree (ART) in our study. ART is a non-parametric approach which recursively partitions a series into mutually exclusive and exhaustive sub-series by binary tree splits. The advantages of ART are that it is computationally fast, can be visualised as a tree diagram, and its performance is not dependent on the length of the series.

Our results from simulations of a typical long memory process shows that the number of breaks identified by ART appears to follow a Poisson process. This could guide the analyst in whether they are dealing with a pure long memory process, or a series with structural breaks. It our results are sound it would be possible to at least identify a structural break model when the number of breaks is more than that would be expected for a pure long memory process.

We have applied ART to two classic time series identified as long memory series. One series showed that, if the number of breaks is Poisson distributed, number of breaks detected by ART was close to the expected number of breaks. It suggested the series was likely to a pure long memory series. However, the other series had more breaks than what was expected. Then, it is possible this is a series with structural breaks rather than a pure long memory process.

Our work provides encouraging results on how ART can distinguish a pure long memory process from a process with structural breaks by using the expected distribution of break points. Further theoretical investigation to confirm the distribution is Poisson, is required.

1. Introduction

The long memory processes in time series, also called long range dependency, strong dependence or Hurst phenomenon, was popularized by Hurst (1951), and Mandelbrot and Van Ness(1968). It has been applied in many areas, such as finance (Granger and Hyung, 2005), network communications (Leland et al, 1993), hydrology (Klemes, 1974), geophysics (Maraun et al, 2004) and climatology (Mills, 2006).

In general, long memory processes exhibit the behavior of a data process after a given time t but not only depend on the circumstance at time *t*, but also on the entire history of the process up to *t*. This process can be described in the time domain as

(1) $\lim_{n\to\infty}\sum_{j=0}^{n}|\rho(j)| \to \infty$

where ρ is the autocorrelation function, j is the lag. The sum of the autocorrelation is not finite. In the frequency domain:

(2) $f(\omega \to 0) \to \infty; f(\omega) = C_f |\omega|^{-\gamma}$

where $f(\omega)$ is the spectral density, ω is the power, C_f is a constant and $\gamma \in (0,1)$. The spectral density tends to be infinite at low frequency or the origin.

Recently, many authors (Ohanissian et al, 2005; Diebold and Inoue, 2001; Granger and Hyung, 2004) have shown that differentiation between true long memory processes and spurious long memory processes is hard to define. The reason is both processes share similar properties.

Spurious long memory processes are series with multiple change/breaks points. That is a non-stationary process which possibly combines several short memory processes, such as ARMA, with probabilistic shifts or breaks. Therefore, we are aware of spurious long memory that does not come from a constant data generating process.

On the other side, true long memory processes always derive from the same data generating mechanism. Then, they will have identical processes at any given size of sampling (Mandelbrot, 1997).

Even though we are clear as to both data generating mechanisms, classification rules are rather unclear. Detection techniques of change/breaks points are the key. In this survey Atheoretical Regression Tree (ART) (Cappelli and Reale, 2005) was only considered as it is computationally inexpensive and easy to manipulate. Then, distributional properties of break points in true long memory processes could allow us to differentiate between a true long memory process and a process with break points.

This paper is organized as follows: The next section discusses the simulation models used to create a true long memory process and the spurious long memory process that can arise with structural breaks; Section 3 covers the process of detecting structural breaks in the simulated series. In the following section we present the empirical results of statistical properties in break points; Two case studies are explained in section 5; Section 6 provides conclusion

2 True and Spurious long memory models

2.1 True long memory models

The most common model for a true long memory processes is an Auto-Regressive Fractionally Integrated Moving Average (ARFIMA),

$$\phi(B)\Delta^d \chi_t = \theta(B)\varepsilon_t ,$$

As in ARIMA models where (ε_t) is White Noise with zero mean, and $\Delta^d = (1-B)^d$, where $d \in (0,0.5)$, is a difference operator, B is a backward operator, $\phi(B)$ are AR parameters, and $\theta(B)$ are MA parameters.

The simplest ARFIMA model, ARFIMA (0,d,0) is termed the Fractional Gaussian Noise (FGN) ,

$$\Delta^d \chi_t = \varepsilon_t$$

In discussions of long memory processes, the Hurst parameter, H, is used as a measure of self-similarity (Hurst, 1951). The higher the value of the Hurst parameter, the more fluctuations the series has. This model we use for our simulation.

That is, the FGN model is

$$\Delta^{(H-\frac{1}{2})} \chi_t = \varepsilon_t$$

Some examples of FGN are given in Figure 1.

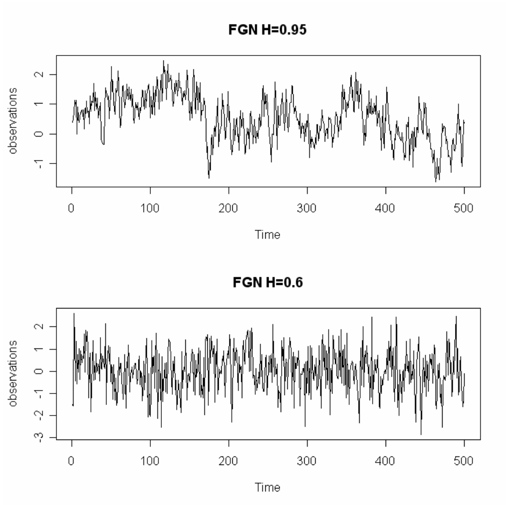

Figure 1. Plots of simulated FGN series.

2.2 Spurious long memory models

It has been shown Markov/Regime switching can appear to be long memory models (Ohanissian et al, 2005; Granger and Terasvirta, 1999).

This model contains sub-time series, stationary ARMA models processes, with probabilistic changes on state levels. Its formulation was presented by Chen and Tiao(1990) is as follows.

(1) $y_t = \mu_t + x_t$

(2) $\mu_t = p_t \eta_t + \mu_{t-1}$

where $(x_t, t \geq 0)$ is local-stationary ARMA model, $(\mu_t, t > 0) \in N(0, c\sigma^2)$, and p_t is binary variable with Prob$(p_t = 1) = \alpha$ and Prob($(p_t = 0) = 1 - \alpha$.

The mean levels change over time according to the probability of p_t. During the changes or breaks, the series are stationary and have different statistical properties. An example of a series from this model is given in Figure 2.

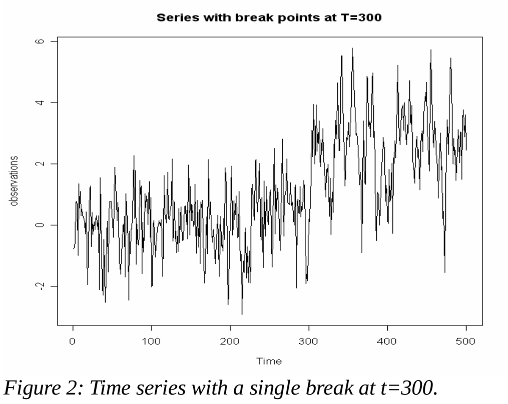

Figure 2: Time series with a single break at t=300.

3 Overview of detection procedures

When detecting change points there are several issues to consider:

1. Expected number of breaks known;
2. Length of the series;
3. The sensitivity of the break detection (i.e. false positives and false negatives);
4. Computational efficiency of the detection process.

This section concentrates on how several detection techniques predict when breaks occur given break dates are unknown and why ART is used.

A common way to detect break points was developed by Bai and Perron (1998, 2003). This utilizes dynamic programming and Fisher's exact optimization. Each serial point is allocated into mutually exclusive and exhaustive sub-groups which maximize the sum of squared errors between groups or minimize the sum of squared errors within groups. The advantage of this procedure is that it is able to find the global minimum. However, it is computational intensive when series are long (e.g > 1,000 points).

In our simulations we will use a different procedure, ART. This is a non-parametric approach, and produces hierarchical tree structures with break dates (Cappelli and Reale, 2003). It makes use of a binary division algorithm with concept of Fishers' contiguous partitions (1958) and Least Square Regression Tree (LSRT) splitting criterion (Breiman et al, 1984). Under this, groups are separated as far as possible.

ART has three advantages over Bai and Perron's procedure:

1. It is computationally considerably faster;
2. One can visualize the results as a tree diagram;
3. It can detect breaks in any length series

The drawbacks are that optimal breaks may not be found though it is expected to be close to optimal; and the number of break points are overestimated in short series.

Overall, ART has many nice features and comparable results to Bai and Perron's procedure. Detailed comparison is studied by Rea et al (2007).

4 Simulations and Empirical results

4.1 Simulations

For the generation of true long memory series and tree construction, fSeries and tree, add-in package for R, were used.

The simulation process was:
(1) A Hurst parameter was selected (H=0.95, 0.9, 0.85, 0.8, 0.75, 0.7, 0.65, 0.6 and 0.55);
(2) 1,000 series of length 5405 were generated for each value of H (Reasons to 5405 are simply because it is reasonably long and also same length as one of our examples);
(3) Each of the 1,000 series for each value of H were broken into sub-groups or regimes, and the results of number of breaks for each of these 1,000 series were examined.

4.2 Results

In this section, empirical distributions of statistics were examined by different theoretical density functions. The best fitted ones were employed to determine statistical properties on sub-series or regimes. The statistics we were interested in:

1. Number of breaks.
2. Regime length.
3. Mean level of series in breaks.
4. Standard deviation of series in breaks.

4.2.1 Number of breaks

The number of breaks is an integer, thus their distribution would be expected to be a discrete distribution. The frequency distributions are plotted in Figure 3. The fitted Poisson has been plotted, as well as the Normal distribution for comparison.

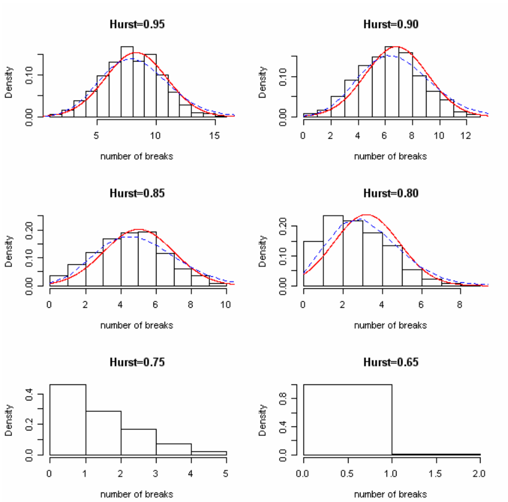

Figure 3: Empirical distribution of the number of breaks with Hurst parameters. Normal (solid, red), Poisson (dash, blue)

H value	Expected number of breaks	Chi-square test. P-value (df)
0.95	8.29	0.185 (14)
0.9	6.82	0.155 (12)
0.85	5.01	0.110 (9)
0.8	3.19	0.078 (7)
0.75	1.76	0.001 (4)
0.7	0.75	0.000 (3)
0.65	0.15	0.000 (1)
0.6	0.0015	0.000 (0)
0.55	0	0.000 (0)

Table 1: Expected number of breaks and Chi-square test statistics on Poisson distribution in FGNs.

Empirically it appears that the number of breaks for a long memory process possibly comes from a Poisson distribution for H ≥ 0.8. As H tends to 0.5, the number of breaks tends to zero (Table 1). That implies that the FGN processes became "stationary" as no breaks were detected. We also have compared the expected number of breaks if they were Poisson distributed with what we obtained from our simulation with a Chi-square test. There is no evidence that the distribution is not Poisson..

4.2.2 Regime length

The distribution of original regime length is highly right-skewed and a natural logarithm transformation was used (figure 4).

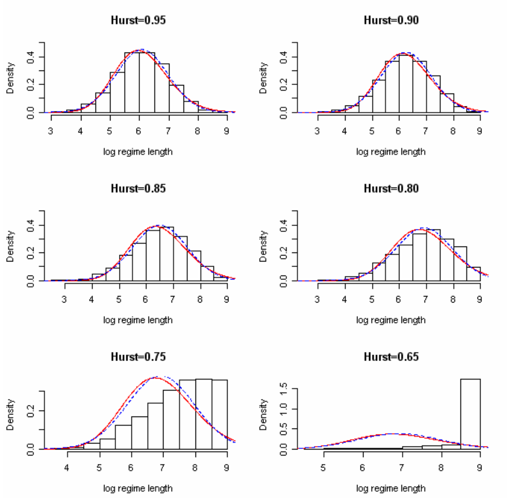

Figure 4: Empirical distributions of log regime length. Gamma (solid,red) and Normal (dash,blue)

After log-transformation, the distribution is less skewed, and it is quite symmetric at $0.9 \le H < 1$. Gamma and Normal distribution were used. The Gamma distribution looks like the better one to capture the changes in the empirical distribution. However, distribution was atypical at small H value ($0.5 \le H < 0.8$).

The expected value of the regime length became larger and larger as H became smaller and smaller. This is consistent with the findings of the previous results as the number of breaks tends to zero. At $0.5 \le H \le 0.65$, the regime length was very likely to be equal to the length of the series (table 2).

H value	Expectation of log regime length	Expectation of regime length
0.95	6.46	637.61
0.9	6.65	771.26
0.85	6.95	1042.03
0.8	7.37	1584.58
0.75	7.81	2470.29
0.7	8.23	3763.93
0.65	8.52	5004
0.6	8.59	5388.83
0.55	8.6	5405

Table 2: Expectations of log regime length and original regime length in FGNs.

4.2.3 Mean levels of series in breaks

Differently from the previous two statistics, the empirical distribution of the mean always appears symmetric. The centre was zero, and the sample standard deviation became smaller as H moved from 0.95 to 0.55. The Logistic distribution was used as well. The distribution looks like either Normal or Logistic distribution.

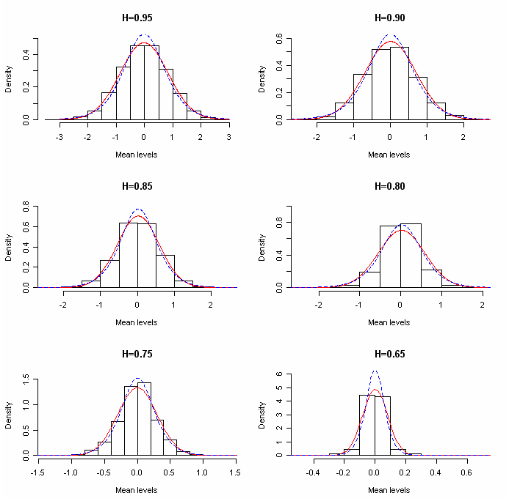

Figure 5: Empirical distributions of mean. Normal (solid,red) and Logistic (dash,blue)

4.2.4 Standard deviations of series in breaks

In term of the standard deviation, the empirical distribution showed symmetry through all H values. Its centre shifts from 0.8 to 1, and the density of point at 1 becomes larger and larger as H becomes smaller. In other words, the standard deviation is more stable where H is small. When H=0.55, the standard deviation was 1. Additionally, densities along the two-sides decayed very quickly.

Because of these features, Logistic and normal distribution worked well where $0.8 \le H < 1$. The Logistic just did a little better than the Normal for this case.

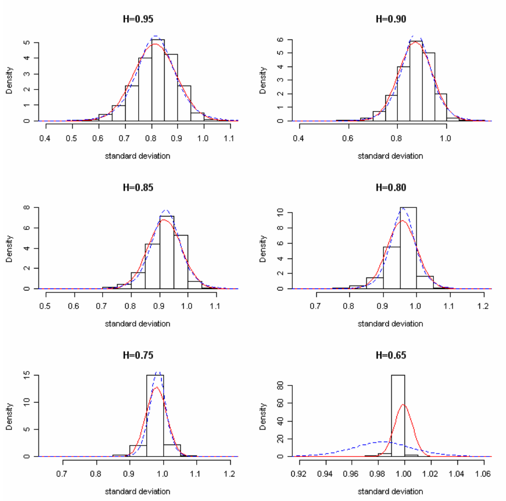

Figure 6: Empirical distributions of standard deviation. Normal (Solid,red) and Logistic (Dash,blue)

4.3 Review

In this section, empirical results provide a good starting point to identify distributions of statistics. Further theoretical investigation is necessary in our future work.

5. Case studies

For our simulations it appears that for a pure long memory process it is possible to predict what would be expected results if one investigated them for structural breaks. This leads us to believe that it should be possible to differentiate between a pure long memory process and a series with structural breaks. For this work we have examined two classic long time series, Nile Minima and Campito Mountain. The background information regarding to data was provided in Rea et al (2006).

5.1 Nile river Minima

This data recorded the yearly minimum water level in Nile River from 662 to 1284 AD (Figure 9). The estimated Hurst parameter for the Nile Minima was estimated 0.837 as by the Whittle estimator in fseries. ART returned 10 breaks i.e. 11 regimes in this dataset. The expected number of breaks in the simulated series for the corresponding H is 11.18. The probability of having 10 breaks is 11.7 percent which was almost the highest through all the densities (Figure 7). Hence, this can be considered a long memory process.

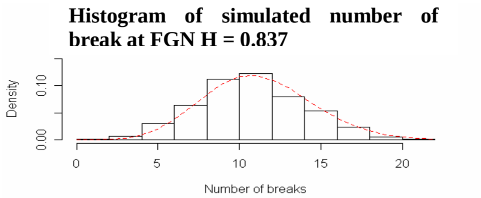

Figure 7: Empirical distribution of number of breaks in Nile yearly minimum water levels 662 to 1284AD.

5.2 Campito Mountain

This data contains 5405 yearly records (3435 BC to 1969 AD) on Tree ring width in Campito Mountain in US (Figure 10). The Whittle estimator returned 0.876 as the estimated value of Hurst parameter. ART detected the whole series with 12 structural breaks, i.e. 13 regimes. From our simulation, the expected number of breaks for the corresponding H is 5.97. Unlikely to the previous case (Nile river minima), the chance of having 12 breaks was very rare, only 1 percent (Figure 8). Therefore, this series cannot be considered as a long memory process.

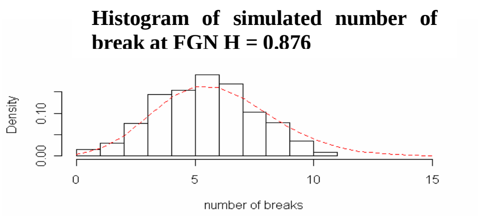

Figure 8: Empirical distribution of number of breaks in tree ring width in Capito Mountain 3435BC to 1969AD.

The Campito Mountain data is considered a textbook example of a FGN process. However the result based on our simulation suggests the series contains structural breaks. This supports the results of Rea et al(2006).

6. Conclusion

In this study, we have proposed a data driven parametric procedure, ART, to distinguish between long memory model and regime switching. The simulations have shown distributional properties on different statistics. Therefore, the results are promising:

1. We have examined a pure long memory process and shown the expected number of (false) breaks that would be identified with their average length;
2. Nile Minima exhibits long memory behavior. However Campito mountain does not;
3. Distributions, such as number of breaks and length of regime, are easy to be recognized for value of Hurst parameter is greater than or equal to 0.8. It also appears that distribution of number of breaks is approximately Poisson distributed. It also implies distributions are difficult to determine if Hurst is less than 0.8. Whereas, it would be unlikely to encounter a series with Hurst less than 0.8 in practice;

It is clear that the value of the Hurst parameter affects expected distributions. Also, there will be differences given the length of the series. We intend to investigate these issues as well as develop theory to support our empirical results.

7. Acknowledgement

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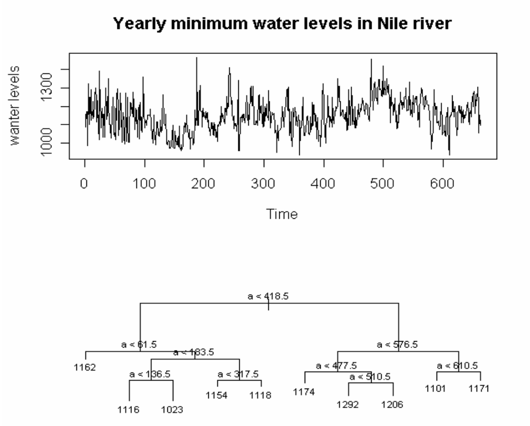

Figure 9: Time series of yearly minimum water levels in Nile river and its ART.

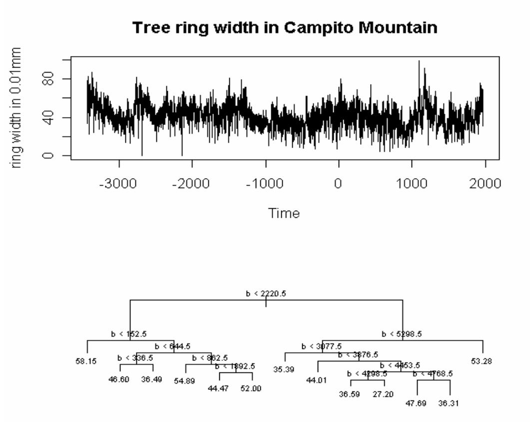

Figure 10: Time series of tree ring width in Campito mountain and its ART.