A Linear Programming Approach to Fitting Rainfall-runoff Models Based on Finite Mixture Hydrographs: a Potential for Flood Forecasting?

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EXTENDED ABSTRACT

It is recognised that the transformation of rainfall to discharge is essentially nonlinear and it would seem logical to construct parsimonious nonlinear models to approximate the runoff process. Lumped nonlinear rainfall-runoff models are simple to construct but difficulties may be created in the calibration process if suboptimal fits to data arise from local minima in the objective function measuring goodness of fit, where small fit values indicate improved fit.

One approach to fitting nonlinear models is to formulate calibration procedures which are hopefully robust against local minima when seeking the global minumum. An alternative approach is advocated here whereby threshold effects and other nonlinearities are approximated in models comprised of a large number of weighted hydrograph components. All these components are of "hydrograph-like" nonlinear form, but the forms themselves are fixed and not some function of nonlinear parameters.

The weights can be obtained via any constrained polynomial function of previous rainfall events, water table heights, or upstream discharge. This allows for dynamic variation of hydrograph form with current catchment state, while with the model itself is still linear and subject only to linear constraints.

It could be argued that linear weighted mixtures are as much "hydrological" in principle as parsimonious nonlinear models. However, the price paid for the simple model structure is the generation of a large number of correlated parameters, corresponding to the coefficients of the functions defining the *N* weight values of the *N* hydrograph components.

A linear programming approach is used to avoid the numerical instability that would arise from least-squares fitting with many correlated parameters and possibly a relatively small calibration set. On the basis of 25 weighted Gumbel distributions, a 100-parameter linear model was constructed with the weights being defined from quadratic functions of the current and previous rainfall in unit time. In the result shown in Figure 1, the calibration data set contained less observations than model parameters and the validation (forecast) discharges were beyond the range of the calibration data. Despite this extreme violation of good modelling practice, the calibration via linear programming was straightforward (and in fact was done using the Excel Solver) and the model fared surprisingly well in this particular instance as a flood-peak forecaster. However, the potential of this interesting procedure is still entirely tentative at this stage and requires further applications to both larger calibration sets and pre-flood forecast calibrations for forecasting purposes.

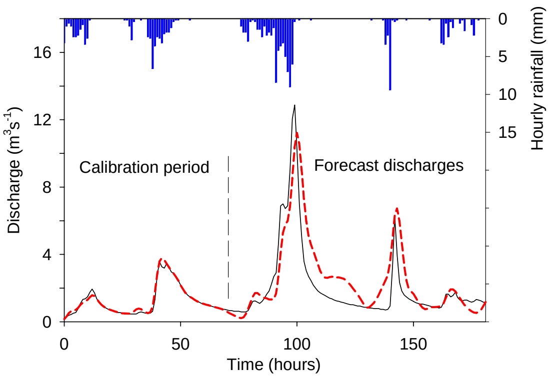

Figure 1. Example application of the 100-parameter model to peak-flow forecasting in a 14 km^2 subcatchment of the Mahurangi River, New Zealand. Calibration fits and forecast discharges are dashed. All data are on an hourly time scale.

1. INTRODUCTION

Lumped nonlinear rainfall-runoff models have a long established history as a means of approximating the nonlinearities in the catchment runoff generating process. However, while such models can be justified as “physically based” to varying degree there can be difficulties with the calibration process terminating at a local minima of the fit objective function. This may have implications for flood forecasting where it could be desirable to carry out frequent model recalibrations to capture the changing state of catchment conditions.

One approach is to develop better general nonlinear calibration techniques that are robust against the effects of local minima, which is an ongoing area of research – see, for example, Tolson and Shoemaker (2007). An alternative may be to modify the structure of the models themselves so they become more numerically amenable while still maintaining their essential features. For example, Kavetski and Kuczera (2007) advocate a smoothing procedure to offset threshold effects, and Yang and Han (2006) propose a generalisation of the unit hydrograph by allowing a number of simpler sub-hydrograph components which can vary with catchment conditions.

The Yang and Han (2006) parameterisation has intuitive appeal but is still a nonlinear model which requires a multi-stage calibration process. A similar but much simpler linear sub-component model is proposed here which can be calibrated easily by linear programming in a single step. Because linear programming is specially suited to over-specified problems, the calibration can even be calculated in the extreme case of more model parameters than calibration data values.

2. MODEL

The proposed model is a generalisation of the finite-mixture approach of Bardsley and Liu (2003), whereby the discharge hydrograph from a given rainfall in time interval Δt is defined to be:

$$Q(t) = \sum_{i=1}^{N} Q_i(t)\, W_i \qquad (1)$$

where the $Q_i(t)$ are N component hydrographs with associated non-negative weights W_i which in turn are defined as a quadratic function of current and previous rainfalls.

This approach is quite flexible in allowing nonlinear forms to be represented by linear mixtures. For example, the forms shown in Figure 2 give the impression of nonlinear functions including location, shape, and scale parameters. In fact, these forms were created from weighted mixtures of the same normal distribution set with a common scale parameter but mean values equally spaced over the 0,1 interval.

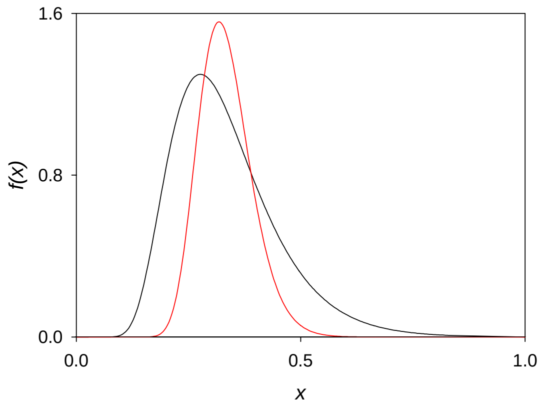

Figure 2. Evident nonlinear forms created as weighted mixtures of normal distributions.

Rather than use normal component hydrographs, 25 equally-spaced Gumbel distribution kernels are set up on the unit interval, standardised to give peak values of 1.0:

$$Q_i(t) = \exp\{1 - (t - \xi_i)/\alpha - \exp[-(t - \xi_i)/\alpha]\} \qquad (2)$$

where the ξ_i are location parameters and α is a scale parameter (set to 0.04). The use of Gumbel (Figure 3) rather than normal kernels was employed simply to better represent the skewed nature of hydrographs (Bardsley, 1989).

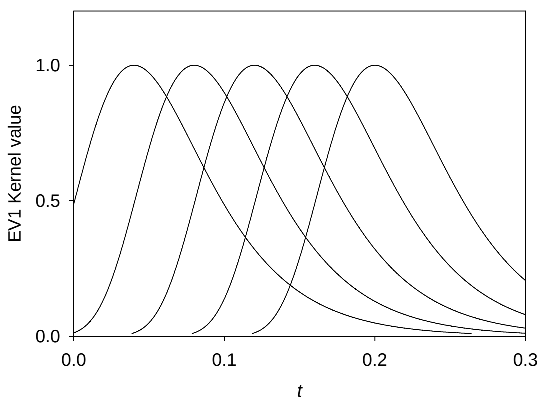

Figure 3. The first five of the 25 Gumbel kernels spaced equally over the 0,1 interval .

Although the terms "location" and "scale" parameter are employed with the distributions of Figure 3, this does not imply their use as nonlinear parameters in the calibration process. Provided there is a sufficient density of distributions along 0,1 then the required hydrograph form in calibration can be met simply by adjusting the linear weights to the numerical values of the component distributions at the *t* values of the calibration data. That is, the nonlinear evaluation is done prior to the calibration and the choice of location and scale parameters is not particularly significant provided a sufficient number of distributions are utilised. Similarly, the 0,1 base length must be scaled up prior to calibration but this is not a model parameter to be determined. Rather, it is simply required that the specified base length be sufficiently large so as not to be a constraint in the calibration process.

The present approach differs from that of Bardsley and Liu (2003), where a smaller number of specific component distributions gave approximate unimodality of the combined hydrograph, but at the expense of two manually adjusted nonlinear parameters in the calibration process. The use instead of a large number of component hydrograph distributions and parameters avoids nonlinear parameters altogether, but at the risk of introducing multimodality as data artefacts in the validation phase (Perrin *et al*. 2001).

The hope is that the simple constrained linear structure of the model might be more forgiving with respect to overfitting if linear programming is employed for model calibration. This is because it is in the nature of linear programming solutions to often have solution values at bounds, so employing a zero lower bound to all parameter values will result in some reduction in the number of nonzero parameters in the final model.

The specific model utilised for illustrative purposes has the Gumbel weights as quadratic functions of the rainfall of just the current and previous Δt, - a fully developed model would use multiple Δt. It is desirable that the weights tend to zero as rainfall amounts become smaller, and the weights always increase as rainfall increases. Therefore the quadratic expressions have zero intercept and all coefficients are constrained to be non-negative. The current and previous rainfall (if non-zero) together give a total of four parameter values as the quadratic coefficients determining a given weight for a hydrograph component arising from a given rainfall in a Δt. Combined with the 25 Gumbel distributions, this yields the final model with 100 potential linear parameters but constrained to non-negative values.

3. EXAMPLE APPLICATIONS

The first application is to hourly rainfall and discharge data from a 14 km^2 subcatchment of the Mahurangi River in Northland, New Zealand. A 50-hour maximum hydrograph base length was specified. The calibration period was deliberately set to a short time interval on the basis that by calibrating to the current catchment state it might be possible to forecast flood events, given the actual "future" rainfalls. Also, it was of interest to see how the model would perform in an over-parameterised situation – there were just 70 calibration points for the 100 parameters. Further, a good test of a hydrological model is to check its ability perform outside the range of the calibration data. In fact, the model did a surprisingly good job as a peak flow forecaster, despite the clear violation of good modelling practice (Figure 1).

Figure 4 gives an indication as to how the model was able to anticipate the 12 m^{3}s^{-1} flow peak of Figure 1. The plotted hydrographs in Figure 4 were simulated from a synthetic rain event over two consecutive hours, with each hour having the rain amount indicated.

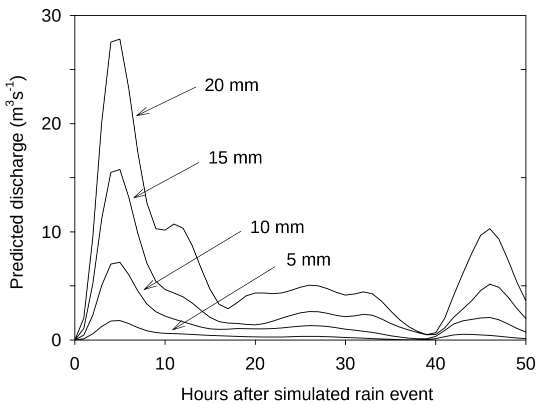

Figure 4. Simulated hydrographs from the Mahurangi model for 2-hour rainfall events.

The secondary peak at about 45 hours is almost certainly an artefact of overfitting, but the main peak shows that a nonlinear relation between peak discharge and rainfall was detected by the model from the limited calibration data. This nonlinear relation, better seen in Figure 5, gave rise to the high 12 m^{3}s^{-1} forecast peak beyond the range of the calibration data.

The origin of the nonlinear effect can be seen in the non-zero values of the calibration parameters listed in Table 1, showing a greater frequency toward the squared terms, particularly in the initial hours of the hydrograph.

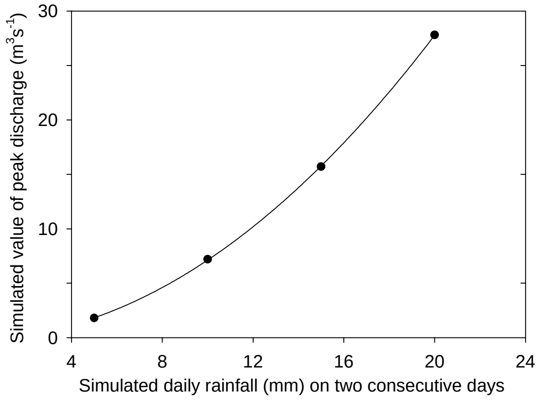

Figure 5. Magnitude of the first peak in the hydrographs of Figure 4, as a function of the simulated rain amount (indicated amount repeats on two consecutive days).

Table 1. The nonzero parameter values for the Mahurangi calibration for the 25 distributions. Column 1 gives distribution number, the 1 and 2 headings denote the linear and squared coefficient values. Columns 2 and 3 are for current rainfall and * denotes coefficients for the previous rainfall.

	1	2	1*	2*
1	-	-	-	-
2	-	0.304	0.006	0.306
3	-	-	-	-
4	-	-	0.016	0.002
5	-	0.007	-	0.072
6	-	0.048	-	-
7	0.025	-	-	-
8	-	0.028	-	0.014
9	-	-	-	0.060
10	-	0.011	0.008	-
11	-	-	-	0.055
12	-	0.010	-	-
13	-	0.045	-	-
14	-	0.019	-	0.024
15	-	-	-	0.032
16	-	-	-	0.023
17	-	0.005	-	-
18	-	-	-	-
19	-	-	-	-
20	-	-	-	-
21	-	-	-	-
22	-	0.129	-	0.006
23	-		-	-
24	-	0.041	-	0.010
25	-	-	-	-

In fact, the main hydrograph peak is likely to be dominated just by the distribution 2 coefficients of 0.304 and 0.306 for the squared rainfalls of the current and previous hour, respectively. The other coefficients are likely to be largely noise although collectively they give a rough approximation to the hydrograph tail, which might be more clearly defined by using a longer calibration set.

The second example (Figure 6) also applies a 50-unit hydrograph base. Time units are days and the discharge is for the Tarawera River at Awakaponga, in the Bay of Plenty in the North Island of New Zealand (catchment area = 900 km^2). Recorded daily rainfall is from Rotorua.

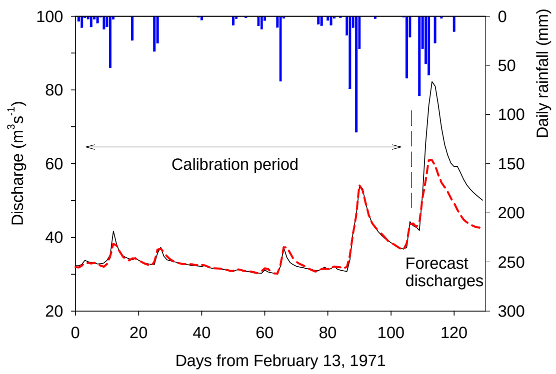

Figure 6. Example calibration and forecast discharges for the Tarawera River at Awakaponga.

It is evident in this case that the good fit of the model to the calibration data does not capture the hydrological processes sufficiently to anticipate the high flood peak in the forecast period. It may have been that the calibration period involved a significant change in the state of the catchment, or there was simply not enough information in the calibration data set relative to the flexibility of the model.

It can be seen in Figure 7 that numerical artefact effects become increasingly evident with higher rainfalls in the Tarawera model. The second high peak here is an artefact derived from coefficients associated with squared terms in the later part of the hydrograph tail (Table 2), while the first peak would probably have led to a better forecast of the actual flood peak if there had been larger coefficient values for the squared terms associated with the earlier distributions Instead, the calibration process anticipated a simple linear increase of flood peak with increased rainfall amount (Figure 8).

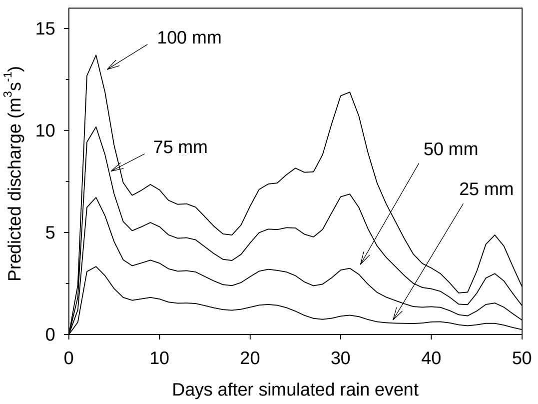

Figure 7. Simulated hydrographs from the Tarawera model for 2-day rainfall events. Each rain event was comprised of two days, each with the rainfall amount indicated.

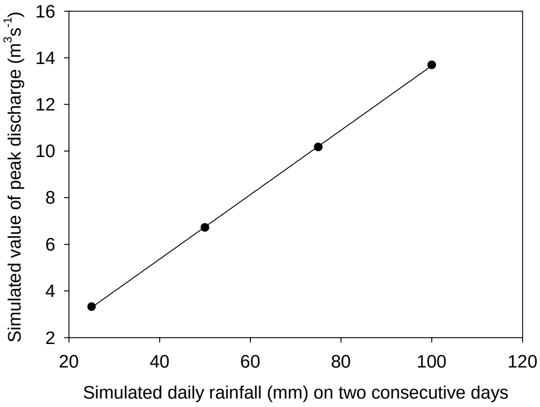

Figure 8. Magnitude of the first peak in the hydrographs of Figure 7, as a function of the simulated rain amount.

4. NOTE ON LINEAR PROGRAMMING CALIBRATION

The usual approach to fitting constrained linear models by linear programming is to minimise the sum of the absolute deviations. If there are K calibration data points then this requires $2K$ fitting variables, additional to the linear parameters of the model itself. This can create a large LP task. An alternative, utilised here, is to use two Fourier series in the LP minimisation of the positive and negative calibration residuals. Model residuals are often correlated so the minimisation operation can be achieved with considerably less Fourier coefficients than the alternative of $2K$ fitting variables. For example, this allowed the Excel solver to be used in the illustrations shown here.

Table 2. The nonzero parameter values for the Tarawera calibration for the 25 distributions. Column headings as for Table 1.

	1	**2**	**1***	**2***
1	0.060		0.177	0.013
2	-	-	-	-
3	-	-	-	-
4	0.032	-	-	-
5	0.042	-	-	-
6	-	0.004	0.004	-
7	0.037	-	0.001	-
8	0.009	-	0.014	-
9	0.009	-	0.008	-
10	0.019	-	-	-
11	0.026	0.026	-	-
12	0.016	-	0.016	-
13	-	-	-	0.117
14	-	-	-	-
15	-	0.068	-	-
16	-	0.073	0.020	-
17	-	0.014	0.002	-
18	-	-	-	0.038
19	0.010	0.008	-	-
20	0.002	-	0.012	-
21	-	-	0.030	-
22	-	-	0.008	-
23	-	-	-	-
24	0.010	-	-	0.093
25	-	-	-	-

5. DISCUSSION AND CONCLUSION

The use of polynomial coefficients to determine positive weights of component hydrographs is quite a powerful linear approach to mimic nonlinear processes. For example, hydrograph lag can be represented by the calibration process selecting zero weights for the first few distributions, while higher-order polynomials would have flexibility to mimic sudden increases in runoff response from rainfall threshold effects.

The coefficients need not be confined to just current and preceding rainfalls. For example, for flood forecasting purposes it could be useful to include catchment water table levels or headwater discharge as factors contributing to the weights via positive polynomial relationships. Seasonal effects might also be incorporated by allowing weights to be influenced by some average of previous temperatures, given an increasing or decreasing linkage between discharge and temperature.

Therefore, a complete "model" as such is not suggested here for immediate application to catchment runoff data. But rather a general many-parameter constrained linear approach is advocated, with calibration using linear programming for both numerical stability and to reduce the number of parameters in the final model. Our use of Gumbel kernels is arbitrary and there is no reason to believe that other unimodal kernels could not serve equally well, given sufficient distribution frequency along the defined hydrograph length.

One inevitable concern with this type of modelling is that it is likely that even with parameter reduction the final model may still contain so many parameters that there is over-flexibility and numerical artefacts in model applications. For example, Tables 1 and 2 indicate that the models still contain 33 and 26 parameters respectively, although a number of the smaller values could probably be set to zero with little effect.

The question of model flexibility is something of a philosophical issue. Yang and Han (2006) make the point that their model structure permits multimodal hydrographs as a useful feature and point to situations where such hydrographs could arise. Similar multimodality applies to the approach discussed here, but the negative aspect is that multiple modes are also free to arise as calibration artefacts. The question then is whether to construct models to minimise artefact risk or allow greater flexibility and increase the possibility of discovery. On balance the latter approach seems better science, noting that a check for artefacts can be achieved by progressively increasing the size of the calibration set.

The issue of absolute parameter numbers is less important because it is quite possible for a nonlinear model with few parameters to be more flexible than a constrained linear model with a much larger number of parameters – discussed also in Bardsley and Liu (2003). The many-parameter approach is not so concerned with accurate determination of the numerous model constants as with providing good validation outcomes. Model error is therefore better quantified by repeated applications to validation data than attempting statistical analysis based on possibly difficult statistical approaches to model estimation errors.

The relative numerical stability of linear programming calibration raises the possibility that many-parameter linear models might be used for flood forecasting with model recalibration prior to each forecast. This is a somewhat more intuitive approach than more sophisticated parameter updating methods like the Kalman filter. However, the many-parameter model essentially creates a new model at each calibration so the forecast is always with an unvalidated model. It would nonetheless be of interest to apply models of this type to past flow records to check their capability for flood forecasting.

There is obviously considerable further work required to verify whether the linear finite mixture approach has any future in rainfall-runoff modelling. However, there is at least potential for the technique to have application over the full range of catchment time and space scales. It would be interesting to establish a universal frequency of component hydrographs capable of describing real-world hydrographs independent of scale.

6. ACKNOWLEDGMENTS

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