Instantaneous Error Term and Yield Curve Estimation

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ABSTRACT

There exist many variations in yield curve modeling based on the Nelson and Siegel's (1987) exponential components framework, but most of them do not care about the generating process of the error term. We often estimate parameters of the Nelson and Siegel's model using an estimation equation that consists of the spot rate model with an independently and identically distributed error term as the traditional method. Usually, nonlinear least squares estimation or maximum likelihood estimation is applied to the model. In this specification, the error term may not be only heteroskedastic but also serially correlated because true covariance matrix of the error term is unknown.

In this paper, to account for the possibility of misspecification, we propose a simple and natural method of yield curve estimation using an instantaneous error term which is generated by a standard Brownian motion process. First, we add the instantaneous error term to the Nelson and Siegel's instantaneous forward rate model. Second, differencing the multiperiod spot rate models transformed from the Nelson and Siegel's instantaneous forward rate model, we obtain a model with serially uncorrelated error terms because they have a property of the independent increment. In our specification, the error term is not serially correlated but heteroskedastic, so we apply weighted least squares estimation to the data. That is, this specification about the error term does not lead to incorrect estimation methods.

We compare the estimated parameters and the shapes of the instantaneous forward rate and yield curves using our specification of the error term and the traditional method in empirical analysis. We use the Japanese Yen Tokyo interbank offered rates (TIBOR) as the spot rate. Some empirical examples in almost periods show that estimated parameters using the model proposed in this paper are similar to those from traditional method. And the probabilities of not rejecting the null hypothesis that the parameter is equal to zero in the proposed modeling are totally larger than those from the traditional estimation.

However, we find that the estimates of the Nelson and Siegel's model are possibly much different when the fluctuations of TIBOR used in the estimation are relatively volatile. Table 1 represents the estimated parameters and Figure 1 plots the yield and forward rate curves in the sample period. We find that the fitted yield curve of our method is better/worse than that of traditional method when the number of months to settlement is long/short. From these results, we find that the shapes of the instantaneous forward rate curves change depending on different specification of the error term such as our specification and the other when the fluctuations of the interest rate are volatile. In other words, it is important to consider the specification of the error term carefully when we investigate the statistical properties of the yield curves or instantaneous forward rate curves.

Table 1. An example of different estimated parameters.

	β_0	β_1	β_2	τ
	Our estimation method			
2000/07/03	0.492	-0.371	-0.053	0.297
	(11.71)	(-6.60)	(-0.09)	(0.61)
	Traditional estimation method			
2000/07/03	0.436	-0.361	0.000	0.200
	(25.65)	(-24.01)	(0.01)	(2.63)

Note: β_0, β_1, β_2 and τ are parameters of the Nelson and Siegel's model defined later to be estimated from the interest rate data. The values in the parentheses represent estimates of t-statistics.

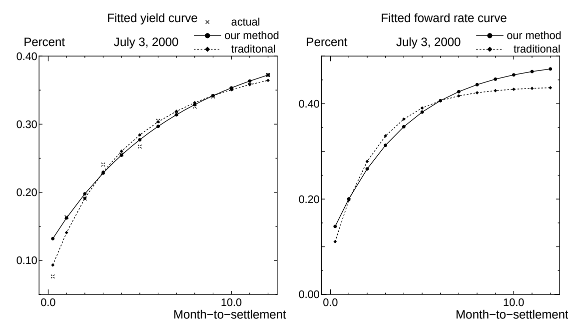

Figure 1. Fitted yield and forward rate curves by our estimation method and traditional estimation method.

1 INTRODUCTION

Various theoretical models for term structures have been developed. Considering these theoretical advancements, term structure models are designed to represent the whole yield curve, including the width and pattern of interest rate fluctuations. From the price theory of derivatives, no-arbitrary model is important. Ho and Lee (1986) and Hull and White (1990) are included in this tradition. The equilibrium models that derive bond and option prices are also important. Vasicek (1977) and Cox et al. (1985) followed this literature. However, McCulloch's (1975) modeling of the discount function with cubic spline approximation is suitable for analyzing the term structure or yield curve. Following his model, Schaefer (1981), Vasicek and Fong (1982) and Steely (1991) proposed estimation with the Bernstein polynomial, exponential spline and B-spline, respectively.

Nelson and Siegel (1987) proposed the exponential components model for the instantaneous forward rate curve. We denote this model as the NS model. This model describes the instantaneous forward rate curve in terms of level, slope and curvature. There exists several exponential components models that estimate the shape of yield curves, for example, Söderlind and Svensson (1995) and Fujiki and Shiratuka (2002). Diebold and Li's (2006) paper presents an alternate method. They introduced time-varying coefficients into the NS model. These models are proposed from a practical viewpoint.

In this paper, we focus on the assumption of the error term introduced in the estimation of the NS model. Despite various extensions to the modeling of the Nelson-Siegel exponential components framework, more discussion about the specification of the error term is required. For instance, we often estimate parameters of the NS model using an estimation equation that consists of the spot rate model with an independently and identically distributed error term. Usually, nonlinear least squares estimation or maximum likelihood estimation is applied to the model. However, the error term may not be only heteroskedastic but also serially correlated.

To account for the possibility of misspecification, we propose an estimation equation with a naturally specified error term by adding an instantaneous error term to the instantaneous forward rate model. Then, differencing the multiperiod spot rate models, the model does not suffer from serial correlation of the error term and its statistical properties are clearly represented. This specification leads to correct estimation of the NS model using nonlinear weighted least squares estimation. Using some empirical examples, we compare the estimated parameters, the spot rate and instantaneous forward rate curves to those using traditional maximum likelihood estimation. First, the estimated parameters are similar to those from traditional maximum likelihood estimation. However, the shapes of the instantaneous forward rate curves are different when the fluctuations of interest rates used in the estimation are volatile. Second, the probabilities of not rejecting the null hypothesis that the parameter is equal to zero in our estimation method are totally larger than those using maximum likelihood.

The paper proceeds as follows. In Section 2, we investigate the problems in the estimation of the NS model and introduce the correct specification of the error term. In section 3, we compare the estimated results in some empirical examples. We conclude the paper in section 4.

2 MODEL

In this section, we consider the estimation of the instantaneous forward rate model of Nelson and Siegel (1987). Let the instantaneous forward rate at maturity m (the remaining periods until maturity being m) be $f(m)$. The NS model is given by:

$$f(m) = \beta_0 + \beta_1 \text{exp}(-m/\tau) + \beta_2 (m/\tau)\text{exp}(-m/\tau), \tag{1}$$

where β_0, β_1, β_2 and τ are parameters to be estimated from the interest rate data. The instantaneous forward rate function is approximated by the sum of the constant term and two exponential functions. As m approaches infinity and zero, the value of $f(m)$ becomes β_0 and $\beta_0 + \beta_1$, which represent Console bonds and the instantaneous spot rate, respectively. That is, the first and second terms represent the contribution of the long- and short-term components on the forward rate curve. β_1 takes a negative value when the shape of the forward rate curve is upward sloping. Because it is reverse yield, β_1 takes a positive value. β_2 is positive and negative when the medium-term component on the forward rate curve creates a hump shape and a U-shape, respectively. τ controls the exponential convergence speeds of the second and third terms. The large value of τ creates a gentle slope and slows down the convergence speed to the shape of the forward rate curve in the long run. Some authors (for instance, Söderlind and Svensson 1995, Fujiki and Shiratuka 2002 and Diebold and Li 2006) proposed various models based on the NS model: Söderlind and Svensson (1995) added another exponential term to this model, Fujiki and Shiratuka (2002) introduced convergence of the second and third terms. We do not consider their models in this paper because we focus on the estimation methods of the NS model.

Using the instantaneous forward rate $f(m)$, we can express the spot rate $r(m)$ as:

$$\begin{aligned} r(m) &= 1/m \int_0^m f(s)ds \\ &= \beta_0 + (\beta_1 + \beta_2)\big[1 - \exp(-m/\tau)\big]/(m/\tau) \\ &\quad - \beta_2 \exp(-m/\tau). \end{aligned} \tag{2}$$

Traditionally, adding an error term, we apply nonlinear least squares (NLS) or maximum likelihood estimation (MLE) for this equation:

$$\begin{aligned} r(m) &= \beta_0 + (\beta_1 + \beta_2)\big[1 - \exp(-m/\tau)\big]/(m/\tau) \\ &\quad - \beta_2 \exp(-m/\tau) + \epsilon(m), \quad \epsilon \sim N(0, \sigma_\epsilon^2), \end{aligned} \tag{3}$$

where ϵ is assumed to be independently and identically normally distributed. However, $\epsilon(m)$ possibly has serial correlation and heteroskedasticity because $r(m)$ does not generate only at time m but through the time period $[0, m)$. To provide an explanation about serial correlation in the error term, we denote the spot rate on the remaining period until maturity m_i as $r(m_i)$, $m_i > m_{i-1}$ at $i > i-1$ and $r(m_0) = r(0) = 0$. Then, $\epsilon(m_i)$ and $\epsilon(m_{i-1})$ are possibly correlated because the time windows of ϵ_{m_i} and $\epsilon_{m_{i-1}}$ are overlapping for $[0, m_{i-1})$. The variances of the estimated parameters are undervalued and the possibility of rejecting a true null hypothesis rises when we use NLS and MLE for (3).

Therefore, we specify the error term in the estimating model for the spot rate. We assume the following process based on the NS model to describe the instantaneous forward rate:

$$f(m) = D(m) + \sigma dW(m), \tag{4}$$
$$D(m) = \beta_0 + \beta_1 \exp(-m/\tau) + \beta_2 (m/\tau)\exp(-m/\tau),$$

where $D(\cdot)$ is expressed by (1) and σ is a constant and $W(\cdot)$ is a standard Brownian motion process. When we consider estimation of the NS model, adding the instantaneous error term to the instantaneous forward rate model is a more natural and simpler specification of the estimating equation than (3). Then, spot rate model is given by:

$$m \cdot r(m) = \int_0^m D(s) + \sigma \int_0^m dW(s). \tag{5}$$

We describe the difference between $m_i \cdot r(m_i)$ and $m_{i-1} \cdot r(m_{i-1})$ as follows:

$$m_i \cdot r(m_i) - m_{i-1} \cdot r(m_{i-1}) = \int_{m_{i-1}}^{m_i} D(s) + \nu_i, \tag{6}$$
$$\nu_i = \sigma\big(W(m_i) - W(m_{i-1})\big) \sim N\big(0, (m_i - m_{i-1})\sigma^2\big),$$

where ν_i is the error term that has properties of independence and heteroskedasticity. ν does not suffer from the serial correlation mentioned above by the independent increment. The heteroskedasticity of the error term also remains because the variance of ν_i depends on the length of $m_i - m_{i-1}$. To cope with this problem, we apply weight $1/\sqrt{m_i - m_{i-1}}$ to (6) in the estimation. Because the properties of the error term are clearly identified, we can apply nonlinear weighted least squares (NWLS) or MLE considering the heteroskedasticity of the error term in (6). It might be obvious that (6) is a more natural form than (3) for estimation of the NS model. In the next section, we compare the estimates of the parameters and the shapes of the instantaneous forward rate and yield curves using NWLS for (6) to those using MLE for (3).

3 EMPIRICAL ANALYSIS

We use the Japanese Yen Tokyo interbank offered rates (TIBOR) as $r(m)$ in (3) and (6) because, by definition, TIBOR are free from coupon effects. TIBOR for m one week and from one month to one year are traded in the Japan Offshore Market. The sample period of the TIBOR data covers the period from August 2000 to December 2005. We focus on a sample period for which fluctuations in TIBOR can be characteristic because we examine relative estimates of the parameters and shapes of the yield curve in various cases.

We select six days within the sample period and describe these results in detail. July 3rd, 2000 (first row in Tables 2 and 3) is the last stage of the zero interest rate policy in Japan. TIBOR at one week maturity is 0.076 percent, that is, it declines to virtually zero. TIBOR at one year maturity takes a larger value, 0.372 percent, than at one week maturity. On August 28th, 2000 (second rows) the Bank of Japan ended the zero interest rate policy. TIBOR at one week and one year maturity where 0.343 percent and 0.491 percent respectively, following their response to the change in monetary policy. However, a reverse yield in TIBOR was observed on December 26th, 2000 (third rows) because analysts forecasted a recession in the future reflecting that growth in the U.S. economy had decelerated from the end of 2000. TIBOR at one year maturity is 0.585 percent, which is smaller than 1.263 percent at one week maturity. August 8th, 2002, July 14th, 2004 and December 10th, 2005 (fourth to sixth rows) are the stages of the quantitative monetary easing policy enforced at the end of 2001. The quantitative monetary easing policy caused a reduction in interest rate volatility and kept the level of interest rates constant. These TIBOR from one week to one year are around 0.05 to 0.12 percent. It is noted that the fluctuation of TIBOR in the first and third rows are relatively volatile in the sample periods; the differences of TIBOR at one week and one year in each period are 0.295 and -0.677 percent, respectively.

Table 2. The parameters estimated by MLE for (3)

	β_0	β_1	β_2	τ
2000/07/03	0.436	−0.361	0.000	0.200
	(25.65)	(-24.01)	(0.01)	(2.63)
2000/08/28	0.568	−0.226	0.000	0.394
	(25.81)	(-11.89)	(0.01)	(1.57)
2000/12/26	0.565	0.889	−0.694	0.075
	(62.78)	(37.04)	(-9.38)	(10.71)
2002/08/05	0.123	−0.076	−0.071	0.070
	(17.57)	(-4.00)	(-0.87)	(1.25)
2004/07/14	0.129	−0.082	−0.062	0.090
	(25.80)	(-8.20)	(-1.38)	(2.14)
2005/12/10	0.130	−0.084	−0.055	0.069
	(43.33)	(-10.50)	(-1.28)	(2.56)

Note: The values in the parentheses represent estimates of t-statistics.

Table 3. The parameters estimated by NWLS for (6)

	β_0	β_1	β_2	τ
2000/07/03	0.492	−0.371	−0.053	0.297
	(11.71)	(-6.60)	(-0.09)	(0.61)
2000/08/28	0.572	−0.226	0.001	0.394
	(8.17)	(-2.83)	(0.01)	(0.01)
2000/12/26	0.585	0.757	−0.758	0.100
	(195.00)	(39.84)	(-31.58)	(33.33)
2002/08/05	0.122	−0.070	−0.066	0.074
	(42.67)	(-2.81)	(-0.99)	(2.12)
2004/07/14	0.130	−0.082	−0.062	0.090
	(32.50)	(-3.15)	(-0.83)	(1.91)
2005/12/10	0.132	−0.082	−0.055	0.069
	(33.25)	(-3.00)	(-0.65)	(1.66)

Note: The values in the parentheses represent estimates of t-statistics.

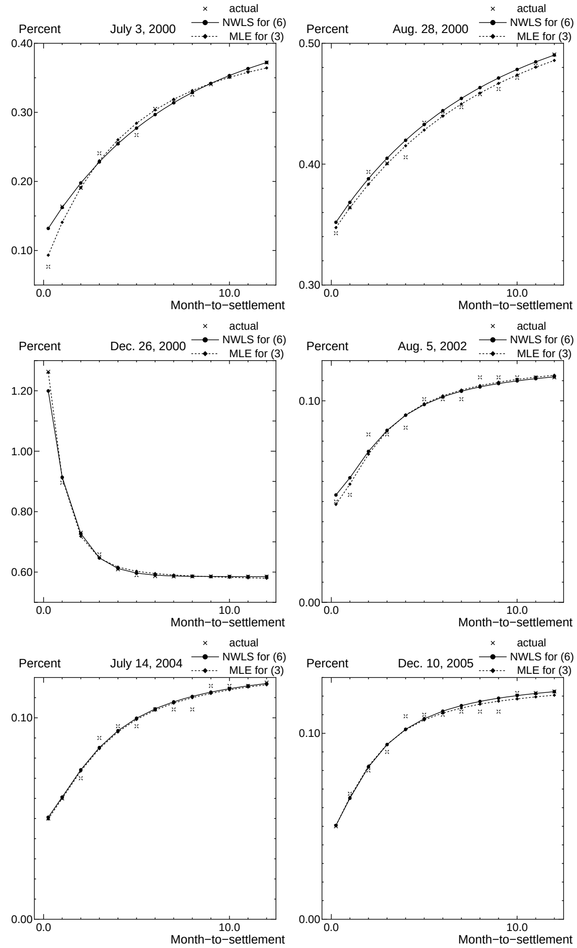

Figure 2. Fitted yield curves by MLE for (3) and NWLS for (6).

Tables 2 and 3 summarize the estimates of parameters β_0, β_1, β_2 and τ in (3) and (6). The signs of the estimated parameters are consistent with theoretical predictions for both MLE for (3) and NWLS for (6). For instance, β_0 and τ take positive values. Except for the case of a reverse yield on December 26th, 2000, the negative estimate of β_1 produces an upward-sloping shape for the forward rate curve. It should be noted that β_1 is significant but β_2 is not significant in almost all of the cases. A reasonable explanation for these results is that the fluctuations of TIBOR until m equals one year are strongly influenced by the second term in (1), which represents the contribution of the short-term components on the forward rate curve. We find that the probabilities of not rejecting the null hypothesis in (6) are totally larger than those in (3) because the standard errors of the estimated parameters in (3) are possibly underestimated.

Figures 2 and 3 plot the spot rate curves and the instantaneous forward curves. The spot rate and the instantaneous forward curves in the first period (first row in Tables 2 and 3) have different shapes from (3) and (6). We find the fitted yield curve of (6) is better/worse than that of (3) when the number of months to settlement is long/short. β_0 and τ using NWLS for (6) are larger in value than those using MLE for (3) and the values of β_1 are negative. Therefore, the instantaneous forward rate curve using NWLS for (6) makes the slope more gentle and converges to a larger value in the long run than MLE for (3). In the third period, the instantaneous forward rates in (3) and (6) fall until two months and increase afterwards. This U-shaped curve is also implied by the significant and negative estimates of β_2 using MLE for (3) and NWLS for (6). The instantaneous forward rate curves in other periods are almost the same shape for (3) and (6). From these results, we find that the shapes of the instantaneous forward rate curves change depending on different specification of the error term such as (3) and (6) when the fluctuations of the interest rate are volatile. In other words, it is important to consider the specification of the error term carefully when we investigate the statistical properties of the yield curves or instantaneous forward rate curves.

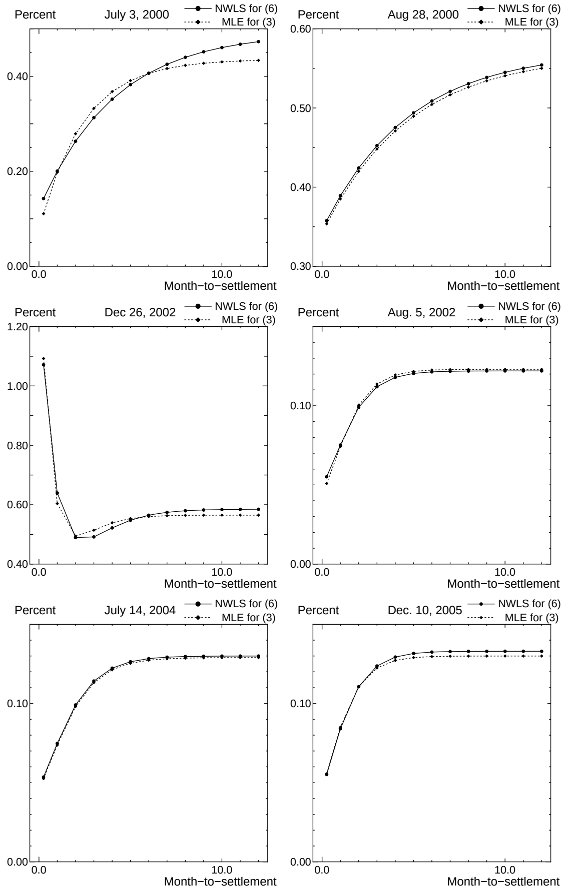

Figure 3. Fitted instantaneous forward curves by MLE for (3) and NWLS for (6).

4 CONCLUSIONS

In this paper, we specify the properties of the error term in the spot rate model transformed from the NS model by adding an instantaneous error term to the instantaneous forward rate model. By taking the difference of the i-th and $i-1$-th spot rate models, the model does not suffer from serial correlation of the error term and its statistical properties are clearly represented. This specification about the error term does not lead to incorrect estimation methods, that is, it is only necessary to apply nonlinear weighted least squares to the spot rate model. In some empirical examples, we estimate the instantaneous forward rate curve on TIBOR by applying the proposed method. First, the estimated parameters are similar to those using traditional maximum likelihood estimation. However, the shapes of the instantaneous forward rate curves are different when the fluctuations of interest rates used in the estimation are volatile. Second, the probabilities of not rejecting the null hypothesis that the parameter is equal to zero in our estimation method are totally larger than those using maximum likelihood. Therefore, we should carefully specify the error term and select the proper estimation method for the NS model.

5 ACKNOWLEDGMENT

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