Testing the Elasticity of Volatility with Respect to the Level of An Integrated Process

Masahito Kobayashi[a], **Michael McAleer**[b]

[a]*Faculty of Economics, Yokohama National University (mkoba@ynu.ac.jp)*
[b]*Department of Economics, University of Western Australia*

Abstract: The volatility, or conditional variance, of some variables in economics and finance can be proportional to a power function of the levels. It is shown that this process, which is known as the constant elasticity of volatility process, can be generated by the inverse Box-Cox transformation of an integrated series with small innovations. A test is proposed for the hypothesis that the power parameter, or volatility elasticity, has a specific value, when the innovation follows a GARCH(p,q) process. The test statistic detects the correlation between the conditional variance and the level of the integrated process, is shown to be a function of Brownian motions under the null hypothesis, and has a nonstandard asymptotic distribution.

Keywords: *Box-Cox transformation, GARCH, Integrated processes, Constant elasticity of volatility.*

1. INTRODUCTION

It is widely accepted in economics and finance that the volatility of the short-term interest rate, namely the conditional variance of interest rate changes, is sensitive to its level. For example, the Cox, Ingersoll, and Ross (1985) model assumes that the conditional volatility of changes in the interest rate is proportional to the level of the interest rate. Although the Cox, Ingersoll, and Ross model was developed to analyze a single-factor general equilibrium term structure, it has been used extensively in the analysis of valuation models for contingent claims that are sensitive to interest rates. Marsh and Rosenfeld (1983) assume that volatility is proportional to the square of the interest rate, so that the interest rate follows a geometric Brownian motion process. Courtadon (1982) uses a similar process to develop a model of discount bond option prices. Assuming that volatility is proportional to the cube of the interest rate, Constantinides and Ingersoll (1984) value bonds in the presence of taxes. A useful comparison of alternative economic models of the dynamics of short-term interest rate volatility is given in Chan et al. (1992). For recent developments of the constant elasticity of volatility (CEV) process, see Conley et al. (1997) and Smith (2002).

The magnitude of the elasticity of volatility with respect to the level of interest rates has been a key empirical issue, but no consensus seems to have yet been reached. Chan et al. (1992) reported that the actual elasticity is higher than those typically assumed in theory. However, Brenner et al. (1996) obtained a lower estimate of the elasticity under the assumption of serially correlated volatility. In particular, they suggested that the reported high elasticity could be explained by neglected time-varying (conditional) heteroscedasticity.

In addition to conditional heteroscedasticity, it is necessary to deal with nonstationarity appropriately in order to estimate the dynamics of volatility. It is natural to propose that short-term interest rates follow a random walk process, as mean reversion is rarely supported in empirical analysis. Owing to the presence of an integrated process, the estimators and test statistics are likely to have nonstandard distributions, so that conventional inferences might be invalid. However, to date there does not seem to have been any development of statistical tools to accommodate such nonstationary data, in which volatility depends upon the level of an integrated process.

In this paper we propose a test for the hypothesis that volatility is proportional to a power transformation of nonstationary interest rates, and derive the asymptotic distribution of the test statistic when the innovation follows a GARCH(p,q) process. The test statistic is expressed as a function of Brownian motions under the null, and has a nonstandard asymptotic distribution.

Although the motivation of the test procedure is based on analyzing short-term interest rates, the method developed in the paper is applicable to a variety of other data. For example, as argued in

Hull (1997, pp.494–497), the volatility of stock prices is likely to be negatively correlated with their levels.

e as that described above, but with opposite sign.

The plan of the paper is as follows. In Section 2 the CEV process and the GARCH model are discussed. In Section 3, the CEV process is derived as the inverse Box-Cox transformation. In Section 4 the associated test statistic is defined, and its asymptotic properties are obtained.

2. THE CONSTANT ELASTICITY OF VOLATILITY MODEL AND THE GARCH MODEL

It is often assumed that short-term interest rates change according to the following model:

$$y_t - y_{t-1} = y_{t-1}^{\lambda}\varepsilon_t, \qquad t=1,...,n, \qquad (1)$$

where y_t is the interest rate at time t and ε_t is the innovation term. This model defines the constant elasticity of volatility (CEV) process, where the conditional variance of changes in the rate of interest is proportional to a power function of its level. In the context of a broad class of short-term interest rate models, Chan et al. (1992) present four alternative values for λ in (1), namely: $\lambda = 0$, in which the conditional volatility of changes in the interest rate is constant; $\lambda = 0.5$, in which the conditional volatility of changes in the interest rate is proportional to its level; $\lambda = 1$ in the geometric Brownian motion of Black and Scholes (1973), in which the conditional volatility of changes in the interest rate is proportional to its square; $\lambda=1.5$, in which the conditional volatility of changes in the interest rate is proportional to its cube; and unspecified λ.

In light of recent developments in the literature, it is natural to assume that ε_t in equation (1) follows a GARCH(p,q) process, that is, the conditional distribution of ε_t, given the information set at time t-1, Ω_{t-1}, is specified as

$$\varepsilon_t \mid \Omega_{t-1} \sim \mathrm{NID}(0, \sigma_t^2),$$

$$\sigma_t^2 \equiv \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2, \qquad (2)$$

and $\omega>0$, $\alpha_i \geq 0$ $(i=1,..,p)$ and $\beta_i \geq 0$ $(i=1,..,q)$ are sufficient conditions for $\sigma_t^2>0$. The ARCH (or α_i) effect indicates the short run persistence of shocks

This problem seems to be essentially the sam

(namely $\sum_{i=1}^{p} \alpha_i$), while the GARCH (or β_i) effect indicates the contribution of shocks to long run persistence (namely $\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i$).

When q=0, the GARCH(p,q) model (2) reduces to Engle's (1982) autoregressive conditional heteroscedasticity (ARCH(p)) model. Bollerslev (1986) showed that the necessary and sufficient condition for the second-order stationarity of the GARCH(p,q) model in (2) is

$$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i < 1, \qquad (3)$$

Ling and McAleer (2002a) established a simple sufficient condition for the strict stationarity and ergodicity of a family of GARCH(1,1) models, and obtained the necessary and sufficient condition for the existence of the moments. The causal expansion of the GARCH(p,q) process and the existence of a unique stationary solution can be used to show that the process starts infinitely far in the past with finite 2m-th moment. Ling (1999) showed that a sufficient condition for the existence of the 2m-th moment of the GARCH(p,q) model is

$$\rho[E(A_t^{\otimes m})] < 1, \qquad (4)$$

where $\rho(A)$ = max{eigenvalues of a matrix A}, $A^{\otimes m} = A \otimes A \otimes ... \otimes A$ (m factors), $\otimes$ is the Kronecker product, and the matrix A_t is a function of η_t and $(\alpha_1,.., \alpha_p, \beta_1,..,\beta_q)'$. . [Note that equation (1) is a special case of (4) when m=1.] Ling and McAleer (2002b) showed that condition (4) is necessary and sufficient for the existence of the 2m-th moment. Therefore, the complete moment structure of the GARCH(p,q) model has now been established.

Defining $\theta = (w, \alpha_1,.., \alpha_p, \beta_1,..,\beta_q)'$, maximum likelihood estimation can be used to estimate θ. Given observations ε_t, t=1,...,n, the conditional log-likelihood can be written as

$$\log f = -(1/2) \sum_{t=1}^{n} (\log\sigma_t^2 + \varepsilon_t^2/\sigma_t^2)$$

in which σ_t^2 is treated as a function of $\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots$. Let $\theta \in \Delta$, a compact subset of R^{p+q+1}, and define

$$\hat{\theta} = \underset{\theta \in \Delta}{\arg\max} \log f \, .$$

As the conditional error ε_t is not assumed to be normal, $\hat{\theta}$ is called the quasi-maximum likelihood estimator (QMLE). For the GARCH(p,q) model, Ling and Li (1998) proved that the local QMLE is consistent and asymptotically normal under fourth-order stationarity. Ling and McAleer (2003) proved the consistency of the global QMLE under only the second moment condition, and derived the asymptotic normality of the global QMLE under the sixth moment condition.

3. THE BOX-COX MODEL AND INTEREST RATE PROCESSES

In this section it is shown that the CEV process y_t defined by (1) can be generated by the inverse Box-Cox transformation of an integrated process. First, it is assumed that the integrated process z_t with initial value z_0 has a GARCH innovation, namely

$$z_t = z_{t-1} + \varepsilon_t , \ \ (t = 1,...,n) \tag{5}$$

where

$$\varepsilon_t \mid \Omega_{t-1} \sim \mathrm{NID}(0, \sigma_t^2), \tag{6}$$

$$\sigma_t^2 \equiv \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2 \tag{7}$$

and ε_t satisfies the following assumption:

Assumption 1: The GARCH(p,q) process ε_t defined by (6) and (7) is stationary and has finite fourth-order moment.

Assumption 1 is satisfied if equation (4) is satisfied for m≥2, as reviewed in the previous section, from the results of Ling (1999), Ling and Li (1998), and Ling and McAleer (2002b). In the problem considered here, the conventional condition (3) is satisfied because it is a special case of (4) for m=1.

Second, it is assumed tends to 0, and hence the volatility of z_t tends to 0 as the sample size n increases**.** The suffix n is suppressed for the sake of notational simplicity, although ω should be written as ω_n and σ_t as $\sigma_{n,t}$.

Assumption 2: ω depends upon the sample size n and $0 < \omega = O(n^{-1})$.

It is easy to see that

$$z_t = z_0 + O_p(1) = O_p(1) \tag{8}$$

from Assumptions 1 and 2, since

$$\varepsilon_t = O_p(n^{-1/2}),$$
$$z_t - z_0 = \varepsilon_t + \cdots + \varepsilon_1 = O_p(1). \tag{9}$$

The small volatility of ε_t is essential in showing that the CEV process can be generated by the inverse Box-Cox transformation of z_t, namely,

$$y_t = \begin{cases} [(1-\lambda)z_t + 1]^{1/(1-\lambda)}, & \lambda \neq 1 \\ \exp(z_t), & \lambda = 1 \end{cases} \tag{10}$$

under Assumptions 1 and 2. The small volatility of y_t, which is implied by the small volatility of z_t, seems realistic since short-term interest rates have small volatility compared with the level, especially in high-frequency data. Note that, from Assumptions 1 and 2, y_t is bounded, namely

$$y_t = y_0 + O_p(1) = O_p(1), \tag{11}$$

since z_t is bounded and the transformation (10) is continuous

The next assumption makes the inverse Box-Cox transformation monotonic for any value of λ.

Assumption 3: z_0 is sufficiently large so that the probability of $z_t > 0$ is negligible for any t.

From the monotonicity of the transformation in (10), the GARCH(1,1) process z_t can be obtained by the Box-Cox transformation of the CEV process y_t, namely

$$z_t = \begin{cases} \dfrac{y_t^{\gamma} - 1}{\gamma}, & \gamma \neq 0, \\ \log(y_t), & \gamma = 0, \end{cases} \tag{12}$$

where $\gamma \equiv 1-\lambda$.

When $\lambda \neq 1$, a Taylor expansion of $y_t - y_{t-1}$ in powers of ε_t gives

$$y_t - y_{t-1} = \left[(1-\lambda)z_{t-1}+1\right]^{\lambda/(1-\lambda)}\varepsilon_t + \tfrac{1}{2}\left[(1-\lambda)z_{t-1}+1\right]^{(2\lambda-1)/(1-\lambda)}\varepsilon_t^2 + \cdots$$
$$= y_{t-1}{}^{\lambda}\varepsilon_t + O_p(n^{-1}),$$

and, when $\lambda = 1$, it gives

$$y_t - y_{t-1} = y_{t-1}\varepsilon_t + O_p(n^{-1}).$$

Thus, we have the following lemma:

Lemma 1: Under Assumptions 1-3, we have that

$$y_t - y_{t-1} = y_{t-1}{}^{\lambda}\varepsilon_t + O_p(n^{-1}) = O_p(n^{-1/2}). \qquad (13)$$

This shows that the interest rate process y_t with volatility proportional to $y_{t-1}{}^{2\lambda}$ can be expressed asymptotically by the inverse Box-Cox transformation of the integrated process z_t. Thus, from the CEV process (13), we can obtain an integrated series whose associated volatility is asymptotically independent of the level by using the Box-Cox transformation.

It should be noted that Assumptions 1 and 2 are sufficient to ensure that z_t is not explosive. We could also have bounded y_t by the assumption of stationarity. However, the low volatility of interest rates compared with their levels seems far more realistic than that of stationarity, which has been shown to be difficult to support empirically (see, for example, Brenner et al. (1996)).

An analysis based upon the assumption that the variance converges to 0 as the sample size increases is known as a small−σ expansion. Bickel and Doksum (1981) used this method in investigating the small sample properties of the Box and Cox (1964) model.

4. TEST STATISTIC

We now propose a test for the hypothesis that the conditional volatility of z_t depends upon the levels, where z_t is obtained by the Box-Cox transformation of y_t in (12). The test is designed to detect the correlation between z_{t-1} and the conditional volatility of $z_t - z_{t-1}$, and is interpreted as a test for the null hypothesis that the value of the Box-Cox transformation parameter λ in (12) is correct. Note that Assumptions 2 and 3 are unnecessary, though harmless, in deriving Theorem 1 below because these assumptions are used only to derive Lemma 1. Only the integrated process defined in Assumption 1 is required to obtain the asymptotic expression of the test statistic. However, Assumptions 2 and 3 are necessary in order to interpret the procedure as an hypothesis test for the constant elasticity of volatility.

The test statistic is defined by

$$S = \frac{1}{n}\sum_{t=1}^{n}\left(\varepsilon_t{}^2 / \hat{\sigma}_t^2 - 1\right) z_{t-1}, \qquad (14)$$

where $\varepsilon_t \equiv z_t - z_{t-1}$ and $\hat{\sigma}_t^2$ is the estimated conditional volatility. This test is, essentially, the sample covariance of z_{t-1} and $\varepsilon_t{}^2 / \hat{\sigma}_t^2$; if the Box-Cox transformation parameter $\gamma \equiv 1-\lambda$ is chosen correctly in (12), $\varepsilon_t{}^2$ has no correlation with z_{t-1}, and hence S would be distributed around zero.

Assumptions 1 and 4 are necessary to obtain the asymptotic expression of the test statistic.

Assumption 4: The MLE of the GARCH parameter $\theta = (\omega, \alpha_1, .., \alpha_p, \beta_1, .., \beta_q)'$ for (7) is consistent and asymptotically normal.

First, expand the estimated volatility as

$$1/\hat{\sigma}_t^2 = 1/\sigma_t^2 - w_t\left(\hat{\omega}-\omega\right)/\sigma_t^4$$
$$-\sum_{i=1}^{p} a_{it}\left(\hat{\alpha}_i - \alpha_i\right)/\sigma_t^4 - \sum_{i=1}^{q} b_{it}\left(\hat{\beta}_i - \beta_i\right)/\sigma_t^4 + ...,$$
(15)

where the derivatives of the conditional volatility are denoted by

$$w_t \equiv \frac{\partial\sigma_t^2}{\partial\omega},\ a_{it} \equiv \frac{\partial\sigma_t^2}{\partial\alpha_i},\ b_{it} \equiv \frac{\partial\sigma_t^2}{\partial\beta_i},$$

and are obtained recursively

Then, from (15), we have the following expansion:

$$S/\sigma = n^{-1}\sum_{t=1}^{n}\left(\frac{\varepsilon_t^2}{\sigma_t^2}-1\right)\frac{z_{t-1}}{\sigma} - \frac{(\hat{\omega}-\omega)}{\sigma^2}n^{-1}\sum_{t=1}^{n}\frac{z_{t-1}}{\sigma}\frac{\varepsilon_t^2}{\sigma_t^4}\sigma^2 w_t$$
$$-\sum_{i=1}^{p}(\hat{\alpha}_i-\alpha_i)n^{-1}\sum_{t=1}^{n}\frac{z_{t-1}}{\sigma}\frac{\varepsilon_t^2}{\sigma_t^4}a_{it} \qquad (16)$$
$$-\sum_{i=1}^{p}(\hat{\beta}_i-\beta_i)n^{-1}\sum_{t=1}^{n}\frac{z_{t-1}}{\sigma}\frac{\varepsilon_t^2}{\sigma_t^4}b_{it} + o_p(1).$$

First, we can easily see that

$$n^{-3/2}\sum_{t=1}^{n}\frac{z_{t-1}}{\sigma}\frac{\sigma^2\varepsilon_t^2}{\sigma_t^4}w_t \Rightarrow E\left[\sigma^2 w_t/\sigma_t^2\right]\int B_0(s)ds,$$
$$n^{-3/2}\sum_{t=1}^{n}\frac{z_{t-1}}{\sigma}\frac{\varepsilon_t^2}{\sigma_t^4}a_{it} \Rightarrow E\left[a_{it}/\sigma_t^2\right]\int B_0(s)ds \qquad (17)$$
$$n^{-3/2}\sum_{t=1}^{n}\frac{z_{t-1}}{\sigma}\frac{\varepsilon_t^2}{\sigma_t^4}b_{it} \Rightarrow E\left[b_{it}/\sigma_t^2\right]\int B_0(s)ds\cdot$$

Noting that the log-likelihood function of ε_t is expressed as

$$\log f(\varepsilon_t\,|\Omega_{t-1}) = -(1/2)\left(\log\sigma_t^2 + \varepsilon_t^2/\sigma_t^2\right),$$

and

$$v_{0t} \equiv \varepsilon_t/\sigma,$$
$$v_{1t} \equiv \varepsilon_t^{\,2}/\sigma_t^2 - 1,$$

$$v_{2t} \equiv (1/2)\sigma^2 v_{1t}w_t/\sigma_t^2 = \sigma^2\partial\log f(\varepsilon_t\,|\Omega_{t-1})/\partial\omega,$$
$$v_{3it} \equiv (1/2)v_{1t}a_{it}\sigma_t^2 = \partial\log f(\varepsilon_t\,|\Omega_{t-1})/\partial\alpha_i,$$
$$v_{4it} \equiv (1/2)v_{1t}b_{it}/\sigma_t^2 = \partial\log f(\varepsilon_t\,|\Omega_{t-1})/\partial\beta_i,$$

are Martingale difference sequences. The partial sum of

$$v_t \equiv \left(v_{0t}, v_{1t}, v_{2t}, v_{31t}, \cdots, v_{3pt}, v_{41t}, \cdots, v_{4qt}\right)$$

converges to a multivariate Brownian motion, that is,

$$n^{-1/2}\sum_{t=1}^{[ns]}v_t$$
$$\Rightarrow BM(\Sigma) \equiv \left(B_0(s), B_1(s), B_2(s), B_{31}(s), \cdots, B_{3p}(s), B_{41}(s), \cdots, B_{4q}(s)\right)$$
(18)

defined upon $0 \le s \le 1$ with covariance matrix Σ, whose elements are given by the unconditional covariance $\Sigma_{ij} = \mathrm{cov}(v_{it}, v_{jt})$, and $[ns]$ is the largest integer not greater than *ns*. Noting that $v_{1t} \equiv \varepsilon_t^{\,2}/\sigma_t^2 - 1$ is orthogonal to ε_t conditionally, the elements of Σ are given as follows:

$\Sigma_{00} = 1$, $\Sigma_{01} = \Sigma_{02} = \Sigma_{0,3i} = \Sigma_{0,4i} = 0$,
$\Sigma_{11} = 2$,
$\Sigma_{1,2} = -E[\sigma^2 w_i/\sigma_t^2]$,
$\Sigma_{1,3i} = -E[a_i/\sigma_t^2]$, $\Sigma_{1,4i} = -E[b_i/\sigma_t^2]$
$\Sigma_{22} = (\sigma^4/2)E[w_t^2/\sigma_t^4]$,
$\Sigma_{2,3i} = (\sigma^2/2)E[a_i w_i/\sigma_t^4]$,
$\Sigma_{2,4i} = (\sigma^2/2)E[b_i w_i/\sigma_t^4]$,
$\Sigma_{3i,3j} = (1/2)E[a_i a_j/\sigma_t^4]$,
$\Sigma_{3i,4j} = (1/2)E[a_i b_j/\sigma_t^4]$,
$\Sigma_{4i,4j} = (1/2)E[b_i b_j/\sigma_t^4]$,
where $\sigma^2 \equiv E[\varepsilon_t^{\,2}]$. Then,

$$n^{1/2}\left((\hat{\omega}-\omega)/\sigma^2, \hat{\alpha}_1-\alpha_1, ..., \hat{\alpha}_p-\alpha_p, \hat{\beta}_1-\beta_1, ..., \hat{\beta}_q-\beta_q\right)$$
$$\Rightarrow\left(B_2(1), B_{31}(1), \cdots, B_{3p}(1), B_{41}(1), \cdots, B_{4q}(1)\right)\Sigma_{(2,4)}^{-1},$$
(19)

where

$$\Sigma_{(2,4)} \equiv Var\left(B_2(1), B_{31}(1), \cdots, B_{3p}(1), B_{41}(1), \cdots, B_{4q}(1)\right)\cdot$$

The first term on the right-hand side of the expansion of S in (16) can be expressed as

$$n^{-1}\sum_{t=1}^{n}\left(\varepsilon_t^{\,2}/\sigma_t^2 - 1\right)z_t/\sigma \Rightarrow \int B_0(t)dB_1(t)\,. \qquad (20)$$

Then, from (17), (19), and (20), we have the expression

$$S/\sigma \Rightarrow \int B_0(t)dB_1(t) \qquad (21)$$
$$-\left(B_2(1), B_{31}(1), \cdots, B_{3p}(1), B_{41}(1), \cdots, B_{4q}(1)\right)\Sigma_{(2,4)}^{-1}$$
$$\left(E\left[\sigma^2 w_t/\sigma_t^2, a_{1t}/\sigma_t^2, \cdots, a_{pt}/\sigma_t^2, b_{1t}/\sigma_t^2, \cdots, b_{qt}/\sigma_t^2\right]\right)'\int B_0(s)ds,$$

where w_t, a_{it}, b_{it} are defined in (15) and the Brownian motion and its variance-covariance matrix are defined in (17). It is easy to see that the distribution of S/σ is independent of σ^2 and ω.

The results are summarized in the following theorem:

Theorem 1: Assume that y_t is generated by the inverse Box-Cox transformation (10) under

Assumptions 1 and 4. Under the null hypothesis that the transformation parameter λ is correct, the test statistic S in (14) has the asymptotic expression (21).

The cases where (p,q)=(0,0) and (p,q)=(1,0) require special attention in applying the formula in (21). First, consider the case where (p,q) = (0,0), namely where

$\varepsilon_t \sim NID(0, \sigma^2)$. As $\sigma_t^2 = \sigma^2 = \omega$, we have only to estimate ω, so that the asymptotic expression for S reduces to

$$S/\sigma = \int B_0(s)dB_1(s) - B_1(s)\int B_0(s)ds,$$

because $B_1(s) \equiv 2B_2(s)$ and $\sigma_t^2 = \sigma^2$.

When ε_t follows an ARCH(1) process, that is,

$\sigma_t^2 \equiv \omega + \alpha\varepsilon_{t-1}^2$,

it is necessary to estimate ω and α. In this case, the asymptotic expression for S is given by

$$S/\sigma \Rightarrow \int B_0(t)dB_1(t)$$
$$-\left(B_2(1), B_3(1)\right)\Sigma_{(2,3)}^{-1}\left(\sigma^2 E\left(w_t/\sigma_t^2\right), E\left(a_{1t}/\sigma_t^2\right)\right)' \int B_0(s)ds$$

and $\Sigma_{(2,3)} \equiv Var\left(B_2(1), B_3(1), \cdots, B_3(1)\right)\cdot$

The null hypothesis is rejected when the deviation of S from 0 is sufficiently large in absolute value.

5. ACKNOWLEDGEMENTS

The second author wishes to acknowledge the financial support of the Australian Research Council and the Center for International Research on the Japanese Economy, Faculty of Economics, University of Tokyo.

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