Bayesian Estimation of Smooth Transition GARCH Model

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Abstract : Research into time series models of changing variance and covariance, which is often called volatility model, has exploded in the last ten years. Financial series are characterised by periods of large volatility followed by periods of relative quietness. This type of clustering led to the idea that volatility is predictable. The ARCH and GARCH models were quite successful in predicting volatility compared to more traditional methods. But better predictions are obtained when asymmetries and nonlinearities in the the response of volatility to news arriving on the market are taken into account. In this paper we propose a new kind of asymmetric GARCH in which the conditional variance obeys two different regimes with a smooth transition function. In this model, the conditional variance reacts differently to negative and positive shocks and its magnitude on shocks have separate effects. As financial data have very often a high frequency of observation, smooth transition seems a priori better than an abrupt transition. The change of regime occurs when the residuals cross the threshold zero. This threshold GARCH models can be generalized using a smooth transition function $F_T(\eta, s_t)$ taking continuous values between zero and one. We treat the joint point t^* and the speed of adjustment η to be two unknown parameters.

Keywords : MCMC; Asymmetric GARCH; Nonlinear modelling; Smooth transition regime; Financial time series

1. INTRODUCTION

Research into time series models of changing variance and covariance, which is often called volatility model, has exploded in the last ten years. Financial series are characterised by periods of large volatility followed by periods of relative quietness. This type of clustering led to the idea that volatility is predictable. The ARCH and GARCH models introduced respectively by Engle [1982] and Bollerslev [1986] were quite successful in predicting volatility compared to more traditional methods. But better predictions are obtained when asymmetries and nonlinearities in the the response of volatility to news arriving on the market are taken into account. The "leverage effect" that commonly represents the asymmetric impact of good news and bad news on volatility has certainly be the most widely explored with Nelson [1991] who proposed his EGARCH model or Glosten et al. [1993] who proposed so called 'GJR' model and also Zakonian [1994] who proposed a threshold GARCH models. Engle and Ng [1993] provided a summary of asymmetric GARCH models and introduced some new formulations. From the specifications to investigate regime switching models with GARCH errors, Teräsvirta (1994) discussed a STAR-GARCH type model from the classical point of view. Chan and McAleer [2000] discussed the quasi-maximum likelihood estimation of these smooth transition autoregressive models with GARCH errors and also Li, Ling, and McAleer [2001] reviewed some recent theoretical results of GARCH errors. Recently, Bayesian treatment of the ARCH model first given by Geweke [1988,1989]. Bayesian inference on GARCH model was performed by Kleibergen and van Dijk [1993] and Bauwens and Lubrano [1998], Bauwens, et al. [1999], and Lubrano [1998] for asymmetric GARCH models with smooth transition regimes.

In this paper we explains how the Gibbs sampler can be used to perform Bayesian inference on GARCH models and proposes a new kind of asymmetric GARCH in which the conditional variance obeys two different regimes with a smooth transition function. In our model, the conditional variance reacts differently to negative and positive shocks and its magnitude on shocks has separate effects. As financial data have very often a high frequency of observation, smooth transition seems a priori better than an abrupt transition.

2. GARCH MODEL WITH STUDENT ERRORS

We wish to conduct Bayesian inference on regression model with GARCH errors. To ease exposition, we consider a parsimonious model which is quite representative of GARCH models used in finance fields. It is a GARCH(1,1) model with Student-t errors (t GARCH). As reported in Bollerslev et al. [1992], the t density, with its fat tails, is more able than the normal to account for excess kurtosis present in the financial data. The model is

$$\begin{cases} y_t = \epsilon_t \sqrt{h_t}, & t = 1, 2, \dots, T \\ \epsilon_t | I_{t-1} \sim t(0, 1, \nu) & \\ h_t = \omega + \alpha y_{t-1}^2 + \beta h_{t-1} & \end{cases} \tag{1}$$

The ϵ_t is assumed independent. The distribution of y_t, given the past information I_{t-1}, is t with mean zero, and variance $h_t \nu/(\nu - 2)$ (assuming $\nu > 2$). The initial variance h_0 is treated as a known constant. To ensure the positivity of h_t, the parameters of the variance equation are restricted by

$$\omega > 0, \;\; \alpha > 0, \;\; \beta > 0. \tag{2}$$

Other restrictions may be considered: for the y_t process to be covariance stationary, one must impose that

$$0 \le \alpha \frac{\nu}{\nu - 2} + \beta < 1 \quad (\text{assuming} \quad \nu > 2). \tag{3}$$

Let θ denote the parameter vector $(\omega, \alpha, \beta, \nu)$. For a sample of T observations, posterior density is

$$\varphi(\theta|y) \propto \varphi(\theta)\ell(\theta|y) \tag{4}$$

where the likelihood function is given by

$$\ell(\theta|y) \propto \prod_{t=1}^{T} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} (\nu h_t)^{-\frac{1}{2}} \left[1 + \frac{y_t^2}{\nu h_t} \right]^{-\frac{\nu+1}{2}}. \tag{5}$$

The prior density $\varphi(\theta)$ should respect at least the positivity restrictions on the parameters, and the condition $\beta < 1$, stationarity condition. One important issue of Bayesian inference is the integrability of the posterio density. If a proper (integrable) prior is used, and the likelihood function is reasonable, the posterior is proper. However, a non-integrability of the posterior may arise with an improper prior. If we consider the behavior of the likelihood function (5) with respect to the parameter ω, α, and β, we see that reasonableness arises if every h_t is strictly positive, as the t density is finite and positive under this condition. A sufficient prior information is needed on ν to force the posterior to tend to zero quickly enough at the tail, in order to be integrable.

3. GRIDDY-GIBBS SAMPLER FOR GARCH MODELS

The Gibbs sampler applied to $\varphi(\theta|y)$ produces a Markov chain $\{\theta^n\}_{n=1}^N$ of draws with equlibrium distribution $\varphi(\theta|y)$. The draw can be used to estimate consistently characteristics of the posterior(moments or marginal densities). The procedure is started using any value θ_i^0 in the support of the posterior distribution. The influence of the starting conditions vanishes after a certain number of draws (burn in). If the algorithm converges after burn in, it produces random draws of the posterior density of θ. The sample is not independent because of the Markov chain structure.

3.1 Griddy-Gibbs Sampler

This algorithm cannot be applied to the GARCH model even if the error term is normal. Consider the simple following ARCH model:

$$\begin{cases} y_t = \mu + \epsilon_t \sqrt{h_t} & t = 1, 2, \dots, T \\ \epsilon_t | I_{t-1} \sim N(0, 1) & \\ h_t = 1 + \alpha (y_{t-1} - \mu)^2 & \end{cases} \tag{6}$$

There are two parameters, μ for the regression function, and α for the skedastic function. Let us assume a flat prior. The conditional posterior density $\varphi(\mu|\alpha, y)$ has a following kernel when α is given fixed value.

$$\kappa(\mu|\alpha, y) = \prod_t [h_t(\alpha, \mu)]^{-1/2} \exp - \frac{1}{2} \frac{(y_t - \mu)^2}{h_t(\alpha, \mu)} \tag{7}$$

If h_t were fixed, (7) would be a normal density. As h_t is the function of both α and μ, the conditional posterior density of μ contains h_t which is also a function of μ. Consequently, it cannot be a normal or any other well known density from which random number could be easily generated. There is no property of conjugacy.

As numerical integration has to be performed on α, the difficulty comes from the fact that the posterior density of α does not belong to a known class of densities. To simulate random draws, several possibilities are open: griddy-Gibbs sampler, Metropolis-Hastings algorithm, and Importance sampling. For the last two methods, an importance function has to be defined for multiplicative heteroscedasticity. We select the first one for this research.

The kernel of $\varphi(\mu|\alpha, y)$, conditionally on the previous draw of the conditioning parameter, can be evaluated over a grid of points. One can compute the corresponding distribution function using a deterministic integration rule. Afterwards, one can generate a draw of μ by inversion of the distribution at a random value sampled uniformly in $[0, 1]$. This technique, called the Griddy-Gibbs sampler, was applied by Ritter-Tanner [1992] to bivariate posterior densities, but the method is really useful if it can deal with higher dimensional posterior densities.

It is easy to incorporate conditioning, (a variance reduction technique), in the algorithm. Conditioning means that to estimate $E(\alpha|y)$, one uses $\sum_{n=1}^{N} E(\alpha|\mu^n, y)/N$ instead of $\sum_{n=1}^{N} \alpha^n/N$. Conditioning is very useful in order to get accurate graphs of the marginal densities for a relatively small number of draws, as one computes $\varphi(\alpha_*|y) = \int \varphi(\alpha_*|\mu, y)\varphi(\mu|y)d\mu$, (where α_* denotes one point of the grid of values of α) by

$$\varphi(\alpha_*|y) \approx \sum_{n=1}^{N} \varphi(\alpha_*|\nu^n, y)/N \tag{8}$$

In our several experiments, it appears that Griddy-Gibbs algorithm is successful in dealing with the skewness of the posterior. This is due to the fact that integration is done on a grid so that every direction can be explored in detail. The main cost of the method is the evaluation of the posterior density kernel.

4. AN ASYMMETRIC GARCH MODEL FOR EXCHANGE RATE DATA

To model an observed series, the simple GARCH model (1) cannot capture some well known stylized facts. First, the model implies that the series has a constant zero mean. Second, the model implies that a shock influences the next conditional variance only by its squared magnitude, not by its sign: the news impact curve, i.e. the graph of h_t on the past shock, y_{t-1}, is symmetrical. This is at odds with the so-called 'leverage effect', which referes to the negative correlation betweeen changes in prices and volatility. Several specifications has been proposed to introduce asymmetry, such as the exponential GARCH (EGARCH) model by Nelson [1991], the 'GJR' Model of Glosten et al.[1993], and threshold ARCH model of Zakonian [1994]. There is no need to put positivity constraint on parameters. GJR type model was used by Engle and Lee [1992], who allow asymmetry to enter the transitory component of volatility, but not the permanent part.

The estimated model allows for a non-zero first-order autocorrelation of the series, and a following GJR-type asymmetric Student-GARCH model.

$$\begin{cases} y_t = \mu + \rho y_{t-1} + u_t \\ u_t = \epsilon_t \sqrt{h_t} \\ \epsilon_t / I_{t-1} \sim t(0, 1, \nu) \\ h_t = \omega + \alpha^+ u_{t-1}^{2+} + \alpha^- u_{t-1}^{2-} + \beta h_{t-1} \\ u_t^{2+} = u_t^2 I_{(u_t > 0)}, \quad u_t^{2-} = u_t^2 I_{(u_t \leq 0)} \end{cases} \tag{9}$$

Using daily data on the exchange rate of Japanese Yen against U.S. dollar for the period 03/04/95–31/03/99, we estimate above model. The data consists of 1010 observations. The dependent variable is the return, measured as the first difference of the logarithm of the exchange rate at time t.

Table 1: ML and posterior results on asymmetric GARCH model for exchange rate

Error dens. Prior(ν)	Student t none(ML)	Student t flat	normal none
μ (.01, .20)	0.0864 [0.0222]	0.0868 [0.0224]	0.0569 [0.0280]
ρ (-.2, .15)	-0.0295 [0.0298]	-0.0300 [0.0305]	0.0456 [0.0289]
ω (.0001, .0055)	0.0019 [0.0016]	0.0027 [0.0014]	0.0931 [0.0518]
α^+ (.01, .04)	0.0216 [0.0087]	0.0257 [0.0073]	0.0885 [0.0325]
α^- (.01, .35)	0.0179 [0.0061]	0.0220 [0.0066]	0.1874 [0.0828]
β (.90, .99)	0.9594 [0.0116]	0.9506 [0.0098]	0.7713 [0.0903]
ν (2.0, M)	3.9281 [0.5245]	4.1110 [0.4243]	∞

Entries in the first column of result (ML) are maximum likelihood estimates and asymptotic standard errors. In the last two columns, they are posterior means and standard deviation (between brackets, under the means). Posterior moments were computed by conditioning (i.e. marginalizing conditional moments). Under the parameter names are the prior intervals (integration ranges). The value of M is 50 for the flat prior.

All the results were computed with the Griddy-Gibbs sampler using 2000 draws after burn in. Three kinds of priors on finite intervals were used. But, the results do not differ a lot. Positive shocks have a stronger impact than negative shocks in Student model. When a symmetric model is estimated, the posterior mean of α

is approximately equal to average of the posterior means of α^+ and α^- in the corresponding asymmetric model. The difference has a posterior mean equal to 0.0037 with a standard deviation equal to 0.0089 (not shown in the table). When a normal density is used ($\nu = \infty$), the estimates of α^*, α^- and β are slightly higher. This adjustment accomodates for the excess kurtosis in the data, since in Student case the degrees of freedom take up this effect.

5. GENERAL CLASS OF SMOOTH TRANSITION GARCH MODEL

The simple GARCH(1,1) is certainly the most widely used model for predicting the volatility of financial series. The regression model with GARCH(1,1) errors can be written as follows:

$$\begin{cases} y_t = x_t'\delta + u_t & t = 1, 2, \dots, T \\ u_t = \epsilon_t\sqrt{h_t} & \epsilon_t \sim t(0, 1, \nu) \\ h_t = \omega + \alpha u_{t-1}^2 + \beta h_{t-1} & \end{cases} \tag{10}$$

Most of the time, x_t contains a constant and lagged values of y_t. In (10), the conditional expectation of y_t is $x_t'\delta$ and the unpredicted part of y_t is $y_t - x_t'\delta$. This represents the "news" arriving on the market. In GARCH model, news have a symmetric impact on volatility, whatever their sign or magnitude and whatever the level of y_t. This model is used as a starting point to introduce asymmetry and level effect on volatility.

A typology of threshold regression models is provided by the nature of the switching function $F_T(\cdot)$, which can be step or smooth. A step transition function is built aound a Dirac function which is zero when a linear combination of x_t is negative and one otherwise. A smooth transition function replaces the Dirac function by a cumulative distribution which is mostly the logistic function, as advocated by Teräsvirta (1994). These function, taking values in $[0,1]$, have the particularity that their value changes not only according to the sign of the linear combination of x_t, but also according to its value.

5.1 Gradual Switching Model

The model of Glosten et al. [1993] represented as follows:

$$\begin{aligned} h_t = &\ \omega + \alpha_1 u_{t-1}^2(1 - S_{t-1}) \\ &+ \alpha_2 u_{t-1}^2 S_{t-1} + \beta h_{t-1} \end{aligned} \tag{11}$$

S_t is an indicator function that is zero when u_t is negative and one otherwise. This formulation introduces asymmetry of reaction for the conditional variance. The change of regime occures when u_t crosses the threshold zero. This threshold GARCH models can be generalized using a smooth transition function $F_T(\eta, s_t)$ taking continuous values between zero and one. We treat the joint point t^* and the speed of adjustment η to be two unknown parameters. Using the transition function that was introduced in Tsurumi [1980] and applied in Tsurumi et al. [1986], the two regime volatility function in (11) becomes:

$$\begin{aligned} h_t &= \omega + \alpha_1 u_{t-1}^2[1 - F_T(\eta, s_t)] \\ &\quad + \alpha_2 u_{t-1}^2 F_T(\eta, s_t) + \beta h_{t-1} \qquad (12) \\ &= \omega + \alpha_1 u_{t-1}^2 + \lambda u_{t-1}^2 F_T(\eta, s_t) + \beta h_{t-1} \end{aligned}$$

where $\lambda = (\alpha_2 - \alpha_1)$. The transition function F_{Trn} satisfies

$$\begin{aligned} \lim_{S_t \to \infty} S_{Trn}(s_t/\eta) &= 1 \\ S_{Trn}(0) &= 0 \\ \lim_{\eta \to 0} S_{Trn}(s_t/\eta) &= 1 \end{aligned} \tag{13}$$

and s_t is given by

$$s_t = \begin{cases} 0 & t \leq t^* \\ t - t^* & t > t^* \end{cases} \tag{14}$$

We use the hyperbolic tangent as the transition function since it is easy to use and since posterior inference on join point t^* and the speed of adjustment η seems to be insensitive to the choice of transition function.

5.2 Smooth Transition Between Negative and Positive Shocks

Among the many possible smooth transition functions, the logistic function was proved to be very convenient in classical nonlinear modelling. For gradual switching GARCH model where the objective is to allow for a possible difference of reaction between negative and positive shocks, this function is, for example like this

$$F(\gamma, u_{t-1}) = \frac{1}{1 + \exp(-\gamma u_{t-1})} \tag{15}$$

The function $F(\cdot)$ tends to zero when $u \to -\infty$ and to one for $u \to +\infty$. So α_1 will characterise negative shocks and α_2 positive ones.

5.3 Smooth Transition Between Small and Big Shocks

Another transition function like the exponential function

$$F(\gamma, u_{t-1}) = 1 - \exp(-\gamma u_{t-1}^2) \tag{16}$$

was made popular for modelling size asymmetrics in models presenting a non-linearity in the mean. Here $F(\pm\infty) = 1$ and $F(0) = 0$. So α_1 will characterise small shocks and α_2 big shocks. For $\gamma \to \infty$, $F(\cdot)$ becomes an indicator function for the point $u = 0$, which makes our model equivalent to symmetric GARCH.

An improvement over this function that says that F goes to zero if u belongs to the interval $[-c, c]$ and goes to one otherwise is give by

$$F(\gamma, u_{t-1}) = \frac{1 - \exp(-\gamma u_{t-1}^2)}{1 + \exp[-\gamma(u_{t-1}^2 - c^2)]} \qquad (17)$$

Introduing an extra parameter, the threshold c, that determines at which magnitude of past errors the change of regime occurs.

6. EMPIRICAL RESULTS

In order to apply smooth transition GARCH model, we use two kinds of daily data on exchange rate for Thailand Baht and Indonesian Rupia both against Japanese Yen. These data are shown for the period 1/3/95-12/31/99 (Baht) (Figure 1), 11/16/95-12/31/99 (Rupia) (Figure 2).

First, using these data we estimate GJR-type asymmetric Student GARCH model (9) by each of two regimes and compare their impact for shocks. The results are shown in Table 2 for Baht and Table 3 for Rupia.

- We can see that after Asian money crisis, volatility of Baht change abruptly between two regimes, but volatility of Rupia change gradualy.
- From the degrees of freedom parameter (ν) we find Student model is appropriate except 2nd regime for Baht.
- Positive shocks have more strong impact than negative shocks in the second regime. The difference gets bigger after the second regime.
- The 2nd regime estimates of positive shocks α^+ are higher than 1st regime and both shocks are the same for Baht and positive shocks are higher than negatives in Rupia.

Second, we estimate the following GJR type threshold normal smooth transition GARCH model.

$$\begin{cases} y_t = \mu + u_t \\ u_t = \epsilon_t \sqrt{h_t} \qquad \epsilon_t / I_{t-1} \sim N(0,1) \\ h_t = \omega + \alpha u_{t-1}^2 + \lambda u_{t-1}^2 * \tanh(S_t/\eta * \pi) \\ \qquad + \beta h_{t-1} \\ S_t = \begin{cases} 0 & t \le t^* \\ t - t^* & t > t^* \end{cases} \end{cases} \qquad (18)$$

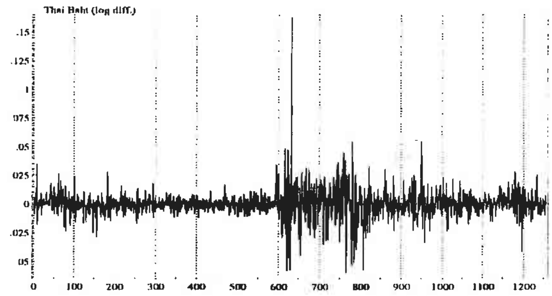

Figure 1: Thailand Baht (log-diff.)

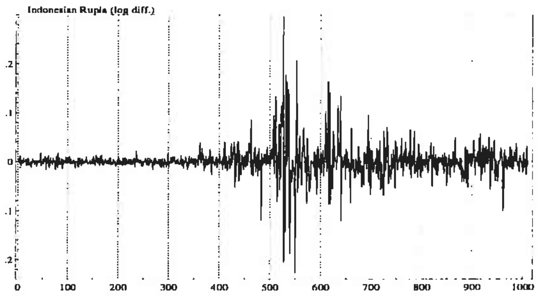

Figure 2: Indonesian Rupia (log-diff.)

The results are shown in Table 4 for Baht and Rupia.

- Both results do not differ a lot, especially for adjustment speed, but joint point is different.
- We can show the marginal disibution for each parameters. From which we can see the figure of distribution. Figure 3 and 4 are posterior distribution of t^* and λ for Baht. Figure 5 and 6 are posterior distribution of t^* and λ for Rupia.

Table 2: Estimation Results for Asymmetric GARCH Model (Baht)

	1st Regime		2nd Regime	
	Baht	Std. Err	Baht	Std. Err
μ	0.05682	0.02170	0.04530	0.03088
ρ	-0.1798	0.04062	0.11360	0.04144
ω	0.03647	0.01565	0.01707	0.00667
α^+	0.09504	0.03696	0.15940	0.04050
α^-	0.09740	0.03974	0.04636	0.02540
β	0.70310	0.08148	0.85130	0.03215
ν	3.32800	0.5998	9.642	4.101

Table 3: Estimation Results for Asymmetric GARCH Model (Rupia)

	1st Regime		2nd Regime	
	Rupia	Std. Err	Rupia	Std. Err
μ	-0.05945	0.02966	0.01973	0.02209
ρ	-0.1351	0.05036	0.00475	0.04546
ω	0.1194	0.04237	0.05970	0.01843
α^+	0.09362	0.04563	0.28050	0.08077
α^-	0.06029	0.03947	0.1219	0.06020
β	0.67650	0.09161	0.44120	0.1143
ν	5.04600	1.5288	3.088	0.4769

Table 4: Smooth Transition GARCH Model (Baht, Rupiah)

	Baht	Std. Err	Rupiah	Std Err
μ	-0.01278	0.01746	-0.01120	0.01094
t^*	505.1	38.09	323.1	17.20
ω	0.02424	0.00807	0.00610	0.00096
α	0.08889	0.02866	0.07276	0.05267
β	0.8274	0.03752	0.8245	0.02039
λ	0.08481	0.01865	0.1707	0.04498
η	21.98	10.06	21.81	8.879
T	1253		1009	

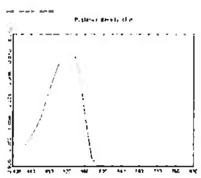

Figure 3: Posterior Density of t^* (Baht)

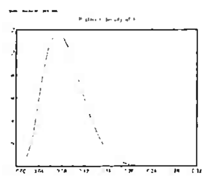

Figure 4: Posterior Density of λ (Baht)

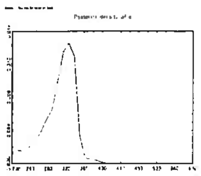

Figure 5: Posterior Density of t^* (Rupia)

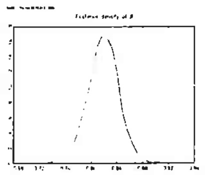

Figure 6: Posterior Density of λ (Rupia)

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