Macroeconomic Policy Development with a Separate Predictive Model

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Abstract: This paper examines the issue of the generation of optimal control policies where there are explicit constraints upon the control values and there is limited knowledge of the complex economic system. The paper develops a methodology where the constrained optimal control is based upon a separate model that predicts the policy targets for the economic system. The methodology as applied to a macroeconomic model.

Keywords: Constrained optimal control; Model based predictive control; Quadratic programming

1. INTRODUCTION

This paper extends the use of optimal control to the situation where there are explicit constraints on the controls. It is also concerned with the case where there is limited knowledge of the economic system. Linear quadratic optimal control has been used extensively in the analysis of macroeconomic policy [see for example, Kendrick, 1981; Neck and Matulka, 1994]. Generally though, economic systems are not linear [see for example, Powell and Murphy, 1997], and social loss functions are not quadratic [see for example, Orphanides and Wieland, 2000]. The linear model and quadratic social loss function are used to approximate the economic system and policy objectives. In this paper we explicitly take into account this approximation as limited knowledge of the economic system.

Linear quadratic optimal control can generate control policies that are not achievable in an economy. To allow for this under linear quadratic optimal control the control is limited through the choice of weights in the social loss function. In this paper we also introduce constraints upon the control variables so that they may be limited to economically realistic values.

The paper develops a methodology whereby the constrained optimal control is based upon a separate learning model that predicts the policy targets of the economic system. A subset of the control is applied to the economic system and the methodology updates the predictive model before developing the next set of constrained optimal controls. The basis of the approach is similar to a range of techniques developed in the control engineering for large complex processing systems. The approach developed here fits with the class of model predictive control and model algorithmic control.

2. THE PROBLEM

The overall approach is to develop a separate model that learns to behave like the economic system with respect to the control and policy target variables. This follows the approach of Herbert [1998], but in this case the model is used to predict future states of the economic system. Using the predicted states an optimal control strategy is developed and a subset of this strategy is applied to the economic system. After the control is implemented the predictive model is updated based on the new observed states of the economic system and a new set of optimal controls is generated. The overall process then continues.

The optimal control is based upon a prediction of the behaviour of the states of the economic system for a window of time. We use a standard discrete-time approach. Let $\hat{\mathbf{x}}(k+j|k)$ be the prediction of the states of the economic system at the time point $k+j$ made at the time point k. The state predictions are made for a fixed length window of time, so that time-points vary from the beginning of the time-window, N_s, to the end of the time-window, N_f. In the control strategy this window will slide over time.

Knowledge is limited about the states of the economic system, and we use predications based upon a standard, state-based, linear, time-varying, model. For the next time-step in a particular time-window, this model is of the form:

$$\begin{aligned} \hat{\mathbf{x}}(k+1|k) &= A(k)\hat{\mathbf{x}}(k|k) + B(k)\mathbf{u}(k) \\ &\quad + C(k)\hat{\mathbf{z}}(k|k) \end{aligned} \tag{1}$$

where

- $\hat{\mathbf{x}}$ represent the vector of predicted states of the economic system;
- $\mathbf{u}$ represent the vector of control variables; and,
- $\hat{\mathbf{z}}$ represent the vector of predicted exogenous variables;

The initial conditions of the prediction model of Equation 1 are known, with $\hat{\mathbf{x}}(1|k) = \mathbf{x_1}$. The predicted values of the exogenous variables are known for the entire time-window. The time-varying parameter matrices, $A(k)$, $B(k)$ and $C(k)$, are also known. They may be time-varying through the introduction of a discount factor so that future time periods are discounted in the control strategy.

The optimal control strategy is based upon the predictive model and is applied to the economic system. The control strategy is based on a subset of the predicted states, and, further, a subset of the control strategy is applied to the economic system.

For the control strategy, desired values of the states of the economic system, $\mathbf{x_d}$, are assumed to be known for the predictive time-window.The aim of the control strategy is to move the economic system to this desired state trajectory, so that the desired values for the controls, $\mathbf{u_d}$, are those that maintain $\mathbf{x_d}$ in the steady-state. These can be calculated.

As is common with many economic systems, we assume that there are more target states than there are control variables. Thus trade-offs are necessary. In the unconstrained case, we use the least squares approach of minimising the Euclidean norm.

In the constrained case, the optimal control strategy is based upon minimising the cost (or social loss) function given by:

$$\begin{aligned} J(k) &= \frac{1}{2}\sum_{i=N_s}^{N_f} ||\mathbf{x_d}(k+i) - \hat{\mathbf{x}}(k+i|k)||^2_{Q(i)} \\ &\quad + \frac{1}{2}\sum_{i=N_s}^{N_f} ||\mathbf{u_d}(k-1+i) \\ &\quad - \mathbf{u}(k-1+i)||^2_{R(i)} \end{aligned} \tag{2}$$

That is, the sum of weighted, squared, deviations from the desired values. The cost function is subject to level constraints on the controls. These are upper and lower bounds and are given by:

$$\underline{\mathbf{u}}(i) \le \mathbf{u}(i) \le \overline{\mathbf{u}}(i) \tag{3}$$

The cost function can also be subject to rate changes on the controls, so that not only is the control level bounded but the amount by which it may move is limited. These constraints are given by:

$$\mathbf{u}(i) - \mathbf{u}(i-1) \le \Delta_u(i) \tag{4}$$

Notice that bounds on the levels and rates of change may vary with time.

In the cost function, the time-varying weighting matrices, $Q(i)$ and $R(i)$ are known. They may, again, include a discount factor for future time-periods.

Of course, the problem is to find the controls that satisfy Equations 1 through 4. The approach taken is to first stack-up the predictive model over time. In the case of the model being time-invariant and for a time-window from $N_s = 1$ to $N_f = n$, the stacked-up model can be written as:

$$X = G\hat{\mathbf{x}}(1) + FU + EZ \tag{5}$$

where

$$X = \begin{bmatrix} \hat{\mathbf{x}}(2) \\ \hat{\mathbf{x}}(3) \\ \hat{\mathbf{x}}(4) \\ \vdots \\ \hat{\mathbf{x}}(n) \end{bmatrix}$$

$$G = \begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^n \end{bmatrix}$$

$$F = \begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ A^2B & AB & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ A^{n-1}B & A^{n-2}B & A^{n-3}B & \cdots & B \end{bmatrix}$$

$$U = \begin{bmatrix} \mathbf{u}(1) \\ \mathbf{u}(2) \\ \mathbf{u}(3) \\ \vdots \\ \mathbf{u}(n) \end{bmatrix}$$

$$E = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ AC & C & 0 & \cdots & 0 \\ A^2C & AC & C & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ A^{n-1}C & A^{n-2}C & A^{n-3}C & \cdots & C \end{bmatrix}$$

$$Z = \begin{bmatrix} \hat{\mathbf{z}}(1) \\ \hat{\mathbf{z}}(2) \\ \hat{\mathbf{z}}(3) \\ \vdots \\ \hat{\mathbf{z}}(n) \end{bmatrix}$$

In the time-varying case, A^n is replaced by:

$$\prod_{i=1}^{n} A(i) \tag{6}$$

and an appropriate time index is used for B and C, with, for example, $B(i)$ and $C(i)$ on the main diagonal.

To obtain a desired trajectory of states so that $X = X_d$, then the desired controls are given by:

$$FU_d = X_d - EZ - G\hat{\mathbf{x}}(1) \tag{7}$$

In the unconstrained case, the optimal control in a least squares sense is given by:

$$U_{opt} = (F^TF)^{-1}F^T(X_d - EZ - G\hat{\mathbf{x}}(1)) \tag{8}$$

For the constrained case, Equation 2 can be re-written as:

$$\begin{aligned} J &= \frac{1}{2}(X_d - X)^T H_1 (X_d - X) \\ &\quad + \frac{1}{2}(U_d - U)^T H_2 (U_d - U) \end{aligned} \tag{9}$$

Substituting for X from Equation 5 and for X_d from Equation 7 into Equation 9 gives:

$$\begin{aligned} J &= \frac{1}{2}(FU_d)^T H_1 FU_d + \frac{1}{2}(FU)^T H_1 FU \\ &\quad - (FU_d)^T H_1 FU + \frac{1}{2}U_d^T H_2 U_d \\ &\quad + \frac{1}{2}U^T H_2 U - U_d^T H_2 U \end{aligned} \tag{10}$$

Now the first and fourth terms in Equation 10 are constants, and thus can be left out of the minimisation process. Thus the cost function to be minimised can be written as:

$$\begin{aligned} \mathcal{J} &= \frac{1}{2}U^T(F^T H_1 F + H_2)U \\ &\quad - U_d^T(F^T H_1 F + H_2)U \end{aligned} \tag{11}$$

Minimising the cost function of Equation 11 subject to the constraints of Equations 3 and 4 is a quadratic programming problem. This generates the constrained optimal control. Once the control is generated for the time-window, a subset is applied to the economic system. The time-window is moved, a new predictive model generated, new controls are generated, and the process continues.

3. NUMERICAL APPLICATION

3.1 A Dynamic Model

The model used in the following examples is based upon the model Extended Dornbush Model (EDBM) of Powell [2000]. The aim of that model is to generate a calibrated, stylised model that has monetary dynamics that behave in the same manner as the larger (100 equation) Murphy macroeconometric model of the Australian economy [Murphy, 1988*a*; Murphy, 1988*b*; Powell and Murphy, 1997]. Both models have rational expectations and rapid short-run adjustments in their financial markets. Both models incorporate key features of the Dornbush [1976] model.

The model used here modifies the EDBM. Firstly, the dynamics of actual aggregate demand (GDP) are expanded as the IS curve is not contemporaneous with actual aggregate demand. Secondly, the dynamics of the domestic price level and exchange rate are developed by discretising the rate

Table 1. Model Equations

Equations of Extended Dornbush Model
Short-run disequilibrium equation
Actual Aggregate Demand
$a(t) = \psi A(t-1) + \varphi(t)(1-\psi)a(t-1)$ $+(1-\varphi(t))(1-\psi)a(t-2)$
Convex weighting coefficient
$1 \leq \psi \leq 0$
Lag weighting function
$\varphi(t) = \chi \exp(-\rho t) \qquad (\rho \geq 0)$
Short-run equilibrium equations
Equilibrium IS Curve
$A = \mu g + \delta(p^* - e - p) - \sigma r + \gamma y + \tau y^*$
LM Curve
$m - p = \phi y - \lambda r$
Uncovered Interest Parity
$r = r^* + x$
Expected rate of currency depreciation
$x = -\dot{e}$
Phillips Curve
$\dot{p} = \pi(a - y)$
Long-run equations
Neutrality of Money
$p(T) = m(T) - y(T)$
Purchasing Power Parity
$p(T) = p^*(T) - e(T)$

of inflation, $\dot{p}$, and the rate of appreciation of the currency, $\dot{e}$. A backwards-difference approximation has been used.

The structural form of the model is given in Table 1; the variables used in the model are given in Table 2; and the parameters used in the following simulations are given in Table 3.

The resultant model is a second-order, linear, time-varying, discrete model in actual aggregate demand, a, the price level, p, and the exchange rate, e. It can be written as:

$$\mathbf{x}(k+1) = A(k)\mathbf{x}(k) + B\mathbf{u}(k) + C\mathbf{z}(k) \qquad (12)$$

with

- $\mathbf{x}(k) = [a(k), p(k), e(k), a(k-1)]^T$ representing the states;
- $\mathbf{u}(k) = m(k)$ representing the controls; and,
- $\mathbf{z}(k) = [y(k), g(k), p^*(k), y^*(k), r^*(k)]^T$ representing the exogenous variables

Table 2. Model Variables

Variables of Extended Dornbush Model	
Variable	Description
a	real aggregate demand (state)
g	real aggregate government expenditure (exogenous)
e	currency spot exchange rate (state) (foreign currency price of a unit of domestic currency)
m	nominal quantity of money (control)
p	price level (state)
p^*	foreign price level (exogenous)
$\dot{p}$	rate of price inflation
r	nominal interest rate
r^*	foreign nominal interest rate (exogenous)
x	expected rate of depreciation of currency
y	real aggregate supply (exogenous) (permanent income or sustainable aggregate demand)
y^*	real income of rest of the world (exogenous)

The parameter matrices are:

$$\varphi(k) = \chi e^{-\rho k} \qquad (13)$$

$$\xi_0 = \varphi(k)(1-\psi) \qquad (14)$$

$$\xi_1 = -\psi\left(\delta - \frac{\sigma}{\lambda}\right) \qquad (15)$$

$$\xi_2 = (1-\varphi(k))(1-\psi) \qquad (16)$$

$$A(k) = \begin{bmatrix} \xi_0 & \xi_1 & -\psi\delta & \xi_2 \\ \Delta\pi & 1 & 0 & 0 \\ 0 & \frac{-\Delta}{\lambda} & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad (17)$$

$$B = \begin{bmatrix} \frac{-\psi\sigma}{\lambda} \\ 0 \\ \frac{\Delta}{\lambda} \\ 0 \end{bmatrix} \qquad (18)$$

$$C = \begin{bmatrix} \xi_1 & \psi\mu & \psi\delta & \psi\tau & 0 \\ -\Delta\pi & 0 & 0 & 0 & 0 \\ \frac{-\Delta\phi}{\lambda} & 0 & 0 & 0 & \Delta \end{bmatrix} \qquad (19)$$

where Δ is the step-size used in the discretisation.

The model has dynamics similar to that of EDBM (and the Murphy Model). It may be considered as

Table 3. Model Parameter Values

Parameters for EDBM	
Parameter	Value
ψ	0.75
χ	3.5
ρ	0.5
μ	0.3
δ	1.1
σ	0.2
γ	0.5
τ	0.25
ϕ	0.5
λ	7
π	0.75
Δ	1

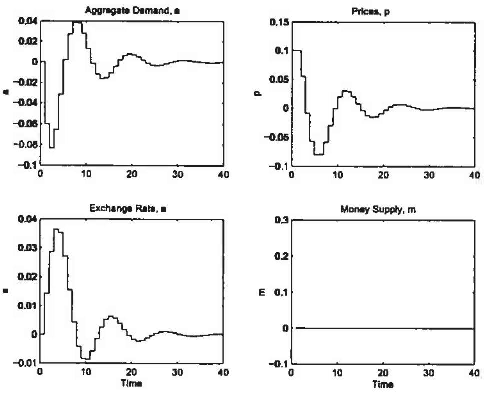

Figure 1. Model response to no control. Open-loop initial condition response.

a calibrated model with time periods of a quarter of a year. With the parameters as in Table 3 the eigenvalues of the state equation driving parameter matrix vary from $\lambda(A(0)) = [0.6091, 0.6408 \pm 0.7711i, 0.7657]$ to $\lambda(A(40)) = [-0.3058, 0.7502 \pm 0.4454i, 0.8053]$ so that the transient dynamics are stable but oscillatory.

3.2 Application

In this application the economic system is the model developed in Section 3.1. The scenario considered is that where a price shock is introduced to the economic system. This is a scenario that has recently occurred in Australia with the introduction of the Goods and Services Tax. In the scenario the economic system is at a steady-state, the price shock occurs and then the economic system moves to a new steady-state.

For this scenario, the model is assumed to be expressed in terms of deviations about the new

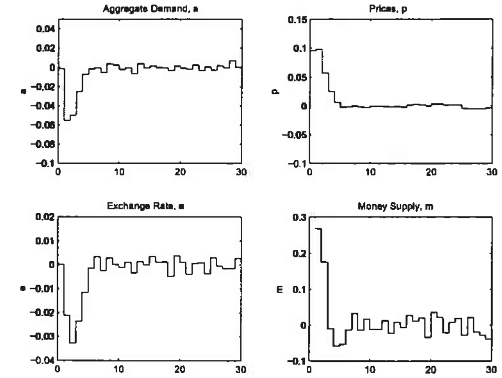

Figure 2. Unconstrained control strategy.

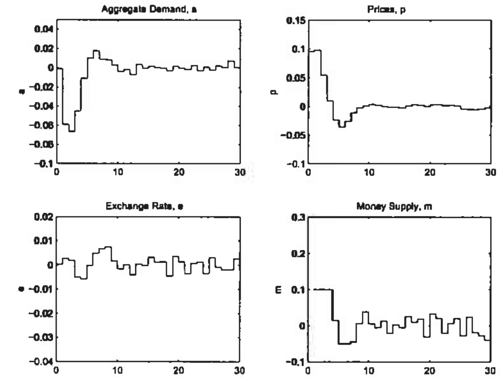

Figure 3. Constrained control strategy.

steady-state. The aim of the control strategy is to move the states and controls to zero from the initial price shock. The price shock occurs at time zero, $k = 0$.

The economic system we use is the model with the addition of random shocks. This can be written as:

$$\mathbf{x}(k+1) = A(k)\mathbf{x}(k) + B\mathbf{u}(k) + \epsilon(k) \qquad (20)$$

where

- $\mathbf{x}$ representing the states as deviations from their respective post-shock steady-state; and,
- $\mathbf{u}$ representing the control as deviations from its post-shock steady-state value.
- ϵ idd$(0, \Sigma)$ represents the random shocks and is white Gaussian noise.

The states of the economic system with no control policy is given in Figure 1.

The model used for the basis of the control strategy is the deterministic, time-invariant version of the model. That is:

$$\hat{\mathbf{x}}(k+1|N_s) = \bar{A}\hat{\mathbf{x}}(k|N_s) + B\mathbf{u}(k) \qquad (21)$$

with

$k \in (N_s, N_f) = (1, 10)$, being the time-window, and

$\bar{A}$, the post-shock, steady-state value of the state driving parameter matrix.

The application compares the constrained and unconstrained optimal control strategies under this scenario. In the unconstrained case the control strategy is created by solving Equation 8 for the time-window.

In the constrained case, the control strategy is developed by minimising Equation 11 over the time-window subject to the constraints on the controls (Equation 3). Future time periods are discounted in the weighting matrices of deviations of states from their desired values. In the application $Q(k) = \varphi(k)[[1,1]^T[1, 0.001]^T]^T$. That is, the time-discount factor is the same as for the EDBM model. Further, the three endogenous variables are weighted equally, the extra delayed state $(a(k-1))$ is weighted much less than its counterpart $(a(k))$. The deviation from the desired control is not weighted, so that $R(k) = 0, \forall k$. The controls are constrained so that $-0.05 \leq \mathbf{u}(k) \leq 0.1$ for the entire time-window.

The predictive model is the time-invariant model of Equation 21, but with a time-horizon, $N_f - N_s$, of 10 time-periods. Thus the predictive model knowledge is limited by the time-horizon limitation, by not knowing the complete model structure as the predictive model is time-invariant, and by not knowing about the random shocks.

In this application the optimal control strategy is based on the predictive model for the window of time. A subset of the control strategy is applied to the economic system. After the control has been applied a new set of data for the states of the economic system is available. The window is then slid forward in time, and a new control strategy is developed. The entire process is then continued.

Figures 2 and 3 show the simulation results of the control strategies for the unconstrained and constrained cases respectively. The random shocks on the states act independently on each state and all have a standard deviation of 0.05. The time-window is 10 periods wide[1]. The first time-period's control is applied to the economic system. That is, the optimal control is developed over predicting the economic system for 10 time-periods and the first control is applied to the economic system before the prediction window slides along a time-period.

[1] Varying the window width between 5 and 15 time periods makes little difference to the results presented in Figures 2 and 3.

Figures 2 and 3 show that the control strategy has the desired effects. Stronger control effort is required in the unconstrained case but the desired values are achieved faster though with greater variability. The control strategy adjusting for the random shocks can also be seen in the Figures.

4. CONCLUSION

This paper has developed a methodology for applying optimal control in the situation where there are explicit constraints upon the control variables and there is limited knowledge of the economic system. The method involved the construction of a linear predictive model for the policy targets of the economic system, developing constrained optimal control policies on the predictive model, applying a subset of these policies to the economic system and then repeating the entire process. The methodology was applied to a small, calibrated macroeconomic model.

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