Using Discrete-time Techniques to Test Continuous-time Models for Nonlinearity in Drift

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Abstract: This paper examines whether or not a discrete-time econometric test for nonlinearity in mean may be used in cases where the data are believed to be generated by a continuous-time model. It is demonstrated that appropriate bootstrapping techniques are required to yield a test statistic with sensible statistical properties.

Keywords: Testing; Continuous-time processes; Bootstrap, Legendre polynomials

1. INTRODUCTION

The notion of financial data as being generated by a continuous-time process is commonplace in theoretical finance. The estimation of the parameters of such processes from discretely observed data, on the other hand, has received relatively less attention. In particular, it is now known that straightforward application of discrete-time methods yield biased parameter estimates in many interesting situations. Subsequently a vast new literature has evolved in this area [see, Sundaresan, 2000]. The major remaining area which now needs to be explored concerns testing. It is important to ascertain whether specification tests, derived from the assumption of discretely observed and generated data, can be adapted to deal with the assumption of continuously-generated data. This paper examines only one potentially important aspect of specification testing, namely, the detection of nonlinearity in mean. The difficulties associated with the application of discrete-time nonlinearity tests to continuously-generated data will be highlighted. It will also be demonstrated that suitable bootstrapping can play a major role in alleviating these problems. These techniques will then be illustrated with reference to the 7-day Eurodolllar spot deposit rate.

2. INFERENCE BASED ON DISCRETE OBSERVATIONS

Most tests for nonlinearity in mean, as defined in Lee et al. [1993], are based on the null hypothesis that the data-generating process is of the form

$$y_t = \alpha_0 + A(L)y_t + B(L)\varepsilon_t \qquad (1)$$

$$\varepsilon_t \sim N(0,\sigma^2) \qquad (2)$$

where $\{y_t\}\, t = 1...T$ is a sequence of discretely-generated and sampled observations and $A(L)$ and $B(L)$ are suitably chosen lag polynomials. Tests for nonlinearity assess the significance of additional variables

$$z_t = g(y_{t-1}, y_{t-2}, \ldots, y_{t-k}),$$

where $g(\cdot)$ is some nonlinear transform of its argument. In this paper the so-called V23 test, due to Teräsvirta *et al.* [1993], is chosen to test for nonlinearities. Here $g(\cdot)$ represents all unique second and third order cross products of $(y_{t-1}, \ldots y_{t-k})$ with their joint significance tested by means of a standard F-test. For example, if $\{y_t\}$ is a Markov process and $A(L) = \alpha_1 L^{(t-s)}$, then

$$E[y_t \mid y_s] = \alpha_0 + \alpha_1 y_s .$$

In this special case the vector $g(\cdot)$ reduces to $g(\cdot) - (y_s^2, y_s^3)$.

In continuous time, a Markov process is usually described by the stochastic differential equation (SDE)

$$dy = \mu(y) + \sigma(y)dW , \qquad (3)$$

where dW is the increment of a standard Wiener process and $\mu(y)$ and $\sigma(y)$ are the drift and diffusion functions respectively. The properties of the discrete-time expectation, $E[y(t) \mid y(s)]$, $s < t$, of the state variable may be examined by using the concept of infinitesimal generators. The infinitesimal, or Dynkin, operator [see Campbell *et al.* 1997] is the time derivative of a conditional expectation of some function $\phi(y)$. By Ito's Lemma

$$d\phi(y) = \phi'(\mu dt + \sigma dW) + \frac{1}{2}\phi''(\mu dt + \sigma dW)^2$$

$$= \left(\phi'\mu + \frac{1}{2}\phi''\sigma^2\right)dt + \phi'\sigma dW$$

and hence

$$A\phi = \frac{dE[\phi(y)]}{dt} = E\left[\phi'\mu + \frac{1}{2}\phi''\sigma^2\right]$$

which is the Dynkin operator for diffusion processes of type (3). Now consider $\phi(y) = y$ and let the drift be given by

$$\mu(y) = \beta_0 + \beta_1 y.$$

It follows directly that $\phi'(y) = 1$ and $\phi''(y) = 0$ so that

$$\frac{dE[y]}{dt} = E[\beta_0 + \beta_1 y] = \beta_0 + E[\beta_1 y]$$

which is a first-order differential equation in $E[y]$. The second-order generator for the case of linear drift may be derived as follows

$$A^2\phi = A(A\phi) = E\left[(A\phi)'\mu + \frac{1}{2}(A\phi)''\sigma^2\right]$$
$$= E[\beta_1\mu] = E_s\left[\beta_1(\beta_0 + \beta_1 y)\right]$$
$$= \beta_1\beta_0 + \beta_1^2 y.$$

Indeed it may be verified that higher-order generators are given by

$$A^n\phi = \beta_1^{n-1}\beta_0 + \beta_1^n y. \qquad (4)$$

It is now possible to derive the expectation of $y(t)$ formed at time s, where $s \le t$. One way to obtain a tractable expression for $E_s[y(t)]$ is to use the Taylor-series approximation

$$E_s[y(t)] = E_s[y(s)] + \frac{dE_s[y(s)]}{dt}(t-s)$$
$$+\frac{1}{2!}\frac{d^2E_s[y(s)]}{dt^2}(t-s)^2$$
$$+\frac{1}{3!}\frac{d^3E_s[y(s)]}{dt^2}(t-s)^3 + \ldots$$
$$= E_s[y(s)] + A\phi(t-s)$$
$$+\frac{1}{2!}A^2\phi(t-s)^2 + \frac{1}{3!}A^3\phi(t-s)^3 + \ldots$$

After making the simplifying assumption that $(t-s) = 1$ and repeated substitution for $A^n\phi$ using (4), this expectation becomes

$$E_s[y(t)] - y(s) = (\beta_0 + \beta_1 y(s)) + \frac{1}{2!}(\beta_0\beta_1 + \beta_1^2 y(s))$$
$$+\frac{1}{3!}(\beta_0\beta_1^2 + \beta_1^3 y(s)) + \ldots$$

or

$$E_s[y(t)] - y(s) = \beta_0\left(1 + \frac{1}{2!}\beta_1 + \frac{1}{3!}\beta_1^2 + \ldots\right)$$
$$+\left(\beta_1 + \frac{1}{2!}\beta_1^2 + \frac{1}{3!}\beta_1^3 + \ldots\right)y(s).$$

Further recognize that

$$e^{\beta_1} = 1 + \beta_1 + \frac{1}{2!}\beta_1^2 + \frac{1}{3!}\beta_1^3 + \ldots$$

and therefore

$$E_s[y(t)] - y(s) = \left(e^{\beta_1} - 1\right)y(s) - \frac{\beta_0}{\beta_1}\left(1 - e^{\beta_1}\right). (5)$$

This is an important result, since it demonstrates that the expected change of discretely-observed, but continuously-generated data can be represented as a discrete-time ARMA model. Note, however, that at this point no verifiable general claim can be made as to the properties of the residuals, particularly the assumption of constant variance. Indeed, analysis of the residuals needs to be done with reference to specific stochastic processes.

Arguably the most popular continuous-time process is the geometric Brownian motion, which forms the core of the popular Black-Scholes model for the pricing of stock options. It is well known that the continuously-compounded returns of this process can be represented in the form specified by (1) and (2). The focus of this paper, however, is on interest-rate processes that are potentially stationary and a more appropriate model, therefore, is the Ornstein-Uhlenbeck (OU) process

$$dy = \kappa(\theta - y)dt + \sigma dW,$$

with θ being the mean to which the process reverts at rate κ. The exact discretisation of the OU process is known to be

$$y_t = \theta(1 - e^{-\kappa}) + e^{-\kappa}y_{t-1} + \varepsilon_t \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$
$$\sigma_\varepsilon^2 = \sigma^2\left(\frac{1 - e^{-2\kappa}}{2\kappa}\right)$$

This is of course a special case of (5) with $\beta_0 = \kappa\theta$ and $\beta_1 = -\kappa$. The discretisation fits the framwork of (1) and (2) by setting $\alpha_0 = \theta\left(1 - e^{-\kappa}\right)$, $A(L) = e^{-\kappa}L$ and $B(L) = 1$, and recognizing that the residuals are normally distributed with constant variance. One caveat, however, should be noted. If the speed of adjustment parameter is zero, the process becomes nonstationary and statistical inference in terms of the V23 test cannot be based on standard distributional theory.

Possibly the most popular stochastic process for modeling interest rates is due to Cox, Ingersoll and Ross [1985], and is given by

$$dy = \kappa(\theta - y)dt + \sigma\sqrt{y}\,dW.$$

This CIR process is a special case of the so-called CKLS model [Chan et al. 1992] given by

$$dy = \kappa(\theta - y)dt + \sigma y^\gamma dW.$$

In both the CIR and CKLS models the linear ARMA model, (1), is an appropriate representation of the mean. On the other hand, as long as $\gamma > 1$, the volatility of the discrete residual, ε_t, will be increasing in the level of the state variable, y_t. This invalidates the use of standard statistical inference in the V23 testing procedure, which is based on the assumption of constant residual variance.

3. TESTING IN CONTINUOUS MODELS

It has been claimed that the discrete ARMA approximation (1) and (2) of a continuous SDE is appropriate when the underlying process is either a geometric Brownian motion or a mean reverting $(\kappa > 0)$ OU process. For the CKLS model difficulties in the application of standard tests are likely to be encountered, due to the non-constancy of the variance in the discrete approximation. To support this conjecture a Monte Carlo experiment is conducted and size results for the V23 test for non-linearity in mean are reported.

Data for three different stochastic processes are generated and tested for non-linearity in mean. The first two models, OU and OU_NUR are variants of the Ornstein-Uhlenbeck process with $\kappa = 0.8$ and 0.002 respectively, $\theta = 0.1$ and $\sigma = 0.006$. The second model with $\kappa = 0.002$ represents very weak mean reversion, making $\{y_t\}$ a near-unit-root process. The third simulation is a CIR model with $\kappa = 0.21459$, $\theta = 0.08571$ and $\sigma = 0.0783$. The V23 test is used to test the null hypothesis

$$H_0 : \Delta y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$$

based on the assumption that $g(\cdot) - \left(y_{t-1}^2, y_{t-1}^3\right)$. Of course, all the simulated models have linear drift and therefore comply with the null hypothesis. The empirical size of the test statistic for sample size of 250 and based on 5000 repetitions of each experiment is reported in Table 1.

Table 1. Empirical size in 5000 repetitions of the V23 test for various linear continuous-time processes with sample size of 250.

	DGP		
Significance Level	**OU**	**OU_NUR**	**CIR**
0.01	0.007	0.035	0.037
0.05	0.042	0.124	0.134
0.10	0.088	0.215	0.232

It is obvious, that the V23 test only has the correct size when the data generating process is an OU process with significant drift. When the drift in the OU model is almost negligible, representing the near-unit root case, the V23 test over-rejects the null hypothesis. Due to the non-stationarity of the data the test no longer has a standard F- distribution, a result analogous to the non-standard distribution of t-tests in Dickey-Fuller regressions. The size distortion in the case of the CIR process is due to the heteroscedasticity in the discretised residuals, which is now a well-known phenomeon.

This problem is likely to be exacerbated in practice as more complex volatility functions than the one in the CKLS process have been proposed for interest rate data. Some extend the CKLS model by introducing a second stochastic factor, stochastic volatility [Andersen and Lund, 1997], others, most notably Aït-Sahalia [1996], specify a more state-dependent form for volatility. The next section takes up the second of these approaches with a view to demonstrating that the size distortion may be overcome by use of a suitable bootstrapping technique.

4. BOOTSTRAPPING COMPLEX VOLATILITY MODELS

Aït-Sahalia [1996] (AS) proposes the following non-linear model for US Eurodollar interest rates

$$dy = \mu(y)dt + \sigma(y)dW$$

$$\mu(y) = a_0 + a_1 y + a_2 y^2 + a_3 / y \qquad (6)$$

$$\sigma^2(y) = b_0 + b_1 y + b_2 y^{b_3}$$

Although this specification has been the subject of recent debate [see, for example, Chapman and Pearson, 2000, and Hurn and Lindsay, 2001], it remains a useful example of a nonlinear SDE. For the purposes of the simulation experiments in this section, both a linear version (ASI) and the unrestricted form of the model (ASII), as in equation (6) are used. Note that the ASI model is linear in drift but has exactly the same diffusion function as ASII. The ASII model is exactly as reported in AS, where the volatility function is quadratic in the state variable with minimum at $y \approx 0.11$. The parameter values used in the experiments are either obtained from a simple OLS regression using the Eurodollar interest rate data or are in fact the values given in Aït-Sahalia [1996].

A bootstrapping mechanism capable of dealing with state-dependent volatility is the pair-bootstrap algorithm. Consider the structural relation

$$x_t = \gamma' \mathbf{w}_t + \varepsilon_t \qquad \varepsilon_t \sim N\left(0, \sigma^2(\mathbf{w}_t)\right) \quad (7)$$

where the residual term is independent but not identically distributed, x_t is a scalar, γ a (kx1) parameter vector and $\mathbf{w}_t$ a (kx1) vector of independent variables. In particular, the residual variance is a function of the exogenous variables w_t. Bootstrap sequences of $\{x_t, w_t\}$, denoted $\{x_t^*, w_t^*\}$, $t=1...T$, can be generated [Flachaire, 1999] by first estimating γ in (7), defining

$$\hat{\varepsilon} - x_t - \hat{\gamma}' w_t$$

and then re-sampling pairs (w_t^*, ε_t^*) and generating

$$x_t^* = \hat{\gamma}' w_t^* + \varepsilon_t^*$$

This approach is unlikely to work for the type of heteroskedasticity encountered in either the CKLS or AS models. A simple application of the pair bootstrap could not yield a set of realisations which captured the autoregressive nature of the original model. It is clear some other bootstrapping approach is needed.

The problem of resampling from a residual distribution, characterised by $E[\varepsilon_t \mid x_{t-1}] = 0$ and $E[\varepsilon_t^2 \mid x_{t-1}] = \sigma_t^2(x_{t-1})$ where the functional dependence in the volatility is unspecified, is closely related to the problem of parameter estimation in the presence of heteroscedasticity of an unknown form. To tackle the latter, several approaches have been proposed [Pagan and Ullah, 1999]. The favoured approach to parameter estimation under these conditions is to apply a generalized-least-squares (GLS) estimate. With reference to equation (7), for example, the estimator is

$$\hat{\gamma} = \left(W'\Omega^{-1}W\right)W\Omega^{-1}x \qquad (8)$$

The (T x k) matrix W is the matrix of independent variables and x is the (T x 1) dependent variable. The variance-covariance matrix, Ω, is a (T x T) matrix with diagonal elements $\{\sigma_t^2\}$ and zeros elsewhere (assuming the absence of autocorrelation). If the form of the heteroskedasticity is unknown, the variance function must be estimated nonparametrically using the OLS residuals $\{\varepsilon_t^2\}$. Kernel estimation and nearest-neighbour methods are two examples. The resultant estimate for Ω, when substituted into (8) yields a semi-parametric estimate for the parameter vector γ.

Returning the problem at hand, namely, that of generating bootstrap replications of $\{x_t^2\}$ under the null hypothesis of a linear drift structure, it is obvious that the same principle can be applied. First estimate the drift parameters, α_0 and α_1 and use the resultant squared residual estimates to obtain a nonparametric estimate of the volatility function $\sigma_t(x_{t-1})$. The bootstrap realizations are then constructed by drawing an appropriate starting value for x_0^* and generate $\{x_t^*\}$ recursively with ε_t^* drawn from the normal distribution $N\left(0, \hat{\sigma}_t(x_{t-1})\right)$.

It now remains to be demonstrated how the variance function is estimated. The method proposed here is an approximation by means of orthogonal polynomials. The approximation of an unknown function by means of orthogonal series takes the following general form [Pagan and Ullah, 1999]

$$\sigma_t^2(x_{t-1}) = \sum_{m=0}^{M} \psi_m z_m(x_{t-1}) + \upsilon_t,$$

where $z_m(x_{t-1})$ is the m-th order orthogonal basis function. Following Hurn and Lindsay [2001], the orthogonal basis functions are the Legendre polynomials

$$z_0(x) = 1, \quad z_1(x) = x, \quad z_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \ldots.$$

The polynomials are only defined in the interval $[-1,1]$ and the mapping

$$x = \left[2x - \left(\bar{x} + \underline{x}\right)\right] / \left(\bar{x} - \underline{x}\right),$$

where $\bar{x}$ and $\underline{x}$ are the maximum and minimum values of the data, is used to map the original observations into the required interval. In this application, the first four Legendre polynomials proved flexible enough to approximate the form of the variance function. There are therefore four parameters $\psi_0 .... \psi_3$ to be estimated using the squared residuals obtained by fitting the linear drift to the data. One appealing feature of this approach is that these parameters are easily obtained by means of a straightforward OLS regression.

Once an estimate of the variance function is available, the bootstrap generation process is given by

$$x_t^* = \hat{\alpha}_0 + \hat{\alpha}_1 x_{t-1}^* + \varepsilon_t^*,$$

where the residual ε_t^* is drawn from $N\left(0, \sigma_t^2(x_{t-1}^*)\right)$. The resulting unconditional distribution of residuals will be a mixture of normals which can mimic some of the features we can see in real life data, most notably leptokurtosis. It is a well-known problem of nonparametric estimates that inference should not be drawn beyond the observed data region. For that reason the re-sampling has to be confined to the observed data range.

As this method of bootstrapping has not previously been proposed in the literature, it is necessary to scrutinize its empirical performance before applying it to a real data set. A small Monte-Carlo experiment is performed to investigate the empirical size and power of the bootstrapped-V23. Since the bootstrap tests are fairly computing intensive, this Monte-Carlo experiment operates with fairly small sample sizes (250, 500 and 1000) and a modest number of replications (1000 for the asymptotic test and 500 for bootstrap test). The results are tabulated in Tables 2 and 3.

From the results in Table 2 one may conclude that there is a substantial size distortion when the asymptotic distribution of the test is used. Furthermore, this distortion is significantly larger than that for the CIR process (see Table 1). By contrast the V23 test based on the Legendre bootstrap appears to have the correct size.

Table 2. Empirical size of the V23 test for a process with linear mean but nonlinear variance.

ASI Model	Asymptotic test		
Significance	**T=250**	**T=500**	**T=1000**
0.01	0.145	0.296	0.513
0.05	0.334	0.506	0.701
0.10	0.445	0.620	0.789
	Bootstrap test		
0.01	0.004	0.008	0.018
0.05	0.034	0.052	0.054
0.10	0.084	0.088	0.134

Table 3. Empirical power of the V23 test for a process with nonlinear mean and variance .

ASII Model	Asymptotic test		
Significance	**T=250**	**T=500**	**T=1000**
0.01	0.260	0.555	0.898
0.05	0.460	0.752	0.956
0.10	0.575	0.834	0.971
	Bootstrap test		
0.01	0.018	0.024	0.080
0.05	0.064	0.134	0.292
0.10	0.146	0.268	0.494

Turning now to the empirical power of the test, it is apparent that in small samples the bootstrapped-V23 test only has modest power to detect the nonlinear drift in the ASII process. It is also clear, however, that the power increases significantly with increasing T.

5. APPLICATION TO INTEREST-RATE DATA

Interest rates have a long history of being modelled as continuous time processes [Chan, 1992]. While many different processes were proposed, most of these are linear in their mean part and possess a level-dependent volatility. The majority of these models are nested within the CKLS process given by

$$dy = \kappa(\theta - y)dt + \sigma y^{\gamma} dW . \qquad (9)$$

The major concern of this paper, however, is with the functional form of the drift. As it is specified in (9) the drift is linear, with κ determining the speed with which the process reverts to its attractor θ. This linear-drift specification has been challenged by Aït-Sahalia [1996] and Stanton [1997]. They argue that that mean reversion is only relevant when the interest rate level is either very high or very low; in the region of the mean the process is seen to behave like a random walk.

The approach of both AS and Stanton is based on estimation of the underlying, potentially non-linear drift without prior testing. The argument here is that hypothesis tests, which do not rely on the estimation of the alternative (nonlinear) model, are the logical starting point for the enquiry. Once again the V23 test will be relied upon. Note that the method of estimating the volatility function by means of Legendre polynomials allows the null hypothesis to be generalized to

$$dy = \kappa(\theta - y)dt + \sigma(y)dW$$

where the form of the volatility function remains unspecified for testing purposes.

The data used here are identical to Aït-Sahalia's [1996] data set, namely, daily observations of the 7-day Eurodollar deposit spot-rate from 1 June 1973 to 25 February 1995 (5505 observations). The V23 test was first used to test the null hypothesis with the significance of the test statistic being evaluated by means of its asymptotic F-distribution. The estimated test statistic is 24.05 with an asymptotic p-value of zero for all practical purposes. There appears no doubt about the nonlinearity of the interest rate process when tested in this fashion.

Armed with overwhelming empirical evidence that the interest rate volatility is a function of the interest rate level, and the simulation results from the previous section (Table 2), however, this conclusion appears dubious given the substantial size distortion of the asymptotic test. It therefore appears to be necessary to apply the Legendre bootstrap. Figure 1 displays the volatility function estimate using the Legendre polynomials. The function is displayed on the observed data range (0.02915 to 0.24333). To avoid the pitfall of obtaining negative estimates of variance in the region of the observed minimum of the data, a constraint

($\psi_0 = \psi_1 - \psi_2 + \psi_3$) was introduced to ensure the variance remained positive. This ensures that the volatility estimate is 0 when the state variable is 0 and positive elsewhere. In order to demonstrate that this does not alter the fundamental characteristics of the volatility function, this estimate is displayed (solid line) next to the unconstrained Legendre estimate. The qualitative similarity between the two estimates is immediately apparent.

When this volatility estimate is used in the bootstrap generation mechanism, the p-value of the V23 test is found to be 0.131. The conclusion is therefore that there is not sufficient evidence to reject the null hypothesis of linearity, constituting a substantial turnaround from the conclusion based on the asymptotic distribution. The question might be asked whether this is due to a lack of power of the testing procedure. Again the experimental evidence provided earlier suggests that this is unlikely. While no simulations with a sample size of T=5505, were performed (due mainly to the computational burden), the available evidence suggests that the bootstrapped-V23 test would have good power in samples of this size and against this type of nonlinear process.

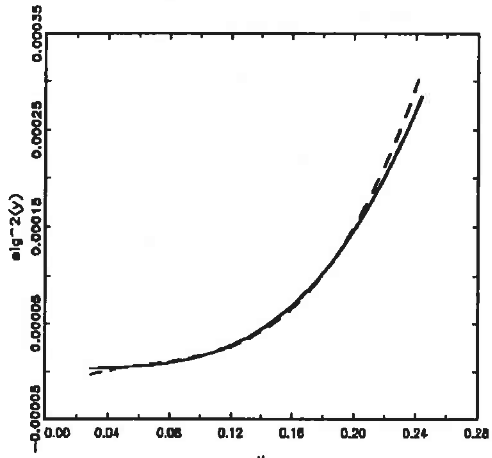

Figure 1. Estimate of the volatility function using Legendre polynomials: unconstrained (dashed line), constrained (solid line).

6. CONCLUSION

Several conclusions can be drawn from this work. In some special cases discrete-time specification tests can simply be applied to discrete observations of continuous-time processes. More generally, however, consistent statistical inference is difficult because the complex diffusion functions commonly encountered in financial data introduce a size distortion in test procedures that are based on asymptotic distributions. The approximation of the diffusion function by means of Legendre polynomials enabled a bootstrapping procedure to be devised, which cures the size distortion. The test should only be regarded as an appropriate diagnostic tool, however, when the diffusion function lends itself to being modelled in this way.

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