Generalised Scaling, Extreme Rainfall and Areal Reduction Factors

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Abstract: The generalised scaling framework is applied to the intensity-duration-frequency (IDF) description of extreme rainfall with the areal reduction factor. We put forward the hypothesis that the cumulative distribution function for the annual maximum series of mean rainfall intensity has a scaling property over the ranges 1 to 24 hours and 1 to 1000 sq. km. This behaviour is demonstrated through an examination of the scaling properties of the moments and the scaling of the parameters of an extreme value distribution fitted to the high-resolution rain gauge, and the areal reduction factor (ARF) curves currently used in design. A simple analytical formula for the IDF relationship is proposed, which embodies the scaling behaviour. Once the scaling parameter has been obtained for a gauge or set of gauges in a region and/or a weather radar data, this formula enables the calculation of rainfall intensities, of a chosen return period and duration shorter than a day, directly from the information obtained from the analysis of daily data.

Keywords: Scaling; Rainfall; Areal Reduction Factor

1. INTRODUCTION

Intensity-duration-frequency (IDF) relations have been widely used in engineering practice for design flood estimations for over a century. One of the first empirical IDF relations was derived by Sherman [1905], who collated measurements taken at various locations in the United States from 1880 to 1905. He concluded that the maximum observed rain rate i, in inches per hour, was related to duration t, in minutes, as $i = \frac{38.64}{t^{0.687}}$. He reported an alternative form for the relation as $i = \frac{420}{t+30}$ for $t < 3$ hours. The frequency of the storms was represented as "ordinary", "extraordinary", and "maximum". In the discussion of Sherman's results, Webster [1905] found that $i = \frac{A}{t^n}$, where A increased from 12 to 18 and 30.585 for ordinary, extraordinary, and maximum rainfall respectively, while n was approximately constant at 0.5-0.6. Bell [1969] proposed the empirical IDF relation $P = \frac{A}{(t+b)^n}$ where P is the average rainfall intensity in mm/minute over t minutes for a particular return period. Bell [1969] found that A was a function of return period and location, whereas b and n were a function of location only. Wenzel [1982] found n to be in the range 0.4 to 0.84 for various locations in the United States. In a recent work, Koutsoyiannis *et al.* [1998] proposed a generalized IDF relation in the form

$$i = \frac{a(T)}{b(d)}, \qquad (1)$$

where i - is the rainfall intensity, T - is the return period, and d - is the duration of the extreme event. The function $a(T)$ can be determined from the probability distribution function of the maximum rainfall intensity, and the function $b(d)$ is sought in the form

$$b(d) = (d + \theta)^{\eta}, \qquad (2)$$

The IDF curves described above apply to point rainfall. More important in practical applications are statistical properties of extreme rainfall over a given area, which usually are calculated by multiplying point intensity by the so-called areal-reduction-factor (ARF). The latter is defined as the ratio or the mean rainfall intensity over a given area to the maximum point intensity of rainfall. By definition, the ARF takes values in the range 0 – 1. The estimation of ARF is a complex problem and different approaches have been developed, based on various statistical techniques, as well as on various assumptions about temporal and spatial correlation structure of rainfall fields [see US Weather Bureau, 1957; Bell, 1976; Rodriguez-Iturbe and Mejia, 1974; Myers and Zehr, 1980; Omolayo, 1993; Srikanthan, 1995; Sivapalan and Bloschl, 1998; Asquith and Famiglietti, 2000; and references therein]. Two kinds of ARF are currently in use: (a) Fixed-area (or geographically-fixed), and (b) Storm-centered. The fixed-area ARF is based on the frequency analysis of a time series of annual maximum precipitation for a given fixed area. An assumption is made about equality of the probability distribution function of the point and areal rainfall. The second type of ARF refers to any given storm. In this paper we consider only fixed-area ARF.

2. SIMPLE SCALING HYPOTHESIS FOR POINT RAINFALL

In contrast to the above mentioned treatments which depend on curve-fitting techniques, a natural source for theories regarding the re-scaling of rainfall statistics is to be found in the scaling hypotheses popularised by Mandelbrot [1982] and Lovejoy and Schertzer [1985]. Burlando and Rosso [1996] in a pioneering paper, sought to apply the scaling hypotheses to annual maximum series of rainfall depth. In their work, the scaling and multiscaling properties of the statistical moments of rainfall depth of different duration were analysed and a lognormal probability distribution was used to model its statistical properties. In a recent work [Menabde *et al.* 1999] we analysed a high resolution (6 minutes) 25-year long record of rainfall intensities from Melbourne, Australia. We have shown that the mean annual maximum rain rate (intensity) has a simple scaling property expressed by:

$$I_d \overset{dist}{=} (d/D)^{-\eta} I_D, \tag{3}$$

where the above equality is understood in the sense of equality of probability distribution functions, and (3) holds for duration over the range from 30 minutes to 24 hours. As a direct consequence of the relation (3) the moments of the mean annual maximum rainfall intensity I_d are scaling as

$$\left\langle I_d^q \right\rangle = f(q) d^{-K(q)}. \tag{4}$$

where $K(q) = \eta q$. The scaling properties of the moments and the dependence of the scaling exponent on moment order are shown in Figures 1 and 2, respectively.

The annual maximum mean rain rate I_d is considered as a random variable with a cumulative distribution function

$$F_d(i) = \Pr(I_d < i). \tag{5}$$

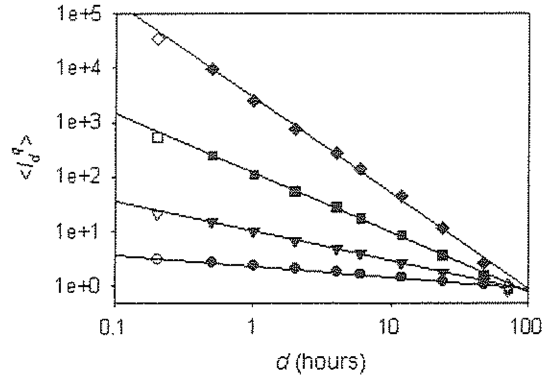

Figure 1. Scaling of the moments of maximum annual rain rate (From top to bottom: q = 3.0, 2.0, 1.0, 0.5).

The return period of an extreme event with an intensity $I_d \geq i$ will be, as usual, given by:

$$T(I_d \geq i) = \frac{1}{1 - F_d(i)}. \tag{6}$$

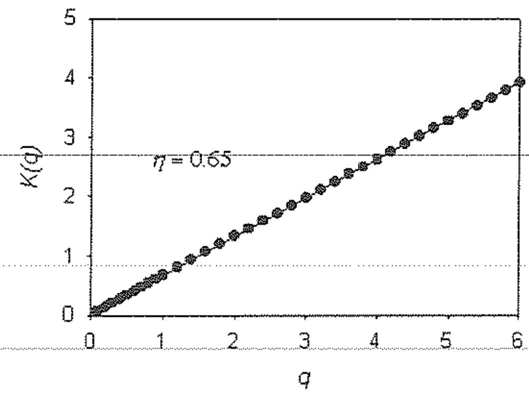

Figure 2. Scaling exponent dependence on the moment order.

The scaling property (3) can be written explicitly in terms of cummulative distribution function (CDF):

$$F_d(i) = F_D((d/D)^{\eta} i). \tag{7}$$

If we assume that the CDF of extreme events has the (standardised) functional form

$$F_d(i) = F\left(\frac{i - \mu_d}{\sigma_d}\right), \tag{8}$$

where F is some function, independent of d, from (7) we get

$$\mu_d = (d/D)^{-\eta} \mu_D, \tag{9}$$

$$\sigma_d = (d/D)^{-\eta}\sigma_D. \quad (10)$$

Substituting (8), (9), and (10) in formula (6) and inverting it in respect to i we get the IDF relation in the general form (1) with $a(T)$ and $b(d)$ given by

$$a(T) = \mu + \sigma F^{-1}(1-1/T), \quad (11)$$

$$b(d) = d^{\eta}, \quad (12)$$

where $\mu = \mu_D D^{\eta}$ and $\sigma = \sigma_D D^{\eta}$ are constants, independent of D (as it follows from (9) and (10)). It can be seen, that the simple scaling hypothesis (3) leads to $\theta = 0$ in the formula (2), proposed by Koutsoyiannis *et al.* [1998].

3. GENERALISED SCALING AND ARF

A simple, but somewhat naïve way to include the areal reduction factor in the scaling framework, is to introduce a new random variable - the annual maximum mean rain rate $I_{d,a}$ over the duration d and the area a, and assume that it has a simple scaling property, analogous to (3), both in time and space:

$$I_{d,a} \overset{dist}{=} (d/D)^{-\eta}(a/A)^{-\lambda} I_{D,A}, \quad (13)$$

with two different exponents η and λ. This relation may be valid (subject to empirical justification) in some range of a and A. However, as the area a tends to zero, the mean rain rate tends to infinity, which obviously contradicts to the common sense and empirical facts, and makes it impossible to connect the properties of point and areal rainfall.

In order to overcome this problem we make a more general assumption about the scaling properties of rainfall, assuming that

$$I_{d,0} \overset{dist}{=} f(d,D) I_{D,0}, \quad (14)$$

and

$$I_{d,a} \overset{dist}{=} \varphi(d,a,A) I_{d,A} \quad (15)$$

where the scale transformation functions f and φ depend on both scales d, D and a, A, and not only on their ratio, as in the case of simple scaling. We assume that there are some empirically defined limits for duration D_0 and area A_0, such that (14) holds for all $a, A < A_0$ and $d, D < D_0$ with the same functional dependence. This type of generalised scaling model was first introduced by Benzi *et al.* [1993].

It is not difficult to find the most general functional form of f and φ. It can be seen from definition (14) that the scale transformation function f has the group property

$$f(d_1,d_3) = f(d_1,d_2) f(d_2,d_3) \quad (16)$$

The most general solution to functional equation (16) is given by

$$f(d,D) = \frac{t(d)}{t(D)}, \quad (17)$$

where $t(d)$ is some arbitrary function. We can find a simple form of $t(d)$ if we mention that for a large duration the annual maximum rain rate has a simple scaling property (3). This means that for d, D much greater then some threshold value d_0,

$$f(d,D) = (d/D)^{-\eta}. \quad (18)$$

A simple form of $t(d)$ satisfying (18) for d, $D >> d_0$ is

$$t(d) = \frac{1}{1+(d/d_0)^{\eta}}, \quad (19)$$

and

$$f(d,D) = \frac{1+(D/d_0)^{\eta}}{1+(d/d_0)^{\eta}}. \quad (20)$$

In the same way for the scale transformation function $\varphi(d,a,A)$ we can get

$$\varphi(d,a,A) = \frac{1+[A/a_0(d)]^{\lambda}}{1+[a/a_0(d)]^{\lambda}}, \quad (21)$$

where $a_0(d)$ is a threshold area, which is supposed to depend on the duration d. As it is shown below this assumption is supported by the empirical data. The generalised scaling hypotheses, expressed by (14), (15), (20), and (21), together with a quite general assumption about the functional form of the CDF of extreme value distribution

$$F_{d,a}(i) = F\left(\frac{i-\mu_{d,a}}{\sigma_{d,a}}\right), \quad (22)$$

leads to a simple IDF relation with an areal reduction factor. Indeed, combining (14) and (15), we get

$$I_{0,0} \overset{dist}{=} f(0,d)\varphi(d,0,a) I_{d,a} \quad (23)$$

and

$$F_{0,0}(i) = F_{d,a}\{f^{-1}(0,d)\varphi^{-1}(d,0,a) i\} \quad (24)$$

$$\mu_{d,a} = f^{-1}(0,d)\varphi^{-1}(d,0,a)\mu_{0,0}, \quad (25)$$

$$\sigma_{d,a} = f^{-1}(0,d)\varphi^{-1}(d,0,a)\sigma_{0,0}, \quad (26)$$

Substituting (25) and (26) into formula (6) and inverting it in respect to i we get the IDF relation with the ARF factor in the most general form:

$$i(d,a,T) = \frac{\mu_{0,0} + \sigma_{0,0} F^{-1}(1-1/T)}{\{1+(d/d_0)^\eta\}\{1+[a/a_0(d)]^\lambda\}} \quad (27)$$

where $\mu_{0,0}$ and $\sigma_{0,0}$ are the values of the parameters $\mu_{d,a}$ and $\sigma_{d,a}$ for zero d and a. Of course, the rainfall intensity at zero duration does not have any physical meaning, but the constant d_0 can be interpolated from empirical data. In fact, as we will see below, for the most practically important cases it is not necessary to know the parameter d_0.

Let us consider some particular cases. First, for a point rainfall, i.e. when $a = 0$, we get

$$i(d,T) = \frac{\mu_{0,0} + \sigma_{0,0} F^{-1}(1-1/T)}{1+(d/d_0)^\eta} \quad (28)$$

This relation is analogous to the general empirical relations, proposed by Bell [1969] and Koustsoyannis *et al.* [1998] (see equations (1) and (2)). As we already mentioned, the parameter d_0 cannot be measured directly, and should be interpolated from empirical data. For the particular case of Melbourne rainfall data, as it is obvious from Figure 1, it is of order of few minutes. From practical point of view, we are mainly interested in estimating the maximum rainfall intensity in the range 1 – 24 hours, i.e when $d >> d_0$. In this case formula (28) can be further simplified and takes the following form

$$i(d,T) = \frac{\mu + \sigma F^{-1}(1-1/T)}{d^\eta} \quad (29)$$

where $\mu = \mu_{0,0} d_0^\eta$ and $\sigma = \sigma_{0,0} d_0^\eta$. This form of IDF relation was first proposed in Menabde *et al.* [1999]. The scaling exponent η can be estimated from a short record of high resolution rain gauge data from the scaling properties of the moments, as in Figure 1 and 2, and it is distribution free. Parameters μ and σ can be estimated from a long sequence of, say, 24-hour rain gauge data and some particular functional form of CDF has to be accepted. As it was shown in [Menabde *et al.*, 1999], in the case of Melbourne rainfall data, Gumbel (EVI) distribution provides a good fit

To estimate the areal reduction exponent λ either a weather radar data or a dense network of rain gauges can be used. First, extreme events should be identified, and then studying the scaling properties of spatial moments of the mean areal rainfall we can estimate the scaling exponent λ and the parameter a_0. *A priory* it is not obvious that the scaling properties will be the same for different accumulations, neither is it obvious that the moments will scale at all. This should be checked empirically in every particular case. (see Figure 3).

Thus, the generalised scaling hypothesis leads to the following IDF relation with the areal reduction factor

$$i(d,a,T) = \frac{\mu + \sigma F^{-1}(1-1/T)}{d^\eta\{1+[a/a_0(d)]^\lambda\}} \quad (30)$$

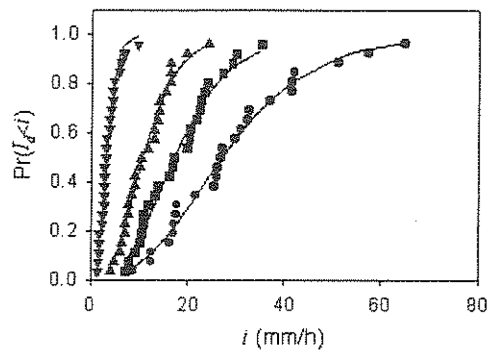

Figure 3. CDF-s for annual maximum rainfall for different accumulations (From left to right: 12, 2, 1, and 0.5 hours).

The areal reduction factor itself, as it can be seen from eq. (30) has the form

$$ARF(a,d) = \frac{1}{1+[a/a_0(d)]^\lambda} \quad (31)$$

As it was mentioned above, to provide a comprehensive justification of our working hypothesis it is necessary to analyse time series of rainfall of a dense network of rain gauges and/or a long continuous records of weather radar. Such data sets were not available at the time of writing. As a surrogate for the comprehensive analysis we analysed the standard depth-area curves [World Meteorological Organisation, 1983], which are currently recommended for use in Australia. The results for the empirically derived ARF for different durations (30min, 1h, 3h, 6h, and 24 h) fitted by formula (31) are shown in Figure 4. The value of the exponent λ is 0.45, and the parameter $a_0(d)$ can itself be fitted by a scaling relation. Thus, $a_0(d)$ can be parametrised as

$$a_0(d) = Cd^\alpha \quad (32)$$

We put forward hypothesis, that the form (31) of the ARF factor is universal, but the scaling exponents, α, and λ, are different for different types of climate. To check this hypothesis, more detailed analysis of Australian data for different locations is necessary.

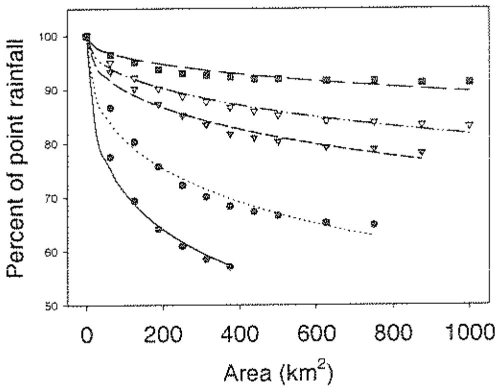

Figure 4. ARF factors for different accumulation periods (From top to bottom: 0.5, 1, 3, 6, and 24 hours).

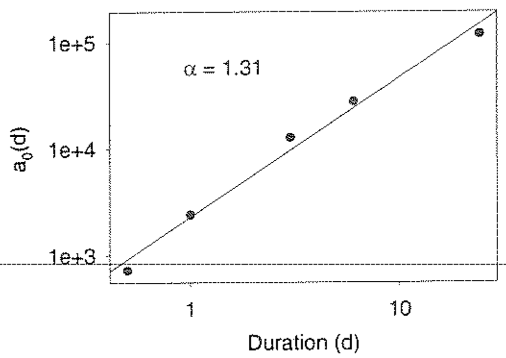

Figure 5. Parameter $a(d)$ fitted by a scaling relation.

4. CONCLUSION

We have presented a hypothesis that the annual maximum rainfall intensities for different durations and areas can be considered as random variables characterised by the same CDF with properly rescaled parameters. The scale transformation functions can be derived if one assumes that the rainfall intensities for different time and space scales are connected through generalized scaling relations. This hypothesis was independently checked for a long record of rain gauge time series (i.e. point rainfall) and the ARF curves recommended for use in Austalia. The proposed methodology allows one to derive a simple analytical IDF relation which includes the ARF. The paper presents an attempt to find a link between the empirical approach, currently used in the engineering practice, and modern scaling theories of time and space properties of rainfall. We consider the results presented here as preliminary. More comprehensive analysis of dense networks of rain gauges and weather radars for different location is needed to further advance the theory.

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