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An improvement of jump diffusion model for Japan Nikkei 225 indexes and its application to estimating the stochastic volatility

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Abstract: We investigate the daily share prices of Nikkei 225 indexes to identify jump times using a jump diffusion model, which consists of the Black-Scholes model with stochastic volatility and a compound Poisson process. We consider how to separate jump times from the observed data. From the obtained results we propose a new type of jump diffusion model for the share prices and a robust scheme to estimate the stochastic volatility.

Keywords: *Jump diffusion model, Nikkei 225 indexes, stochastic volatility, historical volatility*

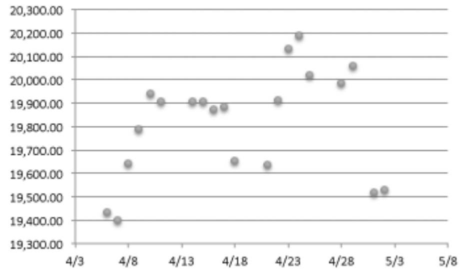

Figure 1. Daily share prices of Nikkei 225 indexes, 2015/2/27∼4/8

1 INTRODUCTION

Fig. 1 shows daily share prices of Nikkei 225 indexes in the period of February 27th ∼ April 8th, 2015. Such real data of daily share prices consist of only jumps without continuous parts. Therefore it is difficult to find real jumps to distinguish precisely between the jump part and the continuous part in the stock price model (1) or (2) defined in the next section.

In this paper we show an algorithm to separate large jumps generated from daily share prices of Nikkei 225 indexes according to Ishida and K. (2015) and K. and Shieh (2019). From the results we can obtain a new type of jump diffusion model for stock prices. Furthermore we would like to propose a new method to estimate the stochastic volatility in the jump diffusion model.

2 MODELING OF SHARE PRICES

The following equation is a jump diffusion model of share prices proposed by Ball and Torous (1983), (1985), Cont and Tankov (2008), Iino and Ozaki (1999), Kou (2002), etc.;

$$dS\left(t\right)=\mu S\left(t\right)dt+\sigma S\left(t\right)dB\left(t\right)+S\left(t\right)dZ\left(t\right),\quad 0\leq t\leq T \tag{1}$$

for small T, where $S\left(t\right)$ is a share price, $B\left(t\right)$ is a standard Brownian motion, μ is a trend parameter, $\sigma_t>0$ is a volatility and $Z\left(t\right)$ is a compound Poisson process. Furthermore the independence of $B\left(t\right)$ and $Z\left(t\right)$ is assumed. Since the volatility σ and the trend μ change in a long period, recently the model (1) was improved as the form

$$dS\left(t\right)=\mu_t S\left(t\right)dt+\sigma_t S\left(t\right)dB\left(t\right)+S\left(t\right)dZ\left(t\right),\quad 0\leq t\leq T, \tag{2}$$

where μ_t is a stochastic trend and σ_t is a stochastic volatility.

On the other hand, in Ishida and K. (2015) and K. and Shieh (2019) we investigated the daily share prices of Nikkei 225 indexes to estimate the jump times of large size jumps. Since the usual model (2) can not fit our results, we would like to propose the following new jump diffusion model;

$$dS\left(t\right)=\mu_t S\left(t\right)dt+\sigma_t S\left(t\right)dB\left(t\right)+\sigma_t S\left(t\right)dZ\left(t\right),\quad 0\leq t\leq T. \tag{3}$$

In the new model (3) the intensity $\lambda>0$ of the compound Poisson process $Z\left(t\right)$ is a constant in the whole period $[0,T]$. Although the risk of the share price $S\left(t\right)$ changes very frequently in $[0,T]$, we would like to show that the cause of the risk change mainly due to σ_t in (2) but not λ from the obtained data.

3 COMPOUND POISSON PROCESS $Z\left(t\right)$

Let $N\left(t\right)$ be a counting process which means a number of jumps until the time $t>0$. The distribution of $N\left(t\right)$ obeys the Poisson distribution such that

$$P\left\{N\left(t\right)=k\right\}=e^{-\lambda t}\frac{\left(\lambda t\right)^k}{k!},\quad k=0,1,2,\cdots \tag{4}$$

for some positive constant $\lambda>0$ which is called "intensity". Let $t_1,t_2,\cdots$ be jump times of $N\left(t\right)$ and $\tau_1,\tau_2,\cdots$ be differences of the jump times defined by

$$\tau_k=t_k-t_{k-1},\quad k=1,2,\ \cdots. \tag{5}$$

Define a compound Poisson process $Z(t)$ by

$$Z(t) = Y_1 + Y_2 + \cdots + Y_{N(t)}, \tag{6}$$

where $Y_1, Y_2, \cdots$ are i.i.d. random variables which mean jump sizes.
It is well known that $\tau_1, \tau_2, \cdots$ are i.i.d. random variables with the exponential distribution $\mathrm{Exp}(\lambda)$ such that

$$P\{\tau_k \leq x\} = \int_0^x \lambda e^{-\lambda s} ds \tag{7}$$

for each $k = 1, 2, \ \cdots$. In the following sections we consider an algorithm to find the jump times $t_1, t_2, \cdots$ by an estimator χ^2_α for the testing fitness to the Poisson distribution of samples of $Z(1)$.

4 Estimation of the Trend μ_t and the Volatility $\sigma_t > 0$

We estimate μ_t and $\sigma_t > 0$ from the Nikkei 225 indexes data. Let

$$r(k) = \log \frac{S(k+1)}{S(k)} \simeq \frac{S(k+1) - S(k)}{S(k)} \tag{8}$$

be the return of the kth day and put

$$s_t = \sqrt{\frac{1}{\ell - 1} \sum_{k=1}^{\ell} (r(t-k) - \bar{r})^2}, \quad \bar{r} = \frac{1}{\ell} \sum_{k=1}^{\ell} r(t-k). \tag{9}$$

s_t is the historical volatility estimating $\sigma_t > 0$ using share prices of ℓ days. Furthermore let

$$\bar{\mu}(t) = \log \frac{S(t)}{S(t-\ell)} \tag{10}$$

be an estimator of μ_t.

Standardize $r(t)$;

$$R(t) = \frac{r(t) - \bar{r}(t)}{s_t} \tag{11}$$

5 Procedure for Identifying Jump Times of Large Jumps

Since the counting process $N(t)$ obeys Poisson distribution

$$P\{N(t) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \cdots, \tag{12}$$

we estimate the intensity λ by the following steps.

We define an *unit period* with q days and divide 7447 days of 30 years from 1986 to 2015 into

$$L = \left[\frac{7450}{q}\right] \tag{13}$$

periods.

We count the number of jumps observed in each period which means a sample of $N(1)$.

We identify jump times of $R(t)$ for large size jumps in each period by the following way.
1st step : Put a threshold level $\alpha > 0$ which controls the jump size.

2nd step: If $|R(k)| \geq \alpha$ in the jth period, then we consider that the kth day is a jump time. Let m_j be the number of all jumps larger than α in the $j$the period. $m_j, j = 1, 2, ..., L$ are samples of $N(1)$.

3rd step: For the samples $m_j, j = 1, 2, ..., L$, put

$$\lambda_\alpha = \frac{1}{L}\left(m_1 + m_2 + \cdots + m_L\right), \tag{14}$$

which is an estimator of intensity of Poisson distribution.

4th step: Let n_k be the number of periods having k jumps in a period. From the definition of n_k

$$n_0 + n_1 + \cdots + n_K = L,$$

where K is the maximum of number of jumps in a period.

5th step: Calculate the following estimator χ^2_α for the test of goodness of fit to Poisson distribution with the intensity λ_α for the samples $n_0, n_1, \cdots, n_K$.

$$\chi^2_\alpha = \sum_{k=0}^{K+1} \frac{\left(n_k - kp_k\right)^2}{kp_k}, \tag{15}$$

where

$$p_k = e^{-\lambda_\alpha t}\frac{\left(\lambda_\alpha t\right)^k}{k!}, \quad k = 0, 1, 2, \cdots, K$$

and

$$p_{K+1} = 1 - \sum_{k=1}^{K} e^{-\lambda_\alpha t}\frac{\left(\lambda_\alpha t\right)^k}{k!}.$$

6th step: Compare χ^2_α for all combinations of number of days in a unit period $q = 20, 30, 40$ and number of days to observe the historical volatility with $\ell = 10, 15, 20, 25, 30$. Calculate χ^2_α for each $\alpha = 1.0, 1.1, \cdots, 2.6$.

7th step: Find returns $r(t)$ which minimize χ^2_α for some q and ℓ. If the minimum χ^2_α is small enough, we conclude that these returns are Poisson-distributed and they are generated by the compound Poisson process $Z(t)$.

6 Tables and Investigation

Table 1. q=20, ℓ=7446 (non-random volatility model)

α	1.9	2.0	2.1	2.2
χ^2_α	4490.7	1416.0	1371.7	907.84
λ_α	1.1374	0.9035	0.8382	0.7250

Since $\chi^2_\alpha > 900$ in Table 1 for every $\alpha \le 2.2$, we can easily understand that the volatility changes randomly.

In Table 2 with q=30, ℓ=25, $\chi^2_\alpha \le 10$ for almost $\alpha \ge 1.0$. Thus we can see that the numbers of jumps $n_0, n_1, \cdots, n_K$ with jump size α obey the Posson distribution with the intensity λ_α for each $\alpha \ge 1.0$.

On the other hand, $\chi^2_\alpha \ge 10$ for any $\alpha \ge 1.0$ in Table 2 with q=30, ℓ=30 . The result means that the combination of q and ℓ is critcal for the quantity of χ_α, where q is the number of days in the unit interval which divides 7447 days in 30 years to $L = \left[\frac{7447}{q}\right]$ intervals and ℓ is the number of observed days used in the

Table 2. q=30, ℓ=25

α	1.0	1.1	1.2	1.3	1.4	1.5
χ^2_α	8.5224	8.00047	8.90629	14.200	8.4489	12.775
λ_α	9.6693	8.3709	7.1572	6.1975	5.4233	4.6209

α	1.6	1.7	1.8	1.9	2.0	2.1
χ^2_α	20.494	7.6199	4.6648	3.2161	7.8219	5.3526
λ_α	3.8750	3.3185	2.8225	2.4475	2.0241	1.7298

α	2.2	2.3	2.4	2.5
χ^2_α	8.5603	8.3653	4.4160	5.2399
λ_α	1.4233	1.2177	1.0322	0.8588

Table 3. q=30, ℓ=30

α	1.9	2.0	2.1	2.2	2.3
χ^2_α	14.228	30.136	31.743	20.925	28.083
λ_α	2.3024	1.9435	1.6572	1.3991	1.1733

Table 4. Frequency distribution table of returns with $|R(t)| \geq 1$, q=30 days, ℓ=25 days

$r(t)$	$\sim$ -0.1	$\sim$ -0.09	$\sim$ -0.08	$\sim$ -0.07	$\sim$ -0.06
frequency	4	1	0	4	8

$r(t)$	$\sim$ -0.05	$\sim$ -0.04	$\sim$ -0.03	$\sim$ -0.02	$\sim$ -0.01
frequency	17	32	108	308	505

$r(t)$	$\sim$ 0.00	$\sim$ 0.01	$\sim$ 0.02	$\sim$ 0.03	$\sim$ 0.04
frequency	236	175	572	283	86

$r(t)$	$\sim$ 0.05	$\sim$ 0.06	$\sim$ 0.07	$\sim$ 0.08	$\sim$ 0.09
frequency	33	14	4	5	0

$r(t)$	$\sim$ 0.1	0.1 $\sim$
frequency	1	3

Table 5. Frequency distribution table of returns with $|R(t)| < 1$, q=30 days, ℓ=25 days

$r(t)$	$\sim$ -0.05	$\sim$ -0.04	$\sim$ -0.03	$\sim$ -0.02	$\sim$ -0.01
frequency	1	1	6	32	393

$r(t)$	$\sim$ 0.00	$\sim$ 0.01	$\sim$ 0.02	$\sim$ 0.03	$\sim$ 0.04
frequency	2201	1974	360	41	6

$r(t)$	$\sim$ 0.05	$\sim$ 0.06	$\sim$ 0.07	0.07 $\sim$
frequency	1	2	2	0

historical volatility s_t^2.

We next consider the reason why the distribution of numbers of $R(t)$ such that $|R(t)| \geq \alpha$ fits Poisson distribution from the view points of jump diffusion models of share prices (2) and (3), respectively.

Under the jump diffusion model (3), the increment $\Delta S(t)$ can be represented by

$$\Delta S(t) = \mu_t S(t)\Delta t + \sigma_t S(t)\Delta B(t) + \sigma_t S(t)\Delta Z(t) \tag{16}$$

where

$$\Delta S(t) = S(t+\Delta t) - S(t),\ \Delta B(t) = B(t+\Delta t) - B(t),\ \Delta Z(t) = Z(t+\Delta t) - Z(t)$$

Let r_t be the return of $S(t)$ defined by

$$r(t) = \frac{S(t+\Delta t) - S(t)}{S(t)} = \frac{\Delta S(t)}{S(t)} = \mu_t \Delta t + \sigma_t \Delta B(t) + \sigma_t \Delta Z(t) \tag{17}$$

Let $R(t)$ be the normalized return defined by

$$R(t) = \frac{r_t - \mu_t \Delta t}{\sigma_t} = \Delta B(t) + \Delta Z(t) \tag{18}$$

When the stock price model (3) is assumed, if the volatility σ_t is small then $r(t)$ is also small and it is difficult to find the jump times in this case. On the other hand, the normalized return $R(t)$ is independent of σ_t, we can find jump times when the volatility σ_t is small.

We should like to present such results from the investigation of stock indexes.

Put $\Delta t = 1$. Then r_t is the daily return and we have

$$P\{|R(t)| \geq 2\,|\Delta Z(t) = 0\} = \frac{P\{|R(t)| \geq 2,\ \Delta Z(t) = 0\}}{P\{\Delta Z(t) = 0\}} \tag{19}$$

$$= \frac{P\{\ |\Delta B(t) + \Delta Z(t)| \geq 2,\ \Delta Z(t) = 0\}}{P\{\Delta Z(t) = 0\}}$$

$$= \frac{P\{\ |\Delta B(t)| \geq 2\}\, P\{\Delta Z(t) = 0\}}{P\{\Delta Z(t) = 0\}} = P\{\ |\Delta B(t)| \geq 2\} \simeq 0.05,$$

which implies that if a jump does not occur at the t-th day, then $|R(t)| < 2$ with probability 0.95. Thus we have the following Remarks 1 and 2.

Remark 1. *From Table 4 we can observe 411 days with $|R(t)| \geq 1$ but $|r(t)| < 0.01$. Normalizing $r(t)$ by historical volatility $s(t)$ picks up these small returns less than 0.01 which are generated by the compound Poisson process $Z(t)$. Since χ_α^2 is very small, large returns picked up by our method fit to Poisson distribution very well.*

Remark 2. *From Table 5 we can observe 92 days with $|R(t)| < 1$ but $|r(t)| \geq 0.02$. Normalizing $r(t)$ by historical volatility $s(t)$ evaluates these large returns more than 0.02 are generated by the continuous process like B-S model. Since χ_α^2 is very small, large returns picked up by our method fit to Poisson distribution very well.*

On the other hand, under the usual jump diffusion model (2) we have

$$R(t) = \frac{r_t - \mu_t \Delta t}{\sigma_t} = \Delta B(t) + \frac{\Delta Z(t)}{\sigma_t}. \tag{20}$$

Thus Table 2 means that $\frac{Z(t)}{\sigma_t}$ obeys the Poisson distribution. The result inconsistent with that $Z(t)$ is a compound Poisson process. Therefore we would like to propose (3) as a financial stochastic model for share prices.

7 Volatility Estimation

As we mentioned in the previous section, $\chi^2_\alpha \leq 10$ for almost $\alpha \geq 1.0$ in Table 2 with q=30, ℓ=25. The result means that the root of the historical volatility s_t defined by (9) with share prices of $\ell = 25$ days optimally estimates the stochastic volatility σ_t. We would like to explain the detail of the volatility estimation in K. and Shieh (2019).

8 Conclusion

1. From Table 1 we can easily find that volatility changes randomly.

2. In Table 2 with q=30, ℓ=25, $\chi^2_\alpha \leq 10$ for almost $\alpha \geq 1.0$. Thus we can see that the numbers of jumps $n_0, n_1, \cdots, n_K$ with jump size α obey the Poisson distribution with the intensity λ_α for each $\alpha \geq 1.0$. The quantity of χ^2_α for $\alpha > 1.0$ is almost same as for Poisson-distributed random numbers generated by computer simulation. Therefore we consider that the returns with these jumps are generated by the compound Poisson process $Z(t)$.

3. From these results we can find not only jump times of large jumps for Nikkei 225 indexes but also optimal historical volatility for the stock index. The root of the historical volatility s_t defined by (9) with share prices of $\ell = 25$ days optimally estimates the stochastic volatility σ_t.

4. The stock price model (3) fits our observations.

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