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Modelling fire line merging using plane curvature flow

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Abstract: The merging of two lines of fire is a relatively common occurrence in landscape fire events. For example, it can arise through the coalescence of two wildfires or when a prescribed fire meets a wildfire as part of suppression efforts. When two fires approach one another, the effects of convective and radiative heat transfer are compounded and high rates of spread can arise as a result. This is particularly the case when two oblique lines of fire meet at some acute angle - the point of intersection on the newly merged fire can advance rapidly. This case was investigated recently by Viegas et al. (2012), who devised a simple analytical model to emulate the effects of energy concentration between the two merging fire lines. In this paper, we present a more geometric approach by considering the evolution of the merged fire as the flow of a plane curve with a normal speed that depends on the curvature of the fire front.

Specifically, we formulate the curvature flow equations in terms of a time-varying graph $y(x,t)$. The resulting evolution equation is a quasilinear degenerate parabolic second-order partial differential equation:

$$\partial_t y = \sqrt{1 + (\partial_x y)^2} + \frac{\epsilon \partial_x^2 y}{1 + (\partial_x y)^2}.$$

The evolution of the merged fire lines is then modelled by solving an associated initial value and boundary value problem, where the initial conditions are taken $y_0(x) \approx |x| \tan\theta$. The θ here controls the initial angle between the two merged fire lines.

Parametric variation of the curvature dependence (via the parameter ϵ) is investigated, and the resultant geometric evolutions of the fire front are compared with the experimental observations of Viegas et al. (2012). The curvature flow simulations were able to capture a number of features that were observed by Viegas et al. (2012). In particular, the model was able to reproduce the 'rotation' of the two fire lines noted by Viegas et al. (2012) and was able to account for the qualititative rate of spread behaviour observed in connection with the rapid advance of the point of intersection of the merged fire lines. This so-called jump velocity profile was characterised by very high initial rates of spread, immediately after the fire lines merged, and then a gradual slowing of the rate of spread approaching a quasi-steady value.

Further theoretical aspects of plane curvature flows and their more general application to fire front modelling are discussed and a number of improvements to the model are suggested.

Keywords: *Plane curvature, evolution equations, normal flow, fire merging*

1 INTRODUCTION

Fire behaviour modelling is traditionally aimed at determining, to within a reasonable degree of accuracy, the rate of spread and fire behaviour characteristics of a single line of fire. This is reflected in the empirically-based models that form the benchmark for fire behaviour modelling in Australia and in many other parts of the world (Pastor et al., 2003; Sullivan, 2009). These models perform reasonably well when used to predict the behaviour of single fires that burn under uniform conditions similar to those upon which the models were derived, but can perform quite poorly when confronted with situations involving more sophisticated dynamics. In some such situations the assumption of quasi-steady spread inherent in empirically-based models, whereby a fire is assumed to spread at a rate of spread uniquely determined by a given set of environmental conditions, can be violated and a fire can behave in ways that are distinctly at odds with the traditional models.

Viegas et al. (2012) consider such an example of dynamic fire spread, namely the merging of two oblique lines of fire. The interaction of these two fire fronts greatly enhances the convective and radiative heat transfer processes that drive fire propagation and can result in the very rapid advance of the merged front. This behaviour was referred to as a 'jump fire' by Viegas et al. (2012) and is illustrated in Figure 1a.

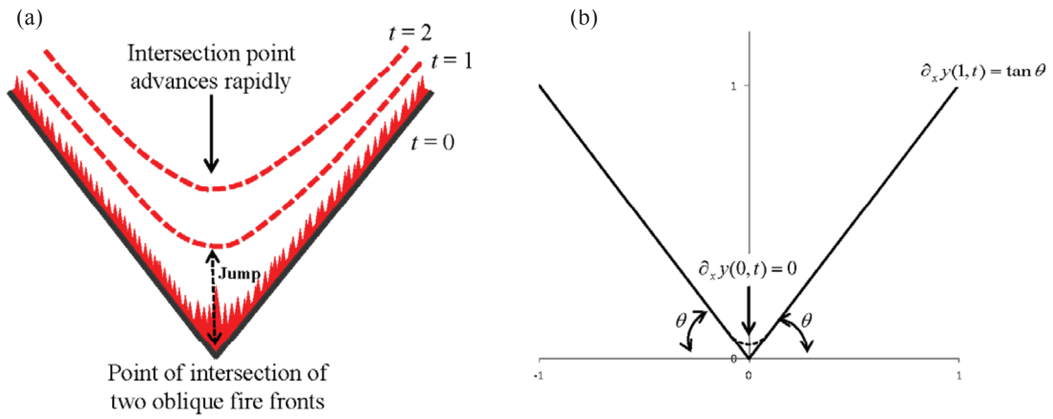

Figure 1. (a) Schematic illustrating the merging of two oblique fire lines and the rapid advance of their intersection point known as the 'jump fire'. The dashed lines represent the approximate position of the fire line 1 and 2 units of time after the fire lines first intersect. (b) Graph of the ideal initial fire front given by $y_0 = |x| \tan\theta$ and the one used in this study (dashed line indicates the interpolating polynomial). The boundary conditions used in the analysis on the half-interval are also indicated.

To account for the behaviour of 'jump fires', Viegas et al. (2012) developed an analytical model that described the accumulation of energy between the two intersecting fire fronts and related it to the resultant rate of spread. Their model was based on an assumed distribution of energy produced by the fire front near the intersection point and included a local quasi-steady spread assumption; that is, despite the intrinsic dynamics of the process, it was broken down into time steps over each of which the fire was assumed to propagate in a quasi-steady fashion.

In this paper we also consider the merging of two oblique fire fronts and propose a model to account for the enhanced rate of propagation of the intersection point. The approach we take considers the curvature of the fire line as a proxy for the accumulation of energy. Curvature has been used classically to approximate the energy of phase interfaces in materials science and in the Bernoulli rod (e.g. Mullins (1956)). The incorporation of curvature permits us to consider the evolution of the merged fire line in purely geometric terms. Indeed, in this framework the propagation of the merged fire front is described by a system of quasilinear degenerate parabolic second-order partial differential equations. The model takes into account the non-linear dynamics at sharp corners and can be analysed without the assumption that the spread is locally quasi-steady.

We consider the propagation of the fire front given a normal speed with a specific (parameterized) functional dependence on the curvature of the fire line. Variation of the parameters involved is investigated to determine the choice of parameters that best reproduces the fire behaviour observed in the experimental work of Viegas et al. (2012). Like Viegas et al. (2012) we only consider the case where the fire lines are arranged symmetrically and the fire spreads through uniform fuel over flat ground.

2 METHODS

2.1 Normal flow of a plane curve with curvature dependent speed

We consider the merged fire front as being described by an evolving plane curve. That is, we consider the two merging fire lines as being described by a mapping $\gamma : (a, b) \times [0, T) \to \mathbb{R}^2$. We assume that for each $t \in (0, T)$ the curve $\gamma(\cdot, t)$ is of class C^2 and regular, so that $|\gamma'(\cdot, t)| \neq 0$. This ensures that there is a well-defined tangent vector at each point. We will refer to the auxilliary mapping $\boldsymbol{\gamma}(\cdot, t) : (a, b) \to \mathbb{R}^2$ as the (time-dependent) curve $\boldsymbol{\gamma}$.

The subsequent evolution of the two merging fire lines is then modelled by the normal flow of $\boldsymbol{\gamma}$ with a speed that depends on the plane curvature of $\boldsymbol{\gamma}$. If, at time t, $\boldsymbol{\gamma}$ is defined parametrically in terms of its (x, y) coordinates as:

$$\boldsymbol{\gamma}(s, t) = (x(s, t), y(s, t)), \quad s \in (a, b), \tag{1}$$

then the plane curvature $k(s, t)$ of $\boldsymbol{\gamma}$ is defined as

$$k(s, t) = \frac{\partial_s x(s, t)\, \partial_s^2 y(s, t) - \partial_s y(s, t)\, \partial_s^2 x(s, t)}{((\partial_s x(s, t))^2 + (\partial_s y(s, t))^2)^{\frac{3}{2}}}. \tag{2}$$

Formally, the evolution of the curve $\boldsymbol{\gamma}$ via normal flow with curvature dependent speed $F(k)$ is then given by the PDE system

$$\boldsymbol{\nu} \cdot \partial_t \boldsymbol{\gamma} = F(k), \tag{3}$$

where $\boldsymbol{\nu}$ is the unit normal vector field to the curve, defined by:

$$\boldsymbol{\nu} = \frac{(-\partial_s y, \partial_s x)}{\sqrt{(\partial_s x)^2 + (\partial_s y)^2}}. \tag{4}$$

We note that (3) implies that

$$\partial_t \boldsymbol{\gamma} = F(k)\boldsymbol{\nu} + W\boldsymbol{\tau}, \tag{5}$$

where $W\boldsymbol{\tau}$ is the tangential component of changes in $\boldsymbol{\gamma}$, with $\boldsymbol{\tau} = \left((\partial_s x)^2 + (\partial_s y)^2\right)^{-\frac{1}{2}} (\partial_s x, \partial_s y)$ the unit tangent vector field. The tangential motion $W\boldsymbol{\tau}$ can be thought of as shuffling around points on the image of $\gamma(\cdot, t)$, amounting to a reparametrisation at each time. Since our main interest is in the qualitative properties of the image and not on the particular parametrisation, the range of choices of W represent a large group of degeneracies intrinsic to the model.

In the following we will consider a speed functional of the form:

$$F(k) = 1 + \epsilon k. \tag{6}$$

We note that for a straight line of fire this speed reduces to $F(k) = 1$, which we take to be the dimensionless (quasi-steady) rate of spread of a fire burning on flat ground in the absence of wind. The analogous expression in the presence of wind and/or slope is given by Roberts (1992). In the presence of (negative) fire line curvature (6) leads to an increase in speed in the normal direction.

If it so happens that the curve $\boldsymbol{\gamma}$ is a graph of a function for all times t, so that $\boldsymbol{\gamma} = \boldsymbol{\gamma}(x, t) = (x, y(x, t))$, then the curvature can be expressed as:

$$k(x, t) = \frac{\partial_x^2 y(x, t)}{(1 + (\partial_x y(x, t))^2)^{\frac{3}{2}}}, \tag{7}$$

and the unit normal vector field reduces to

$$\boldsymbol{\nu} = \frac{(-\partial_x y(x, t), 1)}{\sqrt{1 + (\partial_x y(x, t))^2}}. \tag{8}$$

For the case depicted in Figure 1a and in the experiments of Viegas et al. (2012), it is clear that evolution of the merged fire front can be adequately described by the graph of a function $y(x,t)$. Thus focusing our attention to this case, the PDE system (3) for the normal flow with speed given by (6) reduces to the single nonlinear PDE:

$$\partial_t y = \sqrt{1 + (\partial_x y)^2} + \frac{\epsilon \partial_x^2 y}{1 + (\partial_x y)^2}. \tag{9}$$

Note that the condition $W = F(k)\partial_x y$ has been used in the derivation of (9) to preserve the parameterisation of the graph throughout its evolution; that is, to ensure that $\partial_t x(s,t) = 0$.

2.2 Initial and boundary conditions for the normal flow

We consider an initial fire front which is given by two intersecting straight lines of fire as shown by the solid line in Figure 1b. Such a front can be represented by the graph of the function $y_0 = |x| \tan\theta$, where θ is the angle that each of the fire lines makes with the x-axis in Figure 1b. Ideally, we would like to take this function as our initial condition for the evolution problem (9), considered over the domain $[-1,1]$. For computational purposes we take the closed domain as an approximation of the infinite fire-line considered in the previous section, augmented with appropriate boundary conditions (see (12)). The curvature of this curve cannot be defined at the origin. To make the problem tractable, we smooth the initial front in the vicinity of the origin using polynomial interpolation. The initial conditions are thus:

$$y(x,0) = \begin{cases} -x, & x \in [-1,-\delta], \\ p(x), & x \in [-\delta,\delta], \\ x, & x \in [\delta,1], \end{cases} \tag{10}$$

where $\delta \ll 1$ is some small positive number and $p(x)$ is the fourth-order polynomial satisfying

$$p(\pm\delta) = \delta,\;\; p'(\pm\delta) = \pm 1,\;\; p''(\pm\delta) = 0.$$

The dashed curve in Figure 1b illustrates $p(x)$ for $\delta = 0.1$.

Given the inherent symmetry that we have just imposed on the problem, it is sufficient to consider the problem over the domain $[0,1]$. The evolution of the fire front over $[-1,0]$ is simply the mirror image of that over $[0,1]$. We therefore only need consider the initial condition as

$$y(x,0) = \begin{cases} p(x), & x \in [0,\delta], \\ x, & x \in [\delta,1]. \end{cases} \tag{11}$$

To properly pose the evolution of the fire front in terms of normal flow, the PDE problem given by (9,11) must be augmented with appropriate boundary conditions. If we consider the initial condition (11) as the trunction of a fire line that extends for some distance beyond $x = 1$, then the appropriate Neumann boundary conditions must be:

$$\partial_x y(0,t) = 0,\;\; \partial_x y(1,t) = \tan\theta. \tag{12}$$

3 RESULTS

The nonlinear PDE (9) subject to the initial and boundary conditions given by (11,12) was solved numerically over the interval $[0,1]$ using the method of lines (Schiesser, 1991). Differing values of θ were used and the resulting evolution of the initial fire line was compared with the observations of Viegas et al. (2012). The model output was also used to construct plots of the *jump velocity* against time, where the jump velocity is defined as the velocity of the central point where the two lines of fire initially intersect. These jump velocity profiles are compared with those obtained from the data of Viegas et al. (2012)

3.1 Fire line evolution

Figure 2 shows the resultant evolutions for different values of the parameters ϵ and θ. The vertical progression of the merged fire line is more rapid for larger values of θ, for which the initial fire lines were in closer proximity. Larger values of ϵ also caused more rapid progression of the merged fire fronts and caused them to flatten out more readily.

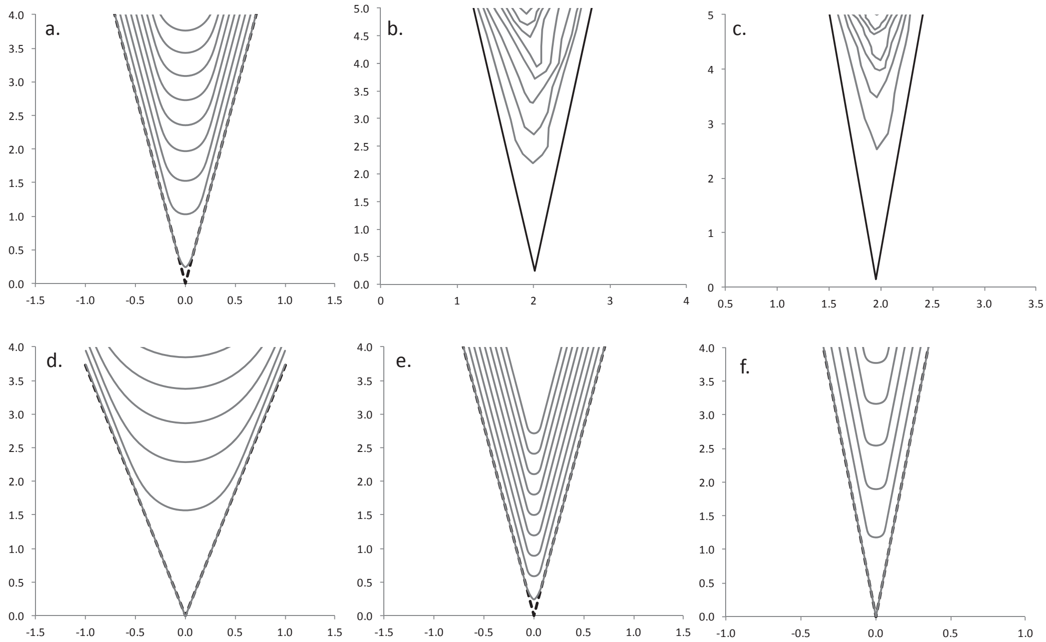

Figure 2. Modelled and experimental evolution of merged oblique fire lines: (a) Modelled fire line evolution with $\epsilon = 1$, $\theta = 80°$; (b) Experimental observations of Viegas et al. (2012) corresponding to the case $\theta = 80°$;(c) Experimental observations of Viegas et al. (2012) corresponding to the case $\theta = 85°$; (d) Modelled fire line evolution with $\epsilon = 5$, $\theta = 75°$; (e) Modelled fire line evolution with $\epsilon = 0.2$, $\theta = 80°$; (f) Modelled fire line evolution with $\epsilon = 0.5$, $\theta = 85°$.

Overall, there is reasonably good agreement between the simulated merged fire line and the behaviour of the experimental fires of Viegas et al. (2012). The model has been able to capture the rapid initial spread of the point of intersection (the 'jump fire') and the subsequent slowing of the progression of the merged fire line.

In particular, Viegas et al. (2012) note that in each of the merged fire line cases they considered, the angle between the two fire lines increased with time, and that this increase was greater for cases when θ (as used in the present paper) was larger. This aspect of the experimental observations of Viegas et al. (2012) is reproduced by our curvature based model. Indeed, it could be argued that treating the experimental fire lines of Viegas et al. (2012) as two straight lines at each step of the evolution is an over-simplification of the situation and that a curved front is more faithful to reality. In any case, the effect of curvature naturally causes the two halves of the merged fire line to 'rotate' as the curve flattens out, in a way consistent with that noted by Viegas et al. (2012).

3.2 Jump velocity profiles

By tracking the evolution of the point corresponding to $x = 0$ in the simulations it was possible to construct profiles of the jump velocity. These can be seen in Figure 3 where the simulated jump profiles are qualitatively compared with those derived from the experimental data of Viegas et al. (2012). To facilitate comparison, the times associated with the data of Viegas et al. (2012) have been rescaled to match the simulation times. Overall, the curvature flow model is able to reproduce the general pattern of behaviour of the jump fire reasonably well. The jump velocity is initially relatively high, but decreases quickly before it asymptotically approaches a limiting rate of spread.

In quantitative terms, the experimental and simulated rates of spread are orders of magnitude apart, so there are clearly some issues of rescaling that would need to be addressed before it could be hoped the model would give reliable predictions.

The jump velocity was also found to increase overall as θ was increased; that is, as the angle between the two initial fire lines was widened the jump velocity of the intersection point decreased. However, the jump velocity obtained when the angle between the fire lines was $30°$ ($\theta = 75°$) was slightly greater than when the angle

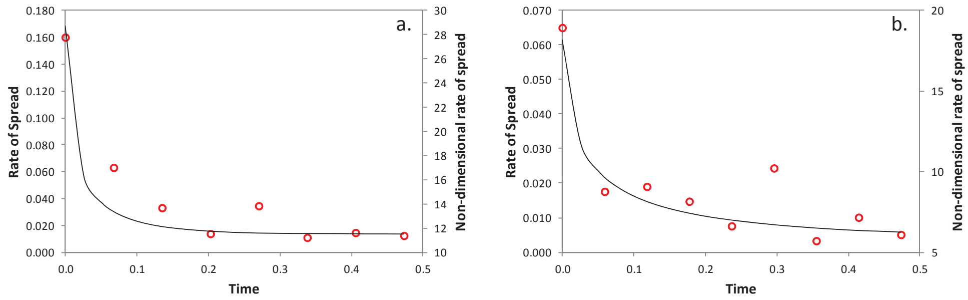

Figure 3. Jump velocity profiles showing model prediction (black line) and experimental values (red circles): (a) $\epsilon = 0.5$, $\theta = 85°$; (b) $\epsilon = 1$, $\theta = 80°$.

between the two lines was 20° ($\theta = 85°$). This can be seen in Figure 4. This is likely an artefact of the way the jump velocity was calculated using crude averaging. As expected the jump velocities are generally lower for the smaller value of ϵ. There is insufficient data in Figure 4 to make further comparison between it and the analytical model derived by Viegas et al. (2012, Fig. 18).

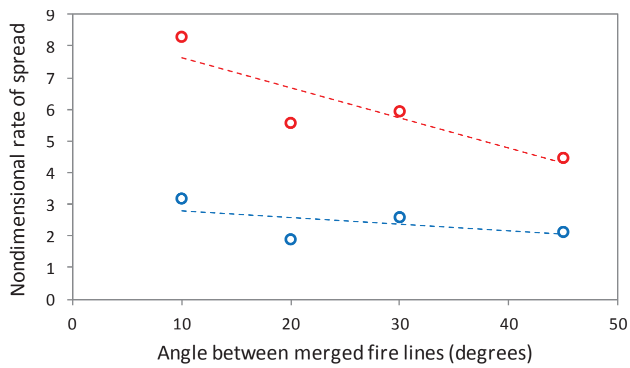

Figure 4. Non-dimensional jump velocity derived from the curvature flow model plotted against the angle between the merged fire lines ($2(90° - \theta)$). The red circles correspond to $\epsilon = 1$, the blue circles to $\epsilon = 0.1$. The dashed straight lines represent the overall linear trend in the values of matching colour.

4 Discussion and Conclusion

Instances of dynamic fire spread pose a significant threat to the safety of firefighters and the community. The rapid rates of spread, like those seen in the 'jump fire' experiments of Viegas et al. (2012), could cause firefighters to be overrun when caught between two merging firelines, for example. In this context it is interesting to note that current operational modelling frameworks are not able to account for the feedbacks that give rise to this type of dynamic behaviour. We have investigated a curvature-based model that accounts for a number of aspects of dynamic fire spread.

The model was conceptualised as a normal flow with curvature dependent speed. The properties of such flows have been well-studied (Sethian, 1985; Osher and Sethian, 1988). In particular, it has been proven that certain normal flows with curvature dependent speed have the property that any sharp corner or cusp point in the initial data will be immediately smoothed (Ecker and Huisken, 1999; Angenent, 1990, 1991; Altschuler and Grayson, 1992). Such a property is consistent with the behaviour of fire perimeters observed during experimental programs (Anderson et al., 1982).

The model was implemented to track the evolution of an initial graph that resembled the initially merged fire lines. The simulated fire line evolution closely resembled the experimental observations of merged fire lines. In particular, the presence of the curvature term in the model (9) was able to account for the variations in angular velocity discussed by Viegas et al. (2012).

The curvature flow model was also able to reproduce the behaviour of the jump velocity of the point of intersection, at least in a qualitative sense. No attempt was made to reconcile the actual magnitudes of the observed and simulated jump velocities. The form of the curvature model taken in (9) was such that as the plane curvature dissipates, the local rate of spread converges to unity, which we would normally interpret as the non-dimensional quasi-steady rate of spread for the particular fuel conditions that are present. However, none of the simulated fire line evolutions had a velocity that approached unity.

The reason for this is the boundary conditions. Recall that we set the boundary condition $\partial_x y(1,t) = \tan\theta$. This was done to try and emulate the motion of two fire lines of infinite extent. The results therefore suggest that these boundary conditions may be too restrictive and do not provide a faithful representation of the evolution of the two *finite* fire lines treated in the experiments of Viegas et al. (2012). In the future we will try to remedy this issue by experimenting with the boundary conditions to better reflect the finite fire lines involved. We will also consider modelling the evolution of the merged fire lines by starting with a simple closed curve, rather than a graph, for initial data and using a level-set method (Sethian, 1985) to numerically determine the evolution. Such an approach will also allow us to incorporate initial data that is only C^0, rather than C^2 as was assumed above.

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