Estimation of Lévy Processes in Mathematical Finance: A Comparative Study

[1]Sueishi, N. and [2]Y. Nishiyama

[1]Graduate School of Economics, Kyoto University,
[2] Kyoto Institute of Economic Research, Kyoto University,
E-Mail: nsueishi@e01.mbox.media.kyoto-u.ac.jp

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EXTENDED ABSTRACT

In the field of mathematical finance, the concern with the applications of Lévy processes has been growing for the last several years. The models are aimed at incorporating *stylized empirical facts*. In the classical Black-Scholes option pricing model, the log return of asset is assumed to follow the normal distribution. However, compared to the normal distribution, the empirical density of log returns typically has more mass near the origin, less in the flanks, and more in the tail. Empirical works also suggest the discontinuity of the sample path of price processes. To account for these features, several models based on Lévy processes have been propounded in the literature. Prominent examples include the Poisson jump model (Merton (1976)), the variance gamma process (Madan and Seneta (1990), Madan et al. (1998)), the normal inverse Gaussian process (Barndorff-Nielsen (1998)), and the CGMY process (Carr et al. (2002)).

Although relevant estimators of the parameters are required for applications of these models, the estimation of Lévy processes is challenging. Since the law of Lévy processes is entirely specified by the infinitely divisible distribution, it suffices to estimate the parameters of corresponding infinitely divisible distributions. However, they are parametrized in terms of the characteristic function and in general we cannot obtain the closed form expression of density function. Thus, to obtain the maximum likelihood estimate, one must rely on the numerical method such as Fourier inversion, which is computationally demanding.

Since the closed form expression of the characteristic function is known, several estimation methods based on the empirical characteristic function can be applied to the estimation of Lévy processes. The basic idea of this approach is to match the theoretical characteristic function with its empirical counterpart. This type of approach was originally proposed by Press (1972) for the estimation of stable distributions. Press (1972) introduced Minimum Distance estimation and Minimum rth-Mean Distance estimation. Feurverger and McDunnough (1981) proposed generalized method of moments (GMM) type estimator. Carrasco and Florens (2000) extended the GMM procedure to the case of a continuum of moment conditions (CGMM). Kunitomo and Owada (2004) recently introduced the maximum empirical likelihood estimator (MELE).

This paper has two aims. Firstly, we propose an efficient and computationally convenient estimation method. We use the framework of Sueishi (2005), which discusses the estimation of stable distributions. We rely on the quasi-likelihood approach. We construct the quasi-likelihood function via the characteristic function. The estimator is characterized as the root of the quasi-likelihood equation. Secondly we compare the finite sample performances of recently proposed estimation methods. Although various estimation methods have been proposed, there have been little study on the comparison of the estimation performance of each method. We employ the CGMM and the MELE as our competitors. A Monte Carlo study is implemented for two well-known Lévy processes: the variance gamma process and the normal inverse Gaussian process. We conduct simulations using different sample sizes and settings. Simulation results show that the QLE outperforms other estimators in many situations, though computational burden is much smaller than the other estimators. Especially, in the case of the normal inverse Gaussian process, the QLE performs on par with the maximum likelihood estimator in moderate sample size.

1 INTRODUCTION

We consider the estimation problem of discretely observed Lévy processes. We first recall the basic properties of Lévy processes. A Lévy process $X = \{X_t : t \geq 0\}$ is a cadlag stochastic process with stationary independent increments. The Lévy Khintchine theorem uniquely determines the characteristic function for Levy processes and infinitely divisible distributions. The characteristic function of the one-dimensional Lévy process X is given by

$$\begin{aligned}\phi(u,t;\boldsymbol{\theta}) &\equiv E[\exp\{iuX_t\}] \\ &= \exp\Big\{t\Big(i\gamma u - \frac{1}{2}\sigma^2 u^2 \\ &\quad + \int_{-\infty}^{\infty}\Big[e^{iux}-1-iux1_{\{|x|<1\}}\Big]\nu(dx)\Big)\Big\},\end{aligned}$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν is a measure on $\mathbb{R}\setminus\{0\}$ verifying

$$\int_{-\infty}^{\infty}(1 \wedge x^2)\nu(dx) < \infty.$$

The measure ν is called the Lévy measure and the triplet (γ, σ^2, ν) is called the characteristic triplet of X. If the Lévy measure is zero, then the process reduces to the Brownian motion with drift. For a detailed discussion of the properties of Lévy processes, see Sato (1999).

Let us denote the price of financial asset by S_t. In the class of exponential-Lévy models, the price S_t is represented as

$$S_t = S_0 \exp\{X_t\},$$

where X_t is a Lévy process. Assume that we observe the price process at equidistant time grid $t_i = 1, 2, \ldots, n$. The log return Y_i is defined as

$$Y_i \equiv \log\frac{S_{t_i}}{S_{t_i-1}} = X_{t_i} - X_{t_i-1}.$$

Thus, this model is based on the assumption that the log returns are independently and identically distributed infinitely divisible random variables. The characteristic function of Y_i is given by

$$\begin{aligned}\phi_\theta(u) = \exp\Big\{&i\gamma u - \frac{1}{2}\sigma^2 u^2 \\ &+\int_{-\infty}^{\infty}\big[e^{iuy} - 1 - iuy1_{\{|y|<1\}}\big]\nu(dy)\Big\}.\end{aligned}$$

The outline of this paper is as follows. In Section 2 we introduce several estimation methods based on the empirical characteristic function. Section 3 describes our estimation procedure. The results of a Monte Carlo comparison are reported in Section 4. Section 5 concludes.

2 ESTIMATION METHODS

In this section we describe how to estimate the parameters of Lévy processes by the alternative methods using characteristic function. Let $Y_1, Y_2, \ldots, Y_n$ be independently and identically distributed infinitely divisible random variables with parameter $\boldsymbol{\theta}$. In the following sections, we denote the characteristic function of Y by $\phi_\theta(u)$, and its real and imaginary part by $\phi_\theta^R(u)$ and $\phi_\theta^I(u)$, respectively. Then we obtain the following condition:

$$E[h(u, Y_i; \boldsymbol{\theta}_0)] = 0, \quad \forall u \in \mathbb{R}$$

where

$$h(u, Y_i; \boldsymbol{\theta}) = \exp(iuY_i) - \phi_\theta(u).$$

and $\boldsymbol{\theta}_0$ is the vector of true parameter values.

Since there is a one-to-one correspondence between the characteristic function and the density function, they have the same information. This fact suggests that estimation based on the empirical characteristic function should be as efficient as the maximum likelihood estimation.

2.1 CGMM

The key observation underlying the estimation based on the empirical characteristic function is that there exist an infinite number of moment conditions. Feuerverger and McDunnough (1981) proposed to choose finite grid $\mathbf{u} = (u_1, \ldots, u_k)'$ and to use $2k$ moment condistions:

$$E[\mathbf{h}_{\boldsymbol{\theta}}(Y)] = \mathbf{0},$$

where

$$\mathbf{h}_\theta(Y_i) = \left(\mathbf{h}_\theta^R(Y_i)', \mathbf{h}_\theta^I(Y_i)\right)', \tag{1}$$

and

$$\mathbf{h}_\theta^R(Y_i) = \Big(\cos(u_1Y_i) - \phi_\theta^R(u_1), ..., \cos(u_kY_i) - \phi_\theta^R(u_k)\Big)',$$

$$\mathbf{h}_\theta^I(Y_i) = \Big(\sin(u_1Y_i) - \phi_\theta^I(u_1), ..., \sin(u_kY_i) - \phi_\theta^I(u_k)\Big)'.$$

The GMM estimator is obtain by

$$\hat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}[\hat{\mathbf{h}}_n(\boldsymbol{\theta})]'\mathbf{W}_n[\hat{\mathbf{h}}_n(\boldsymbol{\theta})], \tag{2}$$

where $\hat{\mathbf{h}}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^n \mathbf{h}_\theta(Y_i)$ and $\mathbf{W}_n$ is a weighting matrix. The expression (2) with optimal weighting matrix is equivalent to

$$\hat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\left\|\mathbf{K}_n^{-1/2}\hat{\mathbf{h}}_n(\boldsymbol{\theta})\right\|, \tag{3}$$

where $\mathbf{K}_n$ is the consistent estimator of the covariance matrix, and $\|\cdot\|$ is the Euclidean norm.

Carrasco and Florens (2000) extended the GMM to the case of a continuum of moment conditions (CGMM). Let π be a probability density function. We introduce the new norm which takes into account the continuum set of moment conditions into the GMM framework. The norm is defined by

$$\|f\|^2 = \int f(u)\overline{f(u)}\pi(u)du,$$

where $\bar{f}$ denotes the complex conjugate of f.

Now we consider how to obtain the CGMM estimator by the analogy of (3). Intuitively, the CGMM estimator is obtained by minimizing the following objective function:

$$\left\|K^{-1}h_n(u,\boldsymbol{\theta})\right\|,$$

where $h_n(u,\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^n h(u, Y_i; \boldsymbol{\theta})$, and K is the covariance operator such that

$$Kf(u) = \int k(u,v)f(v)\pi(v)dv,$$

with $k(u,v) = E[h(u,Y;\boldsymbol{\theta})\overline{h(v,Y;\boldsymbol{\theta})}]$. The covariance operator K is consistently estimated by the operator K_n with kernel

$$k_n(u,v) = \frac{1}{n}\sum_{i=1}^n h(u,Y_i;\tilde{\boldsymbol{\theta}}_n)\overline{h(v,Y_i;\tilde{\boldsymbol{\theta}}_n)},$$

where $\tilde{\boldsymbol{\theta}}_n$ is a preliminary consistent estimator of $\boldsymbol{\theta}$. The problem is that since the inverse of K is not bounded (see Carrasco and Florens (2000) for detail), the operator K^{-1} is not continuous. Therefore, $K^{-1}f$ is not stable against small changes in f. To guarantee the stability, Carrasco and Florens (2000) replace K_n^{-1} by the Tikhonov approximation:

$$(K_n^{\alpha_n})^{-1} = \left(K_n^2 + \alpha_n I\right)^{-1} K_n,$$

where I is the identical operator and α_n is a regularization parameter which goes to zero as n goes to infinity. As a consequence, the CGMM estimator is given by

$$\hat{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \left\|(K_n^{\alpha_n})^{-1/2} h_n(u,\boldsymbol{\theta})\right\|. \quad (4)$$

Carrasco et al. (2004) shows that solving (4) is equivalent to solving

$$\min_{\boldsymbol{\theta}} \overline{\mathbf{v}(\boldsymbol{\theta})}' \left[\alpha_n \mathbf{I}_n + \mathbf{C}^2\right]^{-1} \mathbf{v}(\boldsymbol{\theta}),$$

where $\mathbf{C}$ is a $n \times n$ matrix with (i,j) element c_{ij}, I_n is the $n \times n$ identity matrix, $\mathbf{v}(\boldsymbol{\theta}) = (v_1(\boldsymbol{\theta}), \ldots, v_n(\boldsymbol{\theta}))'$ with

$$\begin{aligned} v_i(\boldsymbol{\theta}) &= \int h_n(u,\boldsymbol{\theta})\overline{h(u,Y_i;\tilde{\boldsymbol{\theta}}_n)}\pi(u)du, \\ c_{ij} &= \frac{1}{n}\int h(u,Y_i;\tilde{\boldsymbol{\theta}}_n)\overline{h(u,Y_j;\tilde{\boldsymbol{\theta}}_n)}\pi(u)du. \end{aligned}$$

Although Carrasco et al. (2004) give a convergence rate of α_n, the asymptotic result does not indicate how to choose α_n in practice.

2.2 Empirical Likelihood Method

Kunitomo and Owada (2004) propose the estimation method for Lévy processes based on the empirical likelihood approach. They extended Qin and Lawless (1994) to the case where the number of restrictions grows with the sample size.

Define the empirical likelihood function by

$$L(F_\theta) = \prod_{i=1}^n dF_\theta(Y_i) = \prod_{i=1}^n p_i, \quad (5)$$

where F_θ is the distribution function of Y. Without restrictions, (5) is maximized by the empirical distribution function $F_n(y) = \frac{1}{n}\sum_{i=1}^n I(Y_i < y)$. The empirical likelihood ratio is defined as

$$R(F) = \frac{L(F_\theta)}{L(F_n)} = \prod_{i=1}^n np_i. \quad (6)$$

The maximum empirical likelihood estimator (MELE) is defined as the maximizer of (6) subject to the following restrictions:

$$p_i \geq 0, \quad \sum_{i=1}^n p_i = 1, \quad \sum_{i=1}^n p_i \mathbf{h}_\theta(Y_i) = \mathbf{0},$$

where $\mathbf{h}_\theta(Y_i)$ is defined by (1). The maximum can be found by Lagrange multiplier method. Write

$$\begin{aligned} L_n(\boldsymbol{\theta}) &= \sum_{i=1}^n \log np_i - \mu\left[\sum_{i=1}^n p_i - 1\right] \\ &\quad - n\boldsymbol{\eta}'\left[\sum_{i=1}^n p_i \mathbf{h}_\theta(Y_i)\right], \end{aligned}$$

where μ and $\boldsymbol{\eta}$ are Lagrange multipliers. Taking derivatives with respect to p_i and setting to zero, we have

$$\frac{\partial L_n(\boldsymbol{\theta})}{\partial p_i} = \frac{1}{p_i} - \mu - n\boldsymbol{\eta}'\mathbf{h}_\theta(Y_i) = 0.$$

Thus we obtain $\mu = n$ and

$$p_i = \frac{1}{n} \frac{1}{1 + \boldsymbol{\eta}(\boldsymbol{\theta})' \mathbf{h}_\theta(Y_i)}, \tag{7}$$

where $\boldsymbol{\eta}(\boldsymbol{\theta})$ is the solution of $\sum_{i=1}^n p_i \mathbf{h}_\theta(Y_i) = \mathbf{0}$. Substituting (7) into (6) yields the log likelihood ratio:

$$\begin{aligned} l_n(\boldsymbol{\theta}) &= \log \prod_{i=1}^n n p_i \\ &= -\sum_{i=1}^n \log\left[1 + \boldsymbol{\eta}(\boldsymbol{\theta})' \mathbf{h}_\theta(Y_i)\right]. \end{aligned}$$

The MELE is obtained by maximizing $l_n(\boldsymbol{\theta})$. Kunitomo and Owada (2004) prove that under some regularity conditions the asymptotic covariance matrix of MELE can attains Cramer-Rao lower bound as the number of grid points goes to infinity $(k \to \infty)$ with sample size.

3 QUASI-LIKELIHOOD ESTIMATOR

Although the evaluation of the density function of Lévy processes is cumbersome, their score function is analytically tractable. In this section, we show the way to approximate the score function $\mathcal{S}(Y; \boldsymbol{\theta})$. To obtain the quasi-score function $\mathcal{S}_*(Y; \boldsymbol{\theta})$, we adopt the orthogonal projection of the score function onto the real valued basis functions, $G = \{g(u_i, y), i = 1, 2, \dots\}$, in Hilbert Space. The important examples include $G_H = \{\sin u_i y, \cos u_i y\}$ and $G_E = \{\exp u_i y\}$. The key insight is that linear combination of the elements of G can approximate square integrable function arbitrarily well. For detail see for example Brant (1984).

Now we consider the approximation method of the score function. Define the vector valued function $\mathbf{g}(Y)$ by $\mathbf{g}(Y) = g(\mathbf{u}, Y) = (g(u_1, Y), \dots, g(u_k, Y))'$. Let $E^*[\mathcal{S}(Y; \boldsymbol{\theta}) | \mathbf{g}(Y)]$ denote the orthogonal projection of $\mathcal{S}(Y; \boldsymbol{\theta})$ onto $\mathbf{g}(Y)$. Including a constant term in the approximation, we obtain the following quasi-score function:

$$\begin{aligned} \mathcal{S}_*(Y; \boldsymbol{\theta}) &= E^*[\mathcal{S}(Y; \boldsymbol{\theta}) | 1, \mathbf{g}(X)] \\ &= E[\mathcal{S}(Y; \boldsymbol{\theta})] \\ &\quad + \boldsymbol{\lambda}(\boldsymbol{\theta})' \Sigma(\boldsymbol{\theta})^{-1} [\mathbf{g}(Y) - \boldsymbol{\gamma}(\boldsymbol{\theta})], \end{aligned} \tag{8}$$

where $\boldsymbol{\lambda}(\boldsymbol{\theta}) = \mathrm{Cov}[\mathbf{g}(X), \mathcal{S}(Y; \boldsymbol{\theta})]$, $\boldsymbol{\gamma}(\boldsymbol{\theta}) = E[\mathbf{g}(Y)]$ and $\Sigma(\boldsymbol{\theta}) = \mathrm{Var}[\mathbf{g}(Y)]$.

The above expression involves unknown true score function $\mathcal{S}(Y; \boldsymbol{\theta})$. However, under mild regularity conditions, we have

$$\mathrm{Cov}[\mathcal{S}(Y; \boldsymbol{\theta}), \mathbf{g}(X)] = \frac{\partial E[\mathbf{g}(Y)]}{\partial \boldsymbol{\theta}'}.$$

In addition, we have $E[\mathcal{S}(Y; \boldsymbol{\theta})] = 0$ by definition of the score function. Therefore, (8) is reduced to

$$\mathcal{S}_*(Y; \boldsymbol{\theta}) = \left[\frac{\partial \boldsymbol{\gamma}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'}\right]' \Sigma(\boldsymbol{\theta})^{-1} [\mathbf{g}(Y) - \boldsymbol{\gamma}(\boldsymbol{\theta})]. \tag{9}$$

Note that the elements of (9) are represented in closed form as the function of $\boldsymbol{\gamma}(\boldsymbol{\theta})$.

In the case of $G = \{\cos u_i y, \sin u_i y\}$, we have

$\boldsymbol{\gamma}(\boldsymbol{\theta}) = (\phi_\theta^R(u_1), \dots, \phi_\theta^R(u_k), \phi_\theta^I(u_1), \dots, \phi_\theta^I(u_k))$,

and the elements of $\Sigma(\boldsymbol{\theta})$ are given by

$$\begin{aligned} &\mathrm{Cov}[\cos(u_i Y), \cos(u_j Y)] \\ &= \frac{1}{2}[\phi_\theta^R(u_i + u_j) + \phi_\theta^R(u_i - u_j)] - \phi_\theta^R(u_i)\phi_\theta^R(u_j), \\ &\mathrm{Cov}[\cos(u_i Y), \sin(u_j Y)] \\ &= \frac{1}{2}[\phi_\theta^I(u_i + u_j) - \phi_\theta^I(u_i - u_j)] - \phi_\theta^R(u_i)\phi_\theta^I(u_j), \\ &\mathrm{Cov}[\sin(u_i Y), \sin(u_j Y)] \\ &= \frac{1}{2}[-\phi_\theta^R(u_i + u_j) + \phi_\theta^R(u_i - u_j)] - \phi_\theta^I(u_i)\phi_\theta^I(u_j). \end{aligned}$$

The quasi-likelihood estimator (QLE) can be obtained as the root of the quasi-likelihood equation:

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n \mathcal{S}_*(Y_i; \boldsymbol{\theta}) \\ &= \left[\frac{\partial \boldsymbol{\gamma}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'}\right]' \Sigma(\boldsymbol{\theta})^{-1} \left[\frac{1}{n}\sum_{i=1}^n \mathbf{g}(Y_i) - \boldsymbol{\gamma}(\boldsymbol{\theta})\right] \\ &= \mathbf{0}. \end{aligned} \tag{10}$$

Sueishi (2005) argues that the asymptotic variance of the QLE can be made arbitrarily close to the Cramer-Rao bound by selecting sufficiently large numbers of grid points.

4 MONTE CARLO

In this section we carry out a Monte Carlo experiment to investigate the finite sample performance of our QLE. We compare the QLE with the CGMM estimator and the MELE. We choose two different DGPs: the variance gamma process and the normal inverse Gaussian process. For each estimator we report the mean (MEAN), the standard error (STD), and the root mean squared error (RMSE). In the experiment of the normal inverse Gaussian process, we also compared all estimators with the maximum likelihood estimator (MLE) to evaluate the efficiency.

The QLE is obtained as the root of the quasi-likelihood equation (10), where $\mathbf{g}$ is given by

$(\cos u_1 Y, \ldots, \cos u Y_k, \sin u_1 Y, \ldots, \sin u_k Y)$. We use the same grid $\mathbf{u}$ when we estimate parameters by the QLE and the MELE. To obtain the CGMM estimator, we use two-step procedure. The preliminary estimator of the CGMM is given by

$$\tilde{\boldsymbol{\theta}}_n = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \|h_n(u;\boldsymbol{\theta})\|^2 = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \int h_n(u;\boldsymbol{\theta})\overline{h_n(u;\boldsymbol{\theta})}\pi(u)du.$$

Since numerical integrations are computationally burdensome, we replace them by the inner product of the vector on equally-spaced grid with intervals 0.1. We select the uniform density on $[-4, 4]$ as the density function π. The regularization parameter α_n is chosen to be 0.001.

4.1 Variance Gamma

The variance gamma (VG) process is introduced in the symmetric case by Madan and Seneta (1990). Madan et al. (1998) discuss the general asymmetric case. The VG process is the time-changed Brownian motion with drift. Let $\gamma(t; 1, \nu)$ be the gamma process with mean rate 1 and variance rate ν. Then VG process with parameter θ, ν and σ can be defined as

$$X_t = \theta\gamma(t; 1, \nu) + \sigma W_{\gamma(t;1,\nu)},$$

where W_t is a standard Brownian motion. The characteristic function of the VG process X_t is given by

$$\phi(u, t; \boldsymbol{\theta}) = \left(\frac{1}{1 - iu\theta\nu + (\sigma^2\nu/2)u^2}\right)^{t/\nu}.$$

Now we assume that the asset price process is given by

$$S_t = S_0 \exp\{\mu t + X_t\},$$

where μ is a drift paramter. Then the log return $Y_i = \log(S_{t_i}/S_{t_i-1})$ has the following characteristic function:

$$\phi_\theta(u) = \exp(i\mu u)\left(\frac{1}{1 - iu\theta\nu + (\sigma^2\nu/2)u^2}\right)^{1/\nu}.$$

Table 1 reports the simulation results of 1000 iterations with 200 observations. The true value of the parameter $\boldsymbol{\theta} = (\theta, \nu, \sigma, \mu)'$ is $(1, 1, 1, 1)'$. The QLE and the MELE are calculated using two types of grids: the coarse grid $\mathbf{u} = (0.5, 2.5, 4.5)'$ and the fine grid $\mathbf{u} = (0.5, 1.5, 2.5, 3.5, 4.5)'$. Table 1 shows that the QLE performs poorly when the coarse grid is used. Compared with the other estimators, the QLE shows severe bias in σ. The MELE outperforms the QLE. In contrast, if the fine grid is used, the QLE performs slightly better than the MELE and is comparable to the CGMM, though the QLE hardly requires the computational effort. Although we do not show the results in the table, we also calculate the CGMM using the standard normal density, which makes little difference in the estimation results.

Table 1. Variance Gamma

	MEAN	STD	RMSE
QLE (coarse grid)			
θ	1.0492	0.2251	0.2305
ν	1.0064	0.2477	0.2478
σ	0.9094	0.2443	0.2606
μ	0.9541	0.1815	0.1872
QLE (fine grid)			
θ	1.0218	0.1810	0.1824
ν	1.0044	0.2323	0.2323
σ	0.9740	0.1338	0.1364
μ	0.9754	0.1318	0.1340
CGMM			
θ	0.9839	0.1735	0.1743
ν	1.0313	0.2640	0.2659
σ	0.9643	0.1433	0.1477
μ	0.9844	0.1292	0.1301
MELE (coarse grid)			
θ	1.0314	0.2045	0.2070
ν	1.0147	0.2451	0.2456
σ	0.9524	0.2037	0.2091
μ	0.9726	0.1638	0.1661
MELE (fine grid)			
θ	1.0272	0.2017	0.2035
ν	0.9918	0.2427	0.2428
σ	0.9718	0.1388	0.1417
μ	0.9683	0.1630	0.1660

4.2 Normal Inverse Gaussian

The normal inverse Gaussian (NIG) process is introduced by Barndorff-Nielsen (1995) as a model for financial data. The NIG process can also be represented as the time-changed Brownian motion. Let $I(t; a, b)$ be an inverse Gaussian process with parameters a, b. Then

$$X_t = \beta\delta^2 I(t, 1, \delta\sqrt{\alpha^2 - \beta^2}) + \delta W_{I(t,1,\delta\sqrt{\alpha^2-\beta^2})}$$

follows the NIG process with parameters α, β and δ. Bandorff-Nielsen (1995) shows that the characteristic function of the NIG distribution is given by

$$\phi_\theta(u) = \exp\left(-\delta\left(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\right)\right),$$

where $0 \leq |\beta| \leq \alpha$ and $\delta > 0$. The parameters can be interpreted as follows: α determines the shape, β the skewness. δ is the scale parameter. It is well known that the NIG distribution has closed form density function given by

$$f(y;\boldsymbol{\theta}) = \frac{\delta\alpha}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta y\right) \times \frac{K_1(\alpha\sqrt{\delta^2+y^2})}{\sqrt{\delta^2+y^2}},$$

where $K_1(\cdot)$ is the modified Bessel function of the third kind and index 1.

The experiments are conducted for 400 and 1000 observations. The true value of the parameter $\boldsymbol{\theta} = (\alpha, \beta, \delta)'$ is $(6, -4, 1)'$. Since the density function is known in closed form for the NIG distribution, we can obtain the MLE. We use the MLE as the efficiency benchmark. We define the efficiency of estimators by the ratio of sample variance of the MLE to that of each estimator.

Table 2 shows the simulation results of 1000 iteration of 400 observations. The generation of random numbers is based on the algorithm of Rydberg (1997). The grid of the QLE and MELE is chosen to be $\mathbf{u} = (1, 2, 3)'$. From Table 2 we see that the QLE dominates the CGMM and the MELE. The MELE of α has a large bias and the estimate of δ is quite inefficient. On the other hand, the QLE is highly efficient even in moderate sample size.

Table 3 reports results using 1000 observations. Since the CGMM estimator requires considerable computational time for large sample size, we repeat simulations only 100 times. From Table 3 we see that all three estimators improve the efficiency when sample size becomes large. In particular, the QLE is comparable to the MLE even though only three grid points are used to construct the quasi-score function. The bias of MELE almost disappears and the efficiency of δ improves remarkably. The CGMM estimates of β and δ are very accurate, though the standard error of α is relatively large.

Table 2. Normal Inverse Gaussian (n=400)

	MEAN	STD	RMSE	Efficiency
	QLE			
α	6.1626	0.5198	0.5446	0.9521
β	-4.0821	0.5586	0.5646	0.8847
δ	1.0230	0.1686	0.1702	0.7873
	CGMM			
α	6.2531	0.5748	0.6280	0.7786
β	-4.1134	0.6081	0.6187	0.7465
δ	1.0308	0.1715	0.1743	0.7609
	MELE			
α	6.3139	0.6048	0.6808	0.7033
β	-4.0663	0.5621	0.5660	0.8737
δ	1.0308	0.2151	0.2249	0.4837
	MLE			
α	6.1511	0.5072	0.5292	—
β	-4.0763	0.5254	0.5309	—
δ	1.0192	0.1496	0.1508	—

Table 3. Normal Inverse Gaussian (n=1000)

	MEAN	STD	RMSE	Efficiency
	QLE			
α	6.0477	0.3011	0.3049	0.9638
β	-4.0240	0.3255	0.3264	0.9695
δ	1.0068	0.0972	0.0974	0.9593
	CGMM			
α	6.0689	0.3321	0.3392	0.7923
β	-4.0356	0.3299	0.3318	0.9438
δ	1.0130	0.0950	0.0958	1.0042
	MELE			
α	6.0809	0.3001	0.3108	0.9702
β	-4.0340	0.3435	0.3452	0.8706
δ	1.0133	0.1052	0.1060	0.8189
	MLE			
α	6.0541	0.2956	0.3005	—
β	-4.0377	0.3205	0.3227	—
δ	1.0032	0.0952	0.0953	—

5 CONCLUSION

In this paper, we propose the estimation method of Lévy processes based on the quasi-likelihood approach. We construct the closed form quasi-score function via the characteristic function. The

QLE is characterized as the root of the quasi-likelihood equation. We carry out a Monte Carlo experiment to examine the performance of our estimator in finite samples. We employ the CGMM estimator and the MELE as competitors. The QLE outperforms the competing methods not only in terms of computational time but also in terms of efficiency. Although asymptotic result shows that the QLE and the MELE are asymptotically efficient, it does not give information about how many grid points we should choose in practice. Monte Carlo results suggest that only five points will suffice for practical purpose. The CGMM estimator does not perform remarkably well. A possible cause is that we choose the density function π and the regularization parameter α arbitrarily. If π and α are properly chosen, it may be possible to improve the efficiency of the CGMM estimator.

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