Comparing Different Approaches for Solving Optimising Models with Significant Nonlinearities

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EXTENDED ABSTRACT

The property of saddle-path instability often arises in economic models derived from optimising behaviour by individual agents. In the case when underlying functional forms are nonlinear, it is likely that the stable and unstable arms defining the saddle-path dynamics will also have nonlinear properties. While closed-form analytic solutions can always be derived for linearised deterministic versions of these models, it will be necessary to use numerical techniques to derive the dynamic properties of calibrated versions of the associated nonlinear models.

There are a range of different approaches by which it is possible to solve the dynamics of nonlinear models with the saddle-path property. In this paper we examine the extent to which the success of alternative approaches can be evaluated. For some models the only information known with certainty about the model are values taken by steady-state solutions. In some special cases it is possible to derive a closed-form analytic solution of the entire path. Any method of evaluation will be dependent upon the amount of information that is known about a particular model solution.

We start by considering the Ramsey (1928) model of a representative consumer. This is a simple two-dimensional dynamic model with the saddle-path property. We are able to demonstrate that the general form of the solution involves nonlinear equations which will lead to nonlinear dynamics and can only be solved using nonlinear solution techniques. We are also able to show how the model can be linearised and a full closed-form solution of the linearised model can be derived using matrix techniques.

Next, alternative solution methods using numerical techniques are discussed. In all cases, the success of the solution can be assessed by evaluating whether or not the chosen solution gives a time-path for each variable that goes from the chosen initial condition to (a small neighbourhood of) the steady-state. Unfortunately, however, there is generally no way of evaluating whether intermediate points on the path between the initial condition and the steady-state are close approximations to the “true” solution. In order to evaluate the “goodness-of-fit” along the entire dynamic path, it is necessary to have a closed-form solution of the entire solution path.

This can only be achieved in special cases. One such special case occurs when the underlying model is linear (and so a solution can be derived using standard matrix techniques) but has complex-valued eigenvalues so that the true solution exhibits oscillatory behaviour. Two linear models with complex-valued eigenvalues are considered.

In our first example, we extend the Ramsey model (discussed earlier) to incorporate habit persistence in consumption. It is argued that, while such a model does have the desired complex-valued eigenvalues, the empirical solution path of the linearised model does not exhibit exceptionally nonlinear behaviour. Thus the linearised version of the model is not particularly suitable for examining the ability of solution algorithms to solve models with highly nonlinear saddle-paths.

In our second example, we consider a perfect foresight version of the Cagan (1956) Model augmented by the introduction of sluggish adjustment for wages and labour. Using this model, it is possible to define an indexing parameter that, when varied, determines a range of values for the stable eigenvalues including generating complex-valued eigenvalues with large imaginary parts and hence significant cycles.

The model can then be employed as a benchmark to compare the properties of solutions derived using a range of solution algorithms. This will allow assessment of the factors that ensure the success or failure of different solution approaches.

1. INTRODUCTION

The property of saddle-path instability often arises in economic models derived from optimising behaviour by individual agents (see, for example, Turnovsky, 2000). In a standard approach formalized by Blanchard and Kahn (1980), solutions to an unanticipated shock are constructed by requiring "jumps" in endogenous variables so that the economy evolves along the stable arm of the saddle (or along the stable manifold in higher-dimensional models). This paper provides further insights into related issues previously discussed by Herbert, Stemp and Griffiths (2005)

In the case when underlying functional forms are nonlinear, it is likely that the stable and unstable arms defining the saddle-path dynamics will also have nonlinear properties. While closed-form analytic solutions can always be derived for linearised deterministic versions of these models, it will be necessary to use numerical techniques to derive the dynamic properties of calibrated versions of the associated nonlinear models.

There are a range of different approaches by which it is possible to solve the dynamics of nonlinear models with the saddle-path property. Forward-shooting and reverse-shooting are two of the most common approaches. In this paper we examine the extent to which the success of alternative approaches can be evaluated.

For some models the only information known with certainty about the model are values taken by steady-state solutions. In some special cases it is possible to derive a closed-form analytic solution of the entire path. Any method of evaluation will be dependent upon the amount of information that is known about a particular model solution.

In the rest of this paper we consider various models with the saddle-path property. We demonstrate how success of alternative solution methods can be evaluated. We start in Section 2 with a basic optimising model of the consumer. In Section 3 we discuss the information available for evaluating alternative solution approaches in this basic model and also in higher dimensional models with the saddle-path property.

Evaluating different solution approaches would be much easier if we had a closed form analytic solution to a model with nonlinear properties. In Sections 4 and 5, we consider linear models with complex-valued eigenvalues. These have the potential to have nonlinear-type oscillatory dynamics but, because the model is actually linear, they also have the desired property of a closed – form analytic solution. We suggest that these models could be used to evaluate the success of alternative solution approaches in solving truly nonlinear models.

2. OPTIMISING MODEL OF CONSUMER

We start by considering the Ramsey (1928) model of a representative consumer whose objective is to choose c so as to maximize

$$V = \int_0^\infty e^{-\delta t} u(c)\, dt \tag{1}$$

subject to

$$\dot{k} = f(k) - (\rho + n)k - c \tag{2}$$

$$k(0) = k_0 \tag{3}$$

$$u' > 0, u'' < 0, u'(0) = \infty, f(0) = 0, f' > 0, f'' < 0$$

This model can be solved in a variety of ways including by using the Pontryagin Maximum Principle. We can derive for the Pontryagin paths by first setting up a Hamiltonian:

$$H = e^{-\delta t} u(c) + \psi\left[f(k) - (\rho + n)k - c \right] \tag{4}$$

thus, yielding,

$$\frac{\partial H}{\partial c} = e^{-\delta t} u'(c) - \psi = 0 \tag{5}$$

$$\dot{\psi} = -\frac{\partial H}{\partial k} = -\psi\left[f'(k) - (\rho + n) \right] \tag{6}$$

$$\dot{k} = f(k) - (\rho + n)k - c \tag{7}$$

Totally differentiate (5) and use it in conjunction with (6) to yield

$$-\delta e^{-\delta t} u'(c) + e^{-\delta t} u''(c)\dot{c} = \dot{\psi} \tag{8}$$

$$\begin{aligned} \dot{\psi} &= -\psi\left[f'(k) - (\rho + n) \right] \\ &= -e^{-\delta t} u'(c)\left[f'(k) - (\rho + n) \right] \end{aligned} \tag{9}$$

Hence

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - (\rho + n + \delta) \right] \tag{11}$$

Hence dynamic system comes down to form

$$\dot{k} = f(k) - (\rho + n)k - c \tag{12}$$

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - (\rho + n + \delta)\right] \quad (13)$$

For typical functional forms of $u(.)$ and $f(.)$, (12) and (13) are nonlinear equations which will lead to nonlinear dynamics and can only be solved using nonlinear solution techniques.

Dynamics with linear properties can be derived by linearising the model around steady state yielding (14).

$$\begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} f'(k^*) - (\rho + n) & -1 \\ -\frac{u'(c^*)}{u''(c^*)} f''(k^*) & 0 \end{pmatrix} \begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix}. \quad (14)$$

The determinant of the system is given by

$$-\frac{u'(c^*)}{u''(c^*)} f''(k^*) < 0. \quad (15)$$

Hence, the eigenvalues are real-valued and have opposite signs so that we have a saddle-path solution. For the linearised model we can then derive a closed-form analytic solutiuon given by:

$$\begin{pmatrix} k - k^* \\ c - c^* \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ \mu_1 - \delta & \mu_2 - \delta \end{pmatrix} \begin{pmatrix} A_1 e^{\mu_1 t} \\ A_2 e^{\mu_2 t} \end{pmatrix}. \quad (16)$$

One of the eigenvalues is unstable. Without loss of generality, assume $\mu_2 > 0$, $\mu_1 < 0$. Then, the solutions along the stable arm of the saddlepath are given by:

$$k - k^* = -A_1 e^{\mu_1 t} \quad (17)$$

$$c - c^* = (\mu_1 - \delta) A_1 e^{\mu_2 t}. \quad (18)$$

Hence the stable path is positively sloped since

$$\left(\frac{dk}{dc}\right)_{STABLE\ ARM} = -\frac{1}{\mu_1 - \delta} > 0. \quad (19)$$

We can then plot the dynamics of c and k in the phase diagram given by Figure 1. With an inherited capital stock of $k(0)$ consumption will start on the stable arm as indicated and move along the stable arm until c and k reach their steady-state values.

3. SOLVING SADDLE-PATH MODELS USING SHOOTING METHODS

This solution defines an infinite number of paths – these are called the Pontryagin paths. In the typical economic problem (such as the model of a representative firm or of a representative consumer), with one state and one co-state variable, each of these paths is saddle-path unstable so that the phase diagram of the Pontryagin paths is as shown in Figure 2.

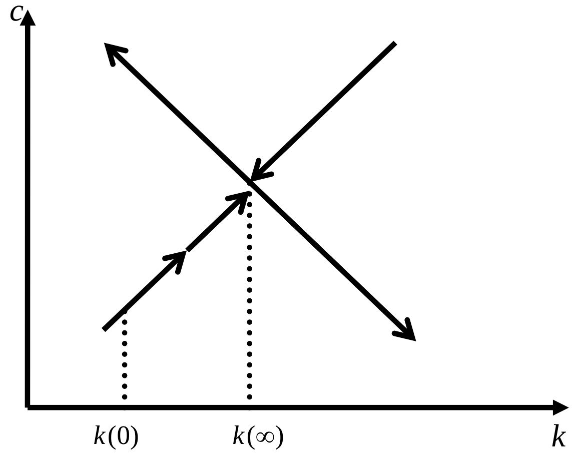

Figure 1
Phase Diagram of Optimal Consumption

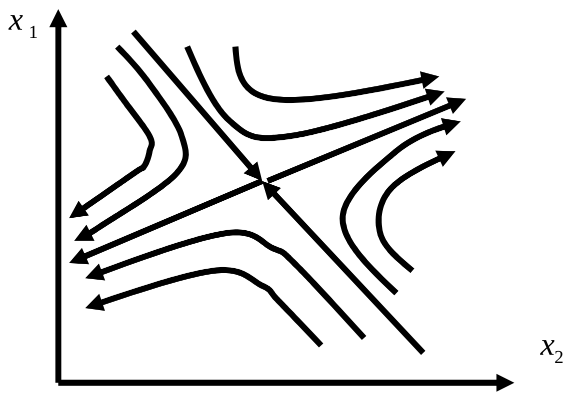

Figure 2
Phase Diagram of Generic Saddle-Path

There is a range of approaches to solving non-linear models with saddlepath properties (see, for example, Judd, 1998). Two of the most well-known approaches are reverse-shooting and forward-shooting. Other approaches are generally a variant on these two. Since all approaches to solving nonlinear models are numerical approaches, in all approaches it is necessary to parametrise the model before using computational techniques to derive solutions for the stable path(s).

In the case of the standard two-dimensional model, reverse-shooting involves just one search in reverse time starting from the neighbourhood of the steady-state. The model dynamics throw the dynamic solution onto the stable arm of the saddle-path. See Figure 3.

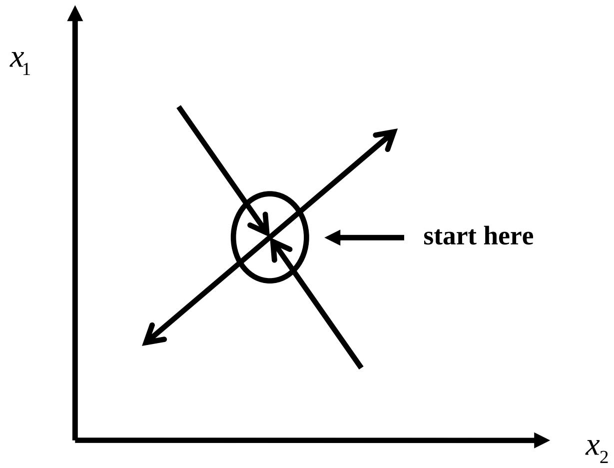

Figure 3
Reverse-Shooting in Two-Dimensional Model

For the same model, forward-shooting requires searching over a grid (see Figure 4).

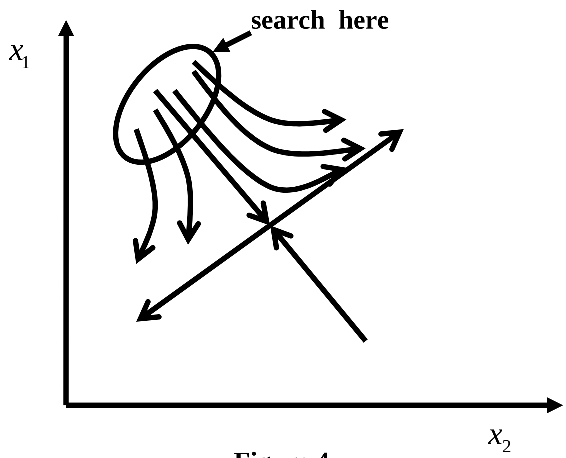

Figure 4
Forward-Shooting in Two-Dimensional Model

Reverse-shooting for higher dimensional models with more than two stable eigenvalues requires searching over a grid (the stable manifold) with the dimension of grid equal to the number of stable eigenvalues (see Figure 5).

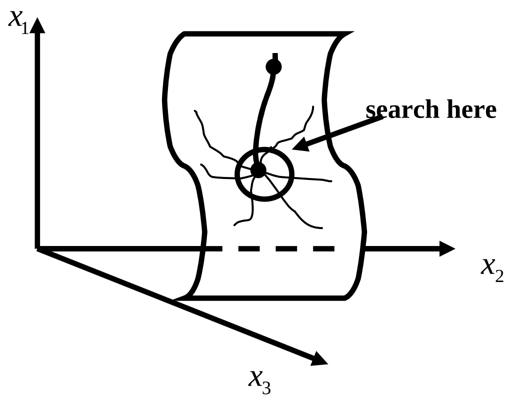

Figure 5
Reverse-Shooting in Higher-Dimensional Models

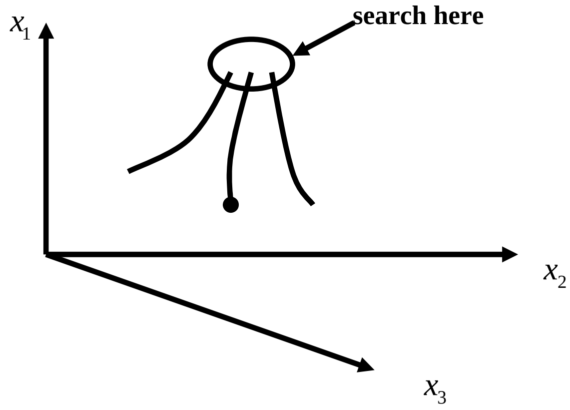

Figure 6
Forward-Shooting in Higher-Dimensional Models

Forward-shooting for the same higher dimensional model requires searching over a larger grid (equal to the entire space) with the dimension of grid equal to the sum of stable and unstable eigenvalues (see Figure 6).

In all cases, the success of the solution can be assessed by evaluating whether or not the chosen solution gives a time-path for each variable that goes from the chosen initial condition to (a small neighbourhood of) the steady-state. Unfortunately, however, there is generally no way of evaluating whether intermediate points on the path between the initial condition and the steady-state are close approximations to the "true" solution.

In order to evaluate the "goodness-of-fit" along the entire dynamic path, it is necessary to have a closed-form solution of the entire solution path. This can only be achieved in special cases. One such special case occurs when the underlying model is linear (and so a solution can be derived using standard matrix techniques) but has complex-valued eigenvalues so that the true solution exhibits oscillatory behaviour. Examples of two such cases are considered in Sections 4 and 5 of this paper.

4. MODEL OF CONSUMER WITH HABIT PERSISTENCE

In our first example of a linear model with complex-valued eigenvalues, we extend the model of Section 2 to incorporate habit persistence in consumption as previously discussed by Stemp (2005).
Then the objective of the representative consumer is to choose c so as to:

$$\underset{c}{\text{Max}}\ V = \int_0^\infty \exp(-\delta t)[u(c) - \tfrac{1}{2}\eta(\dot{c})^2]dt \qquad (20)$$

subject to:

$$\dot{k} = f(k) - (\rho + n)k - c \qquad (21)$$

where
k = capital/labour ratio;
c = consumption/labour ratio;
δ = discount rate;
ρ = rate of capital depreciation; and
n = rate of population growth.
and $u' > 0, u'' < 0, f' > 0$ and $f'' < 0$.

This is the Ramsey (1928) model of optimal saving with an additional term in the criterion function used to model habit persistence, given by: "$-\frac{1}{2}\eta(\dot{c})^2$" .

The solution to these equations can be written as the following four-dimensional equation system, with endogenous variables given by: c, k, ψ and x.

$$\dot{c} = x \qquad (22)$$

$$\dot{k} = f(k) - (\rho + n)k - c \qquad (23)$$

$$\dot{\psi} = \delta\psi + \eta x\{f''(k)[f(k) - (\rho + n)k - c]\}$$
$$-u'(c)[f'(k) - \rho - n] \qquad (24)$$

$$\dot{x} = x[\rho + n + \delta - f'(k)] + \frac{1}{\eta}[\psi - u'(c)] \qquad (25)$$

Using an asterisk, '*', to denote steady-state values, the system can be linearised about its steady-state yielding:

$$\begin{pmatrix} \dot{c} \\ \dot{k} \\ \dot{\psi} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & \delta & 0 & 0 \\ -u''(c^*)\delta & -u'(c^*)f''(k^*) & \delta & 0 \\ -\dfrac{u''(c^*)}{\eta} & 0 & \dfrac{1}{\eta} & 0 \end{pmatrix} \begin{pmatrix} c - c^* \\ k - k^* \\ \psi - \psi^* \\ x - x^* \end{pmatrix} \qquad (26)$$

As $\delta \to 0$, and for an appropriate choice of η , the eigenvalues of the system satisfy:

$$\lambda^2 = \left(\frac{u'(c^*)f''(k^*)}{u''(c^*)}\right)(1 \pm i) \qquad (27)$$

Then, the four eigenvalues of the system are complex-valued and given by:

$$\lambda_1, \lambda_2 = \pm\sqrt{\left(\frac{u'(c^*)f''(k^*)}{u''(c^*)}\right)(1+i)}$$

$$= \pm\left(\sqrt{\frac{u'(c^*)f''(k^*)}{u''(c^*)}}\right)(\sqrt[4]{2})\exp\left(\frac{i\pi}{8}\right) \qquad (28)$$

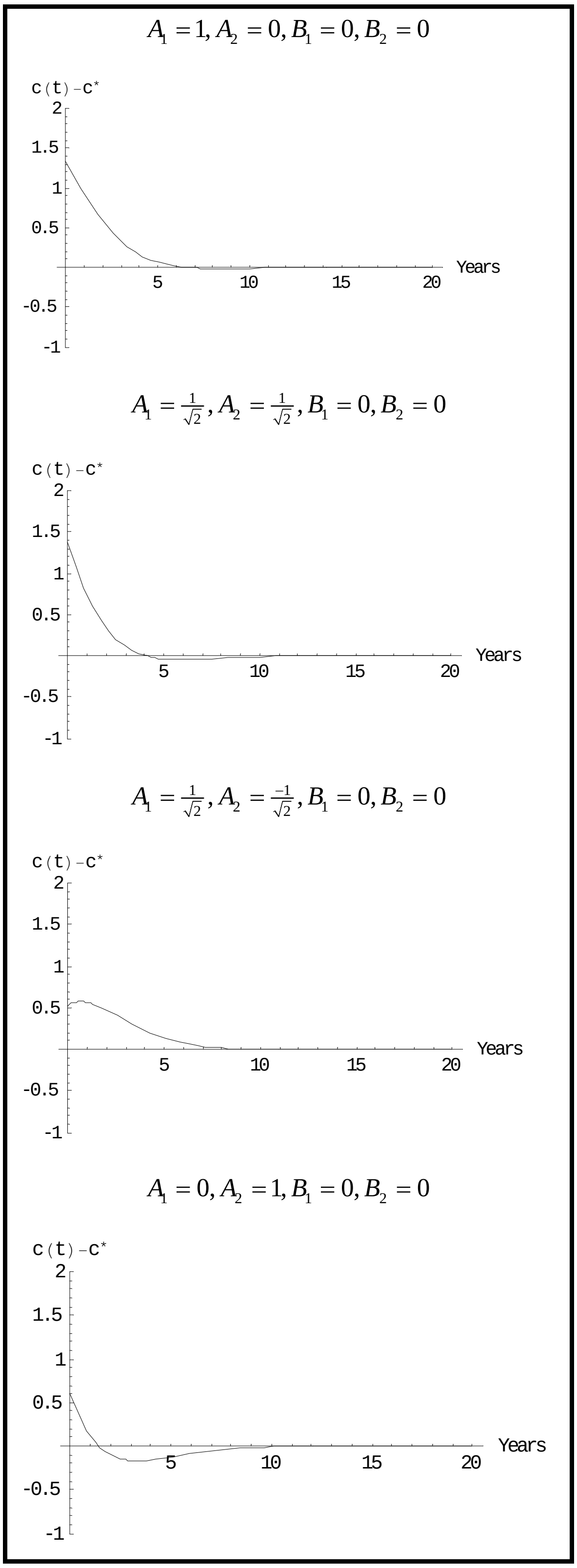

Figure 7
Time-Plots of Consumption for Consumer with Habit Persistence

$$\lambda_3, \lambda_4 = \pm\sqrt{\left(\frac{u'(c^*) f''(k^*)}{u''(c^*)}\right)(1-i)}$$

$$= \pm\left(\sqrt{\frac{u'(c^*) f''(k^*)}{u''(c^*)}}\right)(\sqrt[4]{2})\exp\left(\frac{i7\pi}{8}\right) \qquad (29)$$

For each pair of equations (28-29), one complex-valued eigenvalue has positive real part and one has negative real part. Thus there is total of two stable eigenvalues, given by $\lambda_1(=-\alpha+i\beta)$ and $\lambda_2(=-\alpha-i\beta)$ and two unstable eigenvalues given by $\lambda_3(=\gamma+i\varepsilon)$ and $\lambda_4(=\gamma-i\varepsilon)$.

Restricting our analysis to the case when all eigenvalues are complex-valued, we will write the stable eigenvalues as $\lambda_1(=-\alpha+i\beta)$ and $\lambda_2(=-\alpha-i\beta)$ and the unstable eigenvalues as $\lambda_3(=\gamma+i\varepsilon)$ and $\lambda_4(=\gamma-i\varepsilon)$, where α,β,γ and ε are positive real-valued constants.

The general closed-form solution of the model is then given by:

$$\begin{pmatrix} c-c^* \\ k-k^* \\ \psi-\psi^* \\ x-x^* \end{pmatrix} = \begin{bmatrix} \mathbf{v}(\lambda_1) & \mathbf{v}(\lambda_2) & \mathbf{v}(\lambda_3) & \mathbf{v}(\lambda_4) \end{bmatrix} \begin{bmatrix} (A_1+iA_2)\exp(\lambda_1 t) \\ (A_1-iA_2)\exp(\lambda_2 t) \\ (B_1+iB_2)\exp(\lambda_3 t) \\ (B_1-iB_2)\exp(\lambda_4 t) \end{bmatrix} \qquad (30)$$

where there are two "jump" variables, ψ and $x(=\dot{c})$, so that the constants: A_1, A_2, B_1 and B_2 are determined by initial values for c and k, and by the No-Ponzi game condition.

We can use equation (30) to yield the following solutions for c:

$$c-c^* = \exp(-\alpha t)[2\{A_1(\delta+\alpha)+A_2\beta\}\cos(\beta t)$$
$$+2\{A_1\beta - A_2(\delta+\alpha)\}\sin(\beta t)]$$
$$+\exp(\gamma t)[2\{B_1(\delta-\gamma)+B_2\varepsilon\}\cos(\varepsilon t)$$
$$+2\{B_1\varepsilon - B_2(\delta-\gamma)\}\sin(\varepsilon t)] \qquad (31)$$

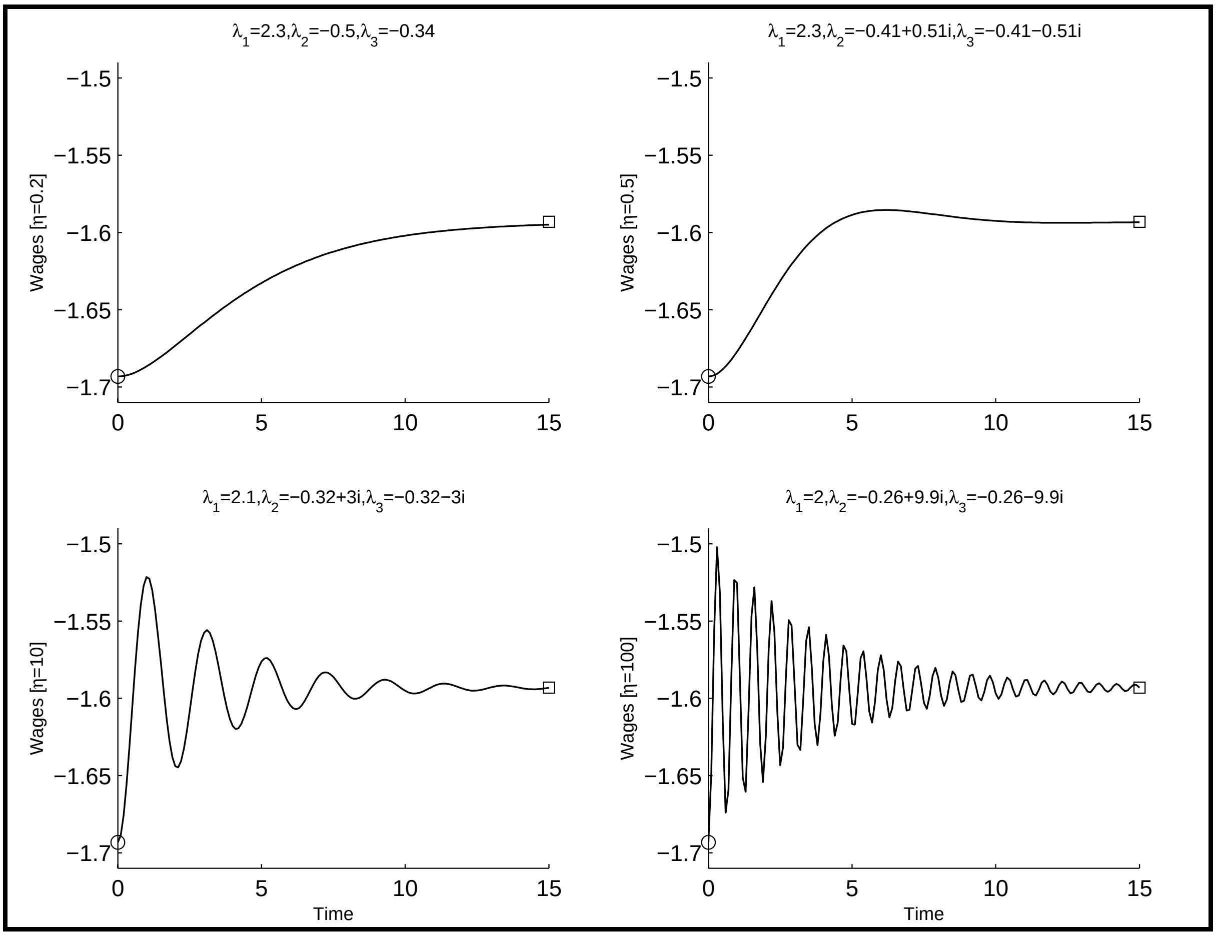

Figure 8
Time-Plots of Wages for Ad-Hoc Model

Stemp (2005) uses a calibration exercise to show that the empirical solution path associated with (31) does not exhibit exceptionally nonlinear behaviour with sample plots as given in Figure 7. Thus the linearised version of the model considered in this paper is not particularly suitable for examining the ability of solution algorithms to solve models with highly nonlinear saddle-paths.

In the next section we provide an alternative model that exhibits more useful properties.

5. AD-HOC MODEL WITH COMPLEX-VALUED EIGENVALUES

Consider the following model:

$$m - p = \alpha_1 y - \alpha_2 \dot{p} \quad (32)$$

$$y = \beta + (1-\gamma)n, 0 < \gamma < 1 \quad (33)$$

$$\dot{n} = \theta(\delta - \gamma n - w + p) \quad (34)$$

$$\dot{w} = \eta(n - \overline{n}) \quad (35)$$

where all variables are functions of time, lower-case letters denote logarithms and
y = output;
n = employment;
p = price level;
m = nominal money stock, assumed to be constant;
w = wage rate;
and
$\overline{n}$ = full employment expressed in logarithms.

This is the perfect foresight version of the Cagan (1956) Model augmented by the introduction of sluggish adjustment for wages and labour.

An investigation of the dynamic properties of this model shows that it is possible to generate complex-valued eigenvalues and associated oscillatory behaviour. The periodicity of the cycles can be controlled by changing just one parameter, η.

Figure 8 employs a calibrated version of the model given by (32)-(35) to show the time-plots of wages for four chosen values of η. The largest cycles are associated with the largest value for η leading to the largest imaginary parts of the complex-valued eigenvalues.

This model can then be employed as a benchmark to compare the properties of solutions derived using a range of solution algorithms as the magnitude of η (and hence of the cycles) is allowed to vary. This will allow assessment of the factors that ensure the success or failure of different solution approaches.

6. CONCLUSION

In this paper we have investigated the extent to which the success of solutions to models with nonlinear saddle-paths can be evaluated. We have argued that in the general case only the steady-state values of a range of variables is available and this contains very limited information about success of the algorithm along the entire path. However, it has been suggested that linear models with complex-valued eigenvalues may provide a benchmark for evaluating the success of solution algorithms when the model is highly nonlinear.

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