Over-Rejections By Weighted Symmetric Unit Root Tests In The Presence Of Multiple Structural Breaks

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EXTENDED ABSTRACT

Pantula et al. (1994) proposed modified Dickey–Fuller (DF) type unit root tests using a weighted symmetric estimator (hereafter, WS tests). In general, these WS tests are considered to have more power than the corresponding DF tests when there is no structural break in a time series.

For the case of the presence of onetime structural change in the sample, Leybourne and Newbold (2000) and Cook (2004) investigated the behavior of the WS tests. Leybourne and Newbold concluded that the weighted symmetric version of the DF t-type test will properly reject the unit root null hypothesis at some specified significance level when the true data-generating process is an I (1) process. On the other hand, Cook's Monte Carlo study suggested that the WS tests might suffer from low power under the stationarity alternative hypothesis where a series is generated by an I (0) process.

However, these considerations regarding the weighted symmetric unit root tests are limited to cases having one or no break. Thus, when a time series has multiple (two or more) breaks, no evidence exists to ascertain the potential effects of these breaks on the hypothesis test by the WS tests. That is, whether the tests remain robust to various possible combinations of parameters expressing locations and magnitudes of breaks needs to be determined. Given these facts, to consider multiple structural changes, this study introduces multiple shifts into the data-generating model of a series and investigates the behavior of weighted symmetric unit root tests under this model.

The limiting distributions of the WS test statistics are derived when multiple structural breaks occur in the sample period. Then, for the case of double breaks, the behavior of the test statistics is examined for some extreme cases of the nuisance parameters that express locations and sizes of the breaks. A Monte Carlo experiment is conducted in order to examine the manner in which in finite samples, the test statistics behave in various combinations of locations of two breaks with increasing break sizes.

The obtained results from this study reveal that the WS tests might be biased toward over-rejections of the unit root null hypothesis because of the presence of multiple structural breaks.

1. INTRODUCTION

Perron (1989) and Leybourne et al. (1998) have indicated that the standard Dickey–Fuller (DF) tests — commonly used in preliminary investigations to observe the non-stationarity of a series — can have very low power and serious size distortion when the true data-generating process has a structural break. Many related studies have been conducted; for example, Montanes and Reyes (1998), Leybourne and Newbold (2000), Lee (2000), and Sen (2001).

However, Pantula et al. (1994) proposed modified DF type unit root tests using a weighted symmetric estimator (hereafter, WS tests). In the absence of a break, these are generally considered to be more powerful than the corresponding DF tests (Pantula et al. (1994), Park and Fuller (1995), Patterson and Heravi (2003)). For these WS tests, in the presence of a structural change in the sample, Leybourne and Newbold (2000) investigated the asymptotic properties of the weighted symmetric version of the DF t-type test under the unit root null hypothesis. They concluded that the spurious rejection of the unit root hypothesis will not be present in the WS t-test even when a break occurs in the level or slope of the trend function of a series. Cook (2004) studied the small-sample performance of the WS and recursive mean-adjusted unit root tests in a stationary series having one mean shift. Cook's Monte Carlo results suggest that these modified tests can suffer from low power under the stationarity alternative hypothesis.

However, these considerations regarding the weighted symmetric unit root tests are limited to cases having one or no break. Thus, when a time series has multiple (two or more) breaks, no evidence exists to ascertain the potential effects of these breaks on the hypothesis test by the WS tests. That is, whether the tests remain robust to various possible combinations of parameters expressing locations and magnitudes of breaks needs to be determined.[1] Given these facts, to consider multiple structural changes, we introduce multiple mean shifts into the data-generating model of a series y_t as follows:

$$y_t = \alpha + \beta t + d_t + z_t$$

$$d_t = T^{1/2} \sum_{h=1}^{H} k_h DU_t^h$$

$$z_t = \rho z_{t-1} + \varepsilon_t,\ \rho = 1,\ t = 0, \ldots, T \qquad (1)$$

where h denotes the order of a break ($h = 1, \ldots, H$), k_h denotes the size of the h th break, ε_t is an independently and identically distributed innovation process with a zero mean and a finite variance σ^2, and the sample size of the series is $T+1$. DU_t^h denotes the h th break in the level of the trend function, where $DU_t^h = 1$ for $t > \tau_h T$ and 0 otherwise, where τ_h is the fraction of the h th break, which is defined as TB_h / T for all T (TB_h is the date of the h th break), and $0 < \tau_1 < \tau_2 < \cdots < \tau_{H-1} < \tau_H < 1$.[2]

Let e_{t-1} denote the residual in the regression of y_{t-1} on an intercept and time trend for $t = 1, \ldots, T+1$; then, the weighted symmetric estimator is obtained by minimizing

$$Q_W(\rho) = \sum_{t=1}^{T} w_t (e_t - \rho e_{t-1})^2 + \sum_{t=0}^{T-1} (1 - w_{t+1})(e_t - \rho e_{t+1})^2$$

where w_t is the weight variable, which is given as $T^{-1}(t-1)$. The weighted symmetric estimator of ρ is obtained as $\hat{\rho}_{WS} = \left(\sum_{t=1}^{T-1} e_t^2 + T^{-1} \sum_{t=1}^{T} e_t^2 \right)^{-1} \sum_{t=1}^{T} e_{t-1} e_t$. The statistics based on this estimator are defined as

$$t_{WS} = \hat{\sigma}_{WS}^{-1} (\hat{\rho}_{WS} - 1) \left\{ \sum_{t=1}^{T-1} e_t^2 + \frac{1}{T} \sum_{t=1}^{T} e_t^2 \right\}^{1/2}$$

$$T(\hat{\rho}_{WS} - 1)$$

where $\hat{\sigma}_{WS}^2 = (T-2)^{-1} Q_W(\hat{\rho}_{WS})$.

In the following sections, we will report only the result of the t_{WS} statistic because the behavior of the $T(\hat{\rho}_{WS} - 1)$ statistic is quite similar to that of the t_{WS} statistic, and refer to the test utilizing the t_{WS} statistic as the weighted symmetric (WS) t-test.

In the next section, we will derive the limiting distribution of the test statistic and explore its asymptotic behavior in the case of double breaks for certain extreme cases with respect to the dates and sizes of the breaks. To examine the small-sample properties of the test, some Monte Carlo experiments are also conducted in Section 3. The conclusion is described in Section 4.

2. LIMITING BEHAVIOR OF THE WS TEST

We show the limiting distribution of the weighted symmetric t-test statistic when multiple mean shifts exist in the time series.

Theorem 1. *For model (1),*

$$t_{WS} \Rightarrow \left\{ (\sigma^2 + c_2) \left(\sigma^2 \int_0^1 W^\tau(r)^2 dr + c_3 + 2 l_3 \right) \right\}^{-1/2}$$

$$\left[\sigma^2\left\{\int_0^1 W^\tau(r)dW^\tau(r) - \int_0^1 W^\tau(r)^2 dr\right\} + c_1 + l_1 + l_2 + s_1 - c_3 - 2l_3\right] \quad (2)$$

where $\Rightarrow$ represents weak convergence, $W^\tau(r)$ is a detrended standard Wiener process defined as $W^\tau(r) \equiv W(r) - a - dr$, where $W(r)$ is a standard Wiener process, $a \equiv 2\int_0^1 (2-3s)W(s)ds$ and $d \equiv 6\int_0^1 (2s-1)W(s)ds$. $c_.$, $l_.$, and s_1 are given by

$$c_1 = -\sum k_h^2(1-\tau_h)(6\tau_h^2 - 3\tau_h + 1) - \sum_{g=1}^{H-1}\sum_{h>g}^{H} k_g k_h \{(1-\tau_g)(6\tau_g\tau_h - 3\tau_g + 1) + (1-\tau_h)(6\tau_g\tau_h - 3\tau_h + 1) - 1\}$$

$$c_2 = \sum k_h^2$$

$$c_3 = \sum k_h^2 \tau_h (1-\tau_h)(3\tau_h^2 - 3\tau_h + 1) + 2\sum_{g=1}^{H-1}\sum_{h>g}^{H} k_g k_h \tau_g (1-\tau_h)(3\tau_g\tau_h - 3\tau_h + 1)$$

$$l_1 = \sigma \sum k_h W^\tau(\tau_h)$$

$$l_2 = \sigma \sum k_h \left\{\tau_h W^\tau(1) - W^\tau(\tau_h) + (1-\tau_h)W^\tau(0) - 6\tau_h(1-\tau_h)\int_0^1 (r - 1/2)dW^\tau(r)\right\}$$

$$l_3 = \sigma \sum k_h \int_{\tau_h}^1 W^\tau(r)dr$$

$$s_1 = \left\{\sigma W^\tau(0) + \sum k_h (1-\tau_h)(3\tau_h - 1)\right\}^2$$

where Σ is the summation from $h = 1$ to H.

This theorem shows that the limiting distribution of the statistic depends on the nuisance parameters that indicate the locations and sizes of the breaks.

In turn, we discuss the asymptotic behavior of the WS test statistic for double breaks ($H = 2$). Three extreme cases of the fractions of breaks are considered: $(\tau_1, \tau_2) \to (0,0)$, $(\tau_1, \tau_2) \to (1,1)$, and $(\tau_1, \tau_2) \to (0,1)$, where both the break fractions lie at the beginning of the sample, both lie at the end of the sample, and the fractions of the first and the second breaks lie at the beginning and the end of the sample, respectively. For simplicity, in the following proposition, we set $\sigma = 1$ without loss of generality.

Proposition 1. *For the case of double breaks ($H = 2$), the limiting distribution of the weighted symmetric t-test statistic using $\hat{\rho}_{WS}$ and $\hat{\sigma}_{WS}$ with $\sigma = 1$ in the three cases of (τ_1, τ_2) is*

as $(\tau_1, \tau_2) \to (0,0)$

$$t_{WS} \to B^{-1/2}\left\{A + k_1 k_2 - (k_1 + k_2)W^\tau(0)\right\}$$

as $(\tau_1, \tau_2) \to (1,1)$

$$t_{WS} \to B^{-1/2}\left\{A + k_1 k_2 + (k_1 + k_2)W^\tau(1)\right\}$$

as $(\tau_1, \tau_2) \to (0,1)$

$$t_{WS} \to B^{-1/2}\left\{A - k_1 W^\tau(0) + k_2 W^\tau(1)\right\}$$

where

$$A = \int_0^1 W^\tau(r)dW^\tau(r) - \int_0^1 W^\tau(r)^2 dr + W^\tau(0)^2$$

and $B = (1 + k_1^2 + k_2^2)\int_0^1 W^\tau(r)^2 dr$.

Regarding the DF t-test, Leybourne and Newbold (2000) have shown that as the break fraction of a break approaches zero, the limiting distribution of the statistic contains the term of the squared break size parameter ($-k^2$) in the numerator (Equation (3)), and this term causes the spurious rejection of the unit root null hypothesis. In Proposition 1, as $(\tau_1, \tau_2) \to (0,0)$ and $(1,1)$, the limiting distribution of the t_{WS}-statistic contains the deterministic term $k_1 k_2$ in the numerator of its equation. Therefore, we can infer that this term can cause over-rejections of the unit root null in the WS t-test if its value is negative and its absolute value is large enough to dominate stochastic terms in the limiting distribution.[3] However, Leybourne and Newbold (2000) also proved that in the case of a single break, the deterministic term corresponding to $k_1 k_2$ in Proposition 1 does not appear in the limiting distribution of the WS t-test (Equation (5)). Whether or not the deterministic term of break size parameters is included in the limiting distribution stems from the difference in the number of breaks considered in the data-generating model (one or multiple break(s)). This implies that multiple breaks in the level can have varying impacts on the WS t-test compared to a single break in the level.

3. MONTE CARLO STUDY

To investigate the small-sample properties of the weighted symmetric unit root t-test, we implement a Monte Carlo simulation. The series, y_t, is generated by model (1) with $\rho = 1$, $H = 2$, and $\alpha = \beta = 1$, and ε_t is $i.i.d.N(0,1)$. The sample size is 200 and there are 5000 replications. The empirical size of the WS t-test at the nominal level (5%) is computed in the region of $0 < \tau_1 < \tau_2 < 1$ at 0.01 intervals. The sizes of two breaks, (k_1, k_2), assume some possible combinations of 0.25, 1.0, –0.25, and –1.0.

Table 1 shows that when one of the two magnitudes of breaks has a negative value, frequent rejections of the null occur at the near dates of two breaks in the test. Moreover, both breaks become significant when their absolute values ((k_1, k_2) = (–1.0, 1.0) and (1.0, –1.0)) are large. This result is consistent with the finding of Proposition 1; it can therefore be concluded that regardless of the sample size of the series, the over-rejection phenomenon occurs in the WS t-test in the presence of multiple mean shifts.

(k_1, k_2)	τ_1	τ_2: 0.02	0.06	0.11	0.31	0.71	0.91	0.96	0.99
(0.25, 0.25)	0.01	2.86	4.44	5.36	5.82	5.62	5.14	5.52	4.86
	0.05		3.70	4.44	5.60	5.98	5.54	5.22	4.94
	0.1			3.18	5.34	6.36	6.02	4.62	5.72
	0.3				3.42	6.22	6.14	5.54	5.70
	0.7					3.94	5.68	5.40	5.60
	0.9						3.60	5.18	5.06
	0.95							3.32	4.50
	0.98								2.98
(1.0, 0.25)	0.01	0.76	1.70	2.14	3.82	4.28	4.06	2.82	3.40
	0.05		0.78	1.20	3.68	3.46	3.80	3.38	3.22
	0.1			0.54	3.94	3.96	4.16	3.66	3.30
	0.3				1.10	4.42	5.02	4.90	4.22
	0.7					0.74	4.02	4.10	4.22
	0.9						0.74	2.04	2.46
	0.95							0.70	1.36
	0.98								0.68
(0.25, 1.0)	0.01	0.62	1.40	2.18	4.68	3.82	3.88	2.82	3.50
	0.05		0.68	1.62	3.80	4.40	3.88	3.62	3.78
	0.1			0.70	4.24	4.54	3.74	3.18	3.96
	0.3				0.74	5.04	4.52	3.70	4.30
	0.7					1.12	4.22	3.16	4.04
	0.9						0.98	1.32	2.42
	0.95							0.56	1.18
	0.98								0.60
(1.0, 1.0)	0.01	0.00	0.18	0.68	3.82	3.80	2.50	2.10	2.62
	0.05		0.02	0.12	3.14	3.94	2.74	2.34	2.04
	0.1			0.00	2.80	4.44	3.28	2.42	2.66
	0.3				0.02	6.78	4.76	3.84	3.66
	0.7					0.04	2.74	2.72	3.68
	0.9						0.02	0.18	0.54
	0.95							0.02	0.00
	0.98								0.00

(k_1, k_2)	τ_1	τ_2: 0.02	0.06	0.11	0.31	0.71	0.91	0.96	0.99
(−1.0, 0.25)	0.01	13.32	7.44	4.68	3.12	3.04	3.06	3.02	4.06
	0.05		10.86	6.02	2.40	2.34	2.60	2.38	2.56
	0.1			12.08	2.88	2.64	2.64	2.76	2.82
	0.3				11.86	2.88	2.86	3.20	3.04
	0.7					12.78	3.68	3.32	3.04
	0.9						11.62	5.94	4.76
	0.95							10.94	7.28
	0.98								12.20
(0.25, −1.0)	0.01	11.96	6.36	4.64	3.58	3.02	3.02	2.46	3.28
	0.05		11.08	6.52	3.18	2.76	2.62	2.56	3.28
	0.1			12.30	3.52	3.08	2.88	2.70	3.38
	0.3				11.72	3.02	2.80	2.60	2.76
	0.7					12.16	2.88	2.70	3.20
	0.9						10.00	5.66	4.52
	0.95							11.34	9.22
	0.98								13.44
(−1.0, 1.0)	0.01	59.10	26.12	6.96	1.32	0.86	1.40	1.72	2.48
	0.05		57.04	19.76	1.10	0.90	1.12	0.96	1.62
	0.1			62.50	1.00	0.74	1.04	0.98	1.28
	0.3				58.92	0.80	0.76	0.62	1.16
	0.7					57.94	0.90	1.20	1.46
	0.9						58.68	17.84	8.36
	0.95							58.86	33.06
	0.98								57.60
(1.0, −1.0)	0.01	60.04	26.44	7.28	1.24	1.28	1.18	1.34	2.64
	0.05		57.80	19.88	0.96	0.82	0.72	0.80	1.52
	0.1			61.36	1.14	0.60	0.50	0.66	1.06
	0.3				58.44	0.98	0.64	1.08	1.34
	0.7					58.92	1.06	1.18	1.42
	0.9						58.82	16.76	8.40
	0.95							57.96	33.00
	0.98								58.54

y_t is generated by model (1) with H = 2, $\alpha = \beta = 1$, and $\varepsilon_t \sim$ i.i.d.N(0,1). The sample size is 200 and there are 5000 replications.

Table.1 The empirical size of the weighted symmetric t-test at the nominal 5% level

4. CONCLUSIONS

We have investigated the behavior of the unit root t-test based on a weighted symmetric estimator when the true data-generating process is integrated of order one with multiple structural breaks in the level. The limiting distribution of the test statistic was derived under the unit root null hypothesis, ignoring the breaks in the process.

As shown by Leybourne and Newbold (2000), in testing the unit root null hypothesis in the presence of one break, the weighted symmetric t-test outperforms the corresponding DF t-test in terms of the absence of the bias toward spurious rejections of the null hypothesis. However, we have revealed that the problem of over-rejections of the null can occur even in the WS t-test when two (multiple) mean shifts exist in the time series.

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6. FOOTNOTES

1. Lumsdaine and Papell (1997) suggested a unit root test that permits two breaks at unknown dates under the alternative hypothesis, and Lee and Strazicich (2003) proposed a minimum Lagrange Multiplier (LM) unit root test that allows for two endogenous breaks under both the null and alternative hypotheses.

2. We have also analyzed three other cases: the cases where $d_t = T^{-1/2} \sum k_h DT_t^h$, where $DT_t^h = t - \tau_h T$ for $t > \tau_h T$ and 0 otherwise (multiple slope changes), $d_t = \sum k_h DU_t^h$, and $d_t = \sum k_h DT_t^h$, where the summation Σ is from $h = 1$ to H. Consequently, it has been found that the WS t-test is asymptotically invariant to the existence of any type of multiple breaks.

3. For the case of more than two breaks, the corresponding deterministic part, which may become the cause of the misjudgment on the hypothesis test, is present in the limiting distribution.

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