Empirical Likelihood Estimation of Continuous-Time Models With Conditional Moment Restrictions

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ABSTRACT

The EL estimators have some favorable higher order asymptotic properties. We extend the EL method proposed by Donald et al. (2003) to estimate non-$i.i.d.$ continuous-time models with the known functional form of the conditional characteristic function. In many cases even the MLE method can not be performed, the EL method can do. More over, not only does the EL method resolve the problem of covariance matrix singularity in the regular GMM but also utilize the information in the conditional moment conditions fully. The EL method can be applied to many popular financial models such as some of diffusion models, jump diffusion models and stochastic volatility models. By way of a Monte Carlo comparison, we show that the EL method has better finite sample properties than C-GMM introduced by Carrasco et al. (2004).

Work in the area of EL has been initiated by Owen (1988). Qin and Lawless (1994) proposed the EL method for general estimation equations. Donald et al. (2003) and Kitamura et al. (2004) propound some estimation methods with conditional moment restrictions. Our method is an extension of the EL method by Donald et al. (2003). According to Newey and Smith (2004), The EL estimators have some favorable higher order asymptotic properties. In particular, the higher order asymptotic bias of the EL will be less than that of the GMM, when there are many moment restrictions. Such theoretical advantage can lead to better results to the empirical analysis with many moment restrictions. Although the method of Kitamura et al. (2004) can also work, we do not select the method of Kitamura et al. (2004), because of its computational burdens.

When we know a closed-form and the tractable expression of the likelihood function of the model, the maximum likelihood estimation method is the best option to estimate parameters of a model. Unfortunately, in many cases we often fail to derive a tractable form of likelihood function, especially when it comes to models with jumps, derivation of the tractable form becomes more difficult. Since in many cases the characteristic function often has a tractable form even when likelihood function does not, the characteristic function may be employed as an available substitute of likelihood function. Once we obtained a tractable form of the characteristic function, we can exploit conditional moment conditions using the conditional characteristic function (CCF) and the empirical conditional characteristic function (ECCF) for non-$i.i.d.$ processes. Based on the conditional moment conditions derived by the way mentioned above, one can perform the EL estimation.

A great deal of effort has been made on the estimation using the GMM approach in this area (see, for example, Singleton 2001, Chacko and Viceira 2003 and Yu 2004). What seems to be a grave drawback, however, is the singularity of the covariance matrix, which occurs when we possess many moment conditions. Carrasco et al. (2003) have introduced the C-GMM (GMM with a continuum of moment conditions) to overcome such a drawback of GMM approach. For the same purpose, we propose a different estimation method adopting the maximum empirical likelihood (EL) approach.

We carry out a Monte-Carlo experiment with the CIR model and a jump diffusion model to compare our method with C-GMM. As a result, the EL method shows us some surpassing finite sample properties, which are shown by the Monte-Carlo simulation.

1 INTRODUCTION

What we are interested in is the estimation of continuous-time models with conditional moment restrictions. We limit our attention on the one dimensional Markov processes, though our EL method works for almost all continuous-time processes with available forms of CCFs. We denote a jump diffusion process as $\{X_t, t \geq 0\}$, which takes values in some open subset $\mathscr{A} \subset \mathbb{R}$ and satisfies the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dZ_t., \qquad (1)$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ is some regular function, W_t and Z_t represent a winner process and a jump process respectively. X_t is adopted to the augmented filtration $\{\mathscr{F}_t, t \geq 0\}$ generated by W_t and Z_t. The jump diffusion process without the term dZ_t is called as the diffusion processes. For some classes taking form as eq(1), closed forms of associated CCF can be derived. We define CCF as

$$\psi(\omega, \tau|\theta, X_t) = E^{\theta}(e^{i\omega X_{t+\tau}}|X_t),$$

where ω, τ, and θ denote the transform variable, increment of time t, and parameter vector respectively, and $i = \sqrt{-1}$. The sample counterpart of CCF, the ECCF is defined as $\exp(i\omega X_{t+\tau})$.

In the following section we show how to derive CCFs of continuous-time stochastic processes, and how to construct conditional moment restrictions for the processes using CCFs and ECCF. Empirical likelihood estimation is described in section 3. In section 4, we summarize another method, the C-GMM. In section 5, we perform a Monte-Carlo simulation to compare EL with C-GMM introduced by Carrasco et al (2003). Section 6 reports conclusions.

2 CCF AND CONDITIONAL MOMENT RESTRICTION

The CCFs of continuous-time stochastic processes is an integral for the construction of our EL method. In this section, we will focus on a class of stochastic processes what is termed as refine processes, though, our method will work whenever we have available forms of CCFs of the processes. As illustrations, we will describe roughly on how to derive the CCFs of some affine processes and how we can construct conditional moment restrictions for the processes using CCFs and ECCF.

2.1 CCFs of affine processes

According to Bakshi and Madan (2000), Duffie et al (2000), and Chacko and Das (2002), the conditional characteristic functions can be derived in closed form for affine stochastic processes. The class of affine processes, defined by Duffie and Kan (1993), encompass most of the popular financial continuous-time models. By solving the associated Kolmogorov backward equation (KBE), we can derive the characteristic functions of a affine process (see, Chacko and Viceira (2003)). Following Chacko and Viceira (2003), we express the KBE by

$$\mathscr{D}\psi(\omega, \tau, |\theta, X_t) = 0,$$

where $\mathscr{D}$ is the infinitesimal generator for the process.

Singleton (2001) defines the affine diffusion processes $\{X_t, t \geq 0\}$ as the a diffusion process, which satisfies the following equations;

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \qquad (2)$$

$$\mu(X_t) = \theta + \kappa X_t$$
$$\sigma(X_t)^2 = h + lX_t,$$

where $\theta, \kappa, h,$and l are some parameters. By solving the associated KBE , we can obtain the CCF of a diffusion process as

$$\psi(\omega, \tau, |\theta, X_t) = \exp(A(\omega, \tau) + B(\omega, \tau)X_t),$$

with A and B satisfying the complex-valued ODEs

$$\dot{B}(\omega, \tau) = -\kappa B(\omega, \tau) - \frac{1}{2} lB(\omega, \tau)^2$$
$$\dot{A}(\omega, \tau) = -\theta B(\omega, \tau) - \frac{1}{2} hB(\omega, \tau)^2$$

with boundary conditions $B(\omega, t+\tau) = u$ and $A(\omega, t+\tau) = 0$.

2.2 Conditional moment restrictions

Using the CCF mentioned above and ECCF, we can construct the following conditional moment conditions:

$$E\left[\psi(\omega, \tau|\theta, X_t) - \exp(i\omega X_{t+\tau})|X_t\right] = 0.$$

We will use the conditional moment condition by real-valued form as follows:

$$E[\mathrm{Re}(\psi(\omega, \tau|\theta, X_t) - \exp(i\omega X_{t+\tau}))|X_t] = 0$$
$$E[\mathrm{Im}(\psi(\omega, \tau|\theta, X_t) - \exp(i\omega X_{t+\tau}))|X_t] = 0, \qquad (3)$$

where Re denotes real part of complex number and Im the imaginary part.

3 EMPIRICAL LIKELIHOOD ESTIMATION

In this section, we will propose an empirical likelihood estimation of continuous-time models with conditional moment restrictions. Our EL method bases on the EL method developed by Donald et al. (2003).

3.1 Unconditional moment restrictions

The main idea of Donald et al. (2003) is to approximate the conditional moment restrictions by exploiting sequences of unconditional moment restrictions. Using the idea of Donald et al. (2003), we approximate the conditional moment restrictions eq(3) by the following equations:

$$\begin{aligned} E[\mathrm{Re}(\psi(\omega,\tau|\theta,X_t)-\exp(i\omega X_{t+\tau}))\otimes q^K(X_t)] = 0 \\ E[\mathrm{Im}(\psi(\omega,\tau|\theta,X_t)-\exp(i\omega X_{t+\tau}))\otimes q^K(X_t)] = 0 \end{aligned} \tag{4}$$

where, $q^K(x)$ is a vector of spline approximating functions. It is defined as

$$\begin{aligned} q^K(x) = (1,x,x^2,x^3,\mathrm{I}(x-s_1>0)(x-s_1)^3,\cdots, \\ \mathrm{I}(x-s_{K-3-1}>0)(x-s_{K-3-1})^3)' \end{aligned} \tag{5}$$

where $\mathrm{I}(\cdot)$ denote an indicator function, $s_i \in \mathbb{R}^+$ for $(i=1,2,\ldots,K-3-1)$ is the knots of spline, which is placed in the support of x.

3.2 Estimation method

Once we obtained moment conditions as eq(4), we can perform an EL estimation using the moment conditions in eq(4). The EL estimation can be accomplished by solving a constrained maximum problem as follows:

$$\max_{p_t>0,\theta\in\Theta}\sum_{t=1}^{T}\ln p_t, \quad s.t. \sum_{t=1}^{T}p_tG_t(\theta)=0, \ \sum_{t=1}^{T}p_t=1, \tag{6}$$

where p_t is a positive weight, for $i.i.d.$ case it can be regarded as probability, and $G_t(\theta)$ is the vector of the form

$$\begin{pmatrix} \mathrm{Re}\left[\psi(\omega,\tau|\theta,X_t)-\exp(i\omega X_{t+\tau})\right]\otimes q^K(X_t) \\ \mathrm{Im}\left[\psi(\omega,\tau|\theta,X_t)-\exp(i\omega X_{t+\tau})\right]\otimes q^K(X_t) \end{pmatrix}.$$

This constrained maximum problem eq(6) is solved using the method of Lagrange multipliers. The associated Lagrangian is

$$F(\theta)=\sum_{t=1}^{T}\ln p_t+\lambda'\left(\sum_{t=1}^{T}p_tG_t(\theta)\right)+\gamma\left(\sum_{t=1}^{T}p_t-1\right).$$

It is straight forward to obtain

$$\hat{p}_t=\frac{1}{n(1-\hat{\lambda}'G_t(\theta))},$$

and $\hat{\lambda}'$ is the solution of

$$\sum_{t=1}^{T}\frac{G_t(\theta)}{1-\lambda'G_t(\theta)}=0.$$

Using the above results, we define our EL estimate as

$$\hat{\theta}=\arg\min_{\theta\in\Theta}\max_{\lambda\in\Lambda(\theta)}\sum_{t=1}^{T}\ln\left[1-\lambda'G_t(\theta)\right]. \tag{7}$$

where

$$\Lambda(\theta)=\{\lambda:\lambda'G_t(\theta)\geq 1, t=i,...,T\}.$$

The optimization problem in eq(7) is a dual saddle point problem. The inner one is a constrained maximization problem with constraints $\lambda'G_t(\theta)\geq 1$ and can be solved by a sequential quadratic programming (SQP) method (see, for example, Gill et al 1981). The outer minimization problem is solved by the simplex search method (to see Lagarias et al 1998). This is a direct search method that does not need numerical or analytic gradients. For the inner problem, Owen (2001) offers an alternative method. Using this method, we can transform the constrained maximization problem into a non-constrained maximization problem. By such a way, the computational burden is reduced, however, we find the solution of the method of Owen (2001) is worse than the method by solving the constrained maximization of inner problem directly in some conditions.

3.3 Asymptotic properties

Donald et al. (2003) investigates the large sample properties of empirical likelihood estimation of $i.i.d.$ data with conditional moment restrictions. They give us the way to approximate the conditional moment restrictions by sequences of unconditional moment restrictions, and show conditions so that the EL estimate can achieve efficiency bound as the number of restrictions grows with the sample size. Although our concern is of the non-$i.i.d.$ case, which is different from the $i.i.d.$ case of Donald et al. (2003). One can expand the results of Donald et al. (2003) to the non-$i.i.d.$ case by modifying some regular conditions of $i.i.d.$ case, and using some asymptotic theory for dependent distributed processes instead of that for $i.i.d.$ processes. Since the moment condition eq(4) is a martingale difference sequence, the Law of Large Number and Central Limit Theorem for martingale difference is useful.

On higher older properties of the EL estimates, Newey and Smith (2004) show us some theoretical advantages. Especially, the EL estimates asymptotic bias does not grow with the number of moment restrictions, while the bias of GMM often does. This property is important for the condition of our method, since many unconditional moment restrictions come out when we approximate the conditional moment restrictions by sequences of unconditional moment restrictions. For this reason, we will perform a Monte Carlo simulation to examine this theoretical advantage in the section 5.

4 GMM WITH A CONTINUUM OF MOMENT CONDITIONS

In the next section, we conduct some simulations for the case of the EL method and the GMM with a continuum of moment conditions (C-GMM). Using the C-GMM we can resolve the problem of covariance matrix singularity in the regular GMM, and utilize the information in the continuum of moment conditions fully.

The regular GMM estimate is defined as

$$\hat{\theta}_T = \arg\min_{\theta} G' W G$$

where G represents a set of sample mean of moment conditions, and W is a weight matrix. The C-GMM estimate is defined similar to the regular GMM, but the set of sample mean of moment conditions G and the weight W are not matrix, they will be in an integration form. C-GMM is based on the arbitrary set of moment conditions

$$\begin{aligned} & E^{\theta_0}\left[h_t(\varkappa;\theta_0)\right] \\ & \equiv E^{\theta_0}\left\{\left[\psi(\omega|\theta_0, X_t) - \exp(i\omega X_{t+\tau})\right]\exp(i\eta X_t)\right\} \\ & = 0 \end{aligned}$$

where $\varkappa = (\omega, \eta) \in \mathbb{R}^2$. Except the instrument $\exp(i\eta X_t)$ and complex number representation, this form is similar to the counterpart of eq.(4). Carrasco et al (2004) defined covariance operator K as

$$K f(\varkappa_1) = \int k(\varkappa_1, \varkappa_2) f(\varkappa_2) \pi(\varkappa_2) d\varkappa_2$$

with

$$k(\varkappa_1, \varkappa_2) = E^{\theta_0}\left[h_t(\varkappa_1;\theta_0)\overline{h_t(\varkappa_2;\theta_0)}\right].$$

They constructed an estimate of the covariance operator K, K_T. Since the inverse operator of K is not bounded, they construct an regularized version of the inverse operator:

$$\left(K_T^{\alpha T}\right)^{-1} = \left(K_T^2 + \alpha_T I_T\right)^{-1} K_T$$

where α_T is a penalizing term, and I_T is the $T \times T$ identity matrix. Then a inner product is defined as

$$\langle f, g\rangle = \int f(\varkappa)\overline{g(\varkappa)}\pi(\varkappa) d\varkappa$$

where $\pi(\cdot)$ is some density function on $\mathbb{R}^2$. Using these definitions, they define the optimal C-GMM estimate as

$$\hat{\theta}_T = \arg\min_{\theta} \left\| \left(K_T^{\alpha T}\right)^{-1/2} \hat{h}_T(\varkappa;\theta) \right\|. \qquad (8)$$

Moreover they introduce a feasible alternative representation of eq. (8) as

$$\hat{\theta}_T = \arg\min_{\theta} \bar{V}'(\theta)\left(\alpha_T I_T + C^2\right)^{-1} V(\theta)$$

where $V = [v_1, \cdots, v_T]'$ with

$$v_t(\theta) = \left\langle \hat{h}_T(\varkappa;\theta), h_t(\varkappa;\hat{\theta}_T^1)\right\rangle,$$

and C is a $T \times T$ matrix with (i, j) element $c_{ij}/(T-1), i, j = 1, \cdots, T$ with

$$c_{ij} = \left\langle h_i(\varkappa;\hat{\theta}_T^1), h_i(\varkappa;\hat{\theta}_T^1)\right\rangle$$

where $\hat{\theta}_T^1$ is a $T^{1/2}$-consistent first step estimate of θ_0.

Due to the regularized version of the inverse and the integration form the operator has, we can resolve the problem of covariance matrix singularity in regular GMM and utilize the information in the continuum of moment conditions fully to increase the efficiency of the estimate.

5 SIMULATION

To examine the theoretical advantage on the higher order properties of EL estimates mentioned above, we perform a Monte Carlo simulation, estimate a scalar CIR process, a scalar Vasicek with exponential jumps process by the Maximum Likelihood Estimate (MLE for CIR only), EL and C-GMM. Since the transition density function of CIR is known, we can perform the Maximum Likelihood Estimation, and check how far the other estimates are from the MLE.

5.1 Model and data generate method

In this subsection, we describe some summaries on the CIR model and Vasicek with exponential jumps model, and show the way to generate the data series.

5.1.1 CIR model

Since the introduction by Cox et al (1985), the CIR model has been used to model interest rates and volatility of asset returns. We represent the CIR model as follows

$$dr_t = (\delta - \kappa r_t)dt + \sigma\sqrt{r_t}dW_t.$$

The transition density function of CIR is known;

$$f(r_{t+\tau}|r_t, t \geq 0) = ce^{-c(u+r_{t+\tau})}\left(\frac{r_{t+\tau}}{u}\right)^{q/2} I_q(2c\sqrt{ur_{t+\tau}}),$$

where $c = 2\kappa/(\sigma^2(1-e^{-\kappa\tau}))$, $u = r_t e^{-\kappa\tau}$, $q = 2\delta/\sigma^2 - 1$ and $I_q(\cdot)$ is the modified Bessel function of first kind of order q. It is known that $2cr_{t+\tau}|r_t \sim \chi^2(2q+2, 2cu)$ is non-central chi square with $2q+2$ degrees of freedom and $2cu$ noncentrality. As mentioned above, we can obtain the CCF of CIR by solving the associated KBE. The CCF of CIR model is as follows:

$$\psi(\omega|\theta,r_t)=\left(1-\frac{i\omega}{c}\right)^{-2\delta/\sigma^2}\exp\left[\frac{i\omega e^{-\kappa}}{1-\frac{i\omega}{c}}r_t\right],$$

here we set $\tau=1$. Following Aït-Sahalia (2002) we set the true values of the parameters as $\delta=0.03$, $\kappa=0.5$, $\sigma=0.15$. According to Aït-Sahalia (2002), these values are realistic for US interest rates. In Monte Carlo simulation, we obtained the data by generating random numbers from $\chi^2(2q+2,2cu)$.

5.1.2 Vasicek with exponential jumps model

Vasicek with exponential jumps model is proposed by Das and Foredi (1996). Generally it is used as a interest rates model. The model is comprised by the Vasicek process and a jump process. The jump size is determined by the absolute exponential distribution variable and the sign of jump by a Bernoulli variable. Carrasco et al (2004) performed a simulation for the scalar Vasicek with exponential jumps process to check the finite sample properties of C-GMM. Following the notation of Carrasco et al (2004), the model equation is

$$dr_t=(\delta-\kappa r_t)dt+\sigma dW_t+J_t dN_t, \tag{9}$$

$$\begin{aligned}|J_t|&\sim EXP(\alpha),\\ sign(J_t)&\sim BIN(\beta),\\ N_t&\sim POI(\lambda).\end{aligned}$$

where $\theta=(\delta,\ \kappa,\ \sigma,\ \alpha,\ \beta,\ \lambda)'$ is the parameter vector. For simplicity we set $\beta=1$ as Carrasco et al (2004). By solving the associated KBE, Das and Foredi (1996) gives the conditional characteristic function $\psi(\omega|\theta,X_t)$

$$\psi(\omega|\theta,r_t)=\exp(A(\omega)+B(\omega)r_t) \tag{10}$$

$$\begin{aligned}A(\omega)&=\frac{i\omega\delta}{\kappa}(1-e^{-\kappa})-\frac{\omega^2\sigma^2}{4\kappa}(1-e^{-2\kappa})\\ &+\frac{i\lambda(1-2\beta)}{\kappa}[\arctan(\omega\alpha e^{-\kappa})-\arctan(\omega\alpha)]\\ &+\frac{\lambda}{2\kappa}\log\left(\frac{1+\omega^2\alpha^2e^{-2\kappa}}{1+\omega^2\alpha^2}\right)\\ B(\omega)&=i\omega e^{-\kappa}.\end{aligned}$$

where τ, which take value 1 in our simulation, is omitted. Following Carrasco et al (2004), we set parameter values, $\delta=0.02949$, $\kappa=0.00283$, $\sigma=0.022$, $\alpha=0.1$, $\lambda=0.28846$.

There are many ways to generate data from this process. We take a way which utilizing the solution of eq(9)

$$\begin{aligned}r(t)&=\frac{\delta}{\kappa}+\left(r(0)-\frac{\delta}{\kappa}\right)e^{-\kappa t}+\sigma\int_0^t e^{-\kappa(t-s)}dW_t\\ &+\sum_0^t e^{-\kappa(t-s)}J_t\triangle N_t,\end{aligned} \tag{11}$$

where $\triangle N_t$ is the increment of N_t at time t. The value of the third term of eq(11) follows Gaussian distribution with mean 0 and variance

$$\left(\sigma\int_0^t e^{-\kappa(t-s)}dt\right)^2.$$

For the forth term, at first we generate a sequence of the time points on which a jump occur, after then generate the size for every time point respectively. It is known that when the jump frequencies follow a Poisson distribution, the jump time points follow an associated exponential distribution.

5.2 Monte Carlo results

The results of CIR are reported in Table 1 and 2. The sample size is $n=200$ for Table 1, $n=500$ for Table 2, and the number of replications is 500. We show mean bias, median bias and RMSE for all sets of estimates. The three methods almost achieved the same efficiency as that of the MLE, and for the parameter σ the EL remarkably outperformed the C-GMM.

Table 1 CIR n=200

	Mean Bias	Median Bias	RMSE
	MLE		
θ	0.0013	0.0007	0.0054
κ	0.0248	0.0147	0.1025
σ	0.0007	0.0007	0.0094
	EL		
θ	0.0015	0.0009	0.0053
κ	0.0347	0.0207	0.1000
σ	-0.0003	-0.0008	0.0099
	C-GMM		
θ	0.0013	0.0006	0.0051
κ	0.0233	0.0099	0.0906
σ	-0.0011	-0.0015	0.0130

Table 2 CIR n=500

	Mean Bias	Median Bias	RMSE
		MLE	
θ	0.0005	0.0002	0.0033
κ	0.0095	0.0032	0.0617
σ	0.0003	0.0005	0.0064
		EL	
θ	0.0004	0.0003	0.0030
κ	0.0149	0.0140	0.0570
σ	-0.0001	-0.0003	0.0061
		C-GMM	
θ	0.0005	0.0001	0.0028
κ	0.0122	0.0040	0.0508
σ	0.0002	0.0006	0.0094

In case of the Vasicek with Jump model, since the transition density function is unknown, we can not perform the MLE estimation. Table 3 and 4 report the results of EL and C-GMM. The sample size are 500 and 1000 for Table 3 and 4 respectively. We set the number of replications to be only 100, due to the simulations of the C-GMM for Vasicek with Jump are extra burdensome. From the values of RMSE, we can see that the EL outperformed the C-GMM for all parameters. More over the mean bias and median bias of the EL method are small than those of the C-GMM for almost all parameters.

Table 3 Vasicek with Jump n=500

	Mean Bias	Median Bias	RMSE
		EL	
θ	-0.0062	0.0020	0.0248
κ	-0.0012	-0.0009	0.0020
σ	-0.0095	-0.0092	0.0136
α	0.1242	-0.0089	0.5074
λ	-0.0376	-0.0716	0.1015
		C-GMM	
θ	0.0128	0.0097	0.0387
κ	-0.0005	-0.0009	0.0026
σ	-0.0203	-0.0220	0.0246
α	0.7177	0.4931	1.3111
λ	0.1056	-0.0860	0.9991

Table 4 Vasicek with Jump n=1000

	Mean Bias	Median Bias	RMSE
		EL	
θ	0.0090	0.0141	0.0196
κ	-0.0007	-0.0004	0.0012
σ	-0.0034	-0.0024	0.0099
α	-0.0676	-0.1096	0.3447
λ	-0.0312	-0.0521	0.1052
		C-GMM	
θ	0.0035	0.0004	0.0312
κ	-0.0012	-0.0014	0.0018
σ	-0.0142	-0.0220	0.0170
α	0.7292	-0.0707	1.1298
λ	0.1760	-0.0767	0.5001

These results suggest that the EL method has superior performance for finite sample, and surpasses the C-GMM in some conditions.

6 CONCLUSION

We proposed a Maximum Empirical Likelihood (EL) estimation method for non-$i.i.d.$ continuous-time models with known functional form of the conditional characteristic function by expanding the EL method of Donald et al (2003). Our EL method fully take one of the theoretical advantages on the higher order properties that the MLE estimators have a smaller asymptotic bias than some other methods in some conditions. The Monte Carlo simulations showed some evidences.

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