Temporal Aggregation of Equity Return Time-Series Models

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EXTENDED ABSTRACT

In many economic and financial applications, the problem of temporal aggregation arises when data are observed at a lower frequency than the data generation frequency of the underlying model. The resulting observed data, which is referred to as the aggregate series, contain less information, and may lead to a distorted view of the true model, leading to potential errors in decision making. Therefore, the study of the effects of temporal aggregation is important for making proper decisions that are based on aggregate data. This paper examines the effect of temporal aggregation on five classes of stochastic equity return models that are commonly used in actuarial practice. Analytical formulae for the linkages between some aggregated and disaggregated stochastic models are presented. If a model is closed under temporal aggregation, the parameters of the lower frequency model can be directly implied by the higher frequency (i.e., more data) model.

With large volatility observed in stock markets around the world over the last few years, many actuaries are now being urged to employ stochastic models to measure the solvency risk generated from insurance products with equity-linked guarantees. There are a large number of potential stochastic models for equity returns. Insurance regulators, both in Europe and North America, normally do not restrict the use of any stochastic model that reasonably fits the historical baseline data. However, in the U.S. and Canada, the final model must be calibrated to some specified distribution percentiles. The emphasis of the calibration process remains on the tails of the equity return distribution over different holding periods. The results in this paper are particularly useful when we need to study distributions and dynamics of longer term (for instance, more than 10 years) accumulation factors for equity-linked insurance products. The data on 10-year (or longer) accumulation factors is often very limited, even for the mature markets in Europe and North America. The advantages of choosing a closed (under temporal aggregation) class of processes for modelling asset returns and equity-linked guarantees are discussed. Actuarial applications of temporal aggregation using S&P500 total return data are given.

1. INTRODUCTION

With large volatility observed in stock markets around the world over the last three years (see Table 1), many actuaries are now being asked to employ stochastic models to measure the solvency risk generated from insurance products with equity-linked guarantees.

Let S_t be the monthly total return index value at t, for $t = 0, 1, 2, \ldots, n$. Define

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right) \tag{1}$$

as the *log return* for the t-th month. The log return series for a m-month non-overlapping holding period can be constructed by

$$R_T = \log\left(\frac{S_{mT}}{S_{m(T-1)}}\right) = \sum_{t=m(T-1)+1}^{mT} r_t \tag{2}$$

for $T = 1, 2, \ldots, N$, and we assume that $N = [n/m]$ is an integer. The accumulation factor for the period $[m(T-1), mT)$ (in months) is given by

$$A_T = \left(\frac{S_{mT}}{S_{m(T-1)}}\right) = \exp(R_T). \tag{3}$$

Equation (2) is called temporal aggregation in the time series econometric literature. The parameter m is termed the order of aggregation. Thus, we can set $m = 12, 60$ and 120 to obtain the one-year, five-year and ten-year accumulation factors from the disaggregated r_t series.

In recent years there has been growing interest in studying the effect of temporal aggregation on financial asset returns (e.g., see Meddahi and Renault, 2003). In general, we say that a model is *closed* under temporal aggregation if the model keeps the same structure, with possibly different parameter values, for any data frequency. This property has also been discussed in the actuarial literature (Klein, 2002; and Hardy, 2002).

The rest of this paper is organised as follows. In Section 2, we examine the effect of temporal aggregation on classes of stochastic equity return models that are commonly used in actuarial practice. The advantages of choosing a closed (under temporal aggregation) class of processes for modelling asset returns are discussed. Section 3 illustrates the application of temporal aggregation results by some actuarial examples. Concluding remarks are given in Section 4.

2. TEMPORAL AGGREGATION

In this section, temporal aggregation of commonly used equity return models is studied. It should be noted that if the monthly stochastic equity return model is closed under temporal aggregation, the parameters of the monthly model can be directly calibrated to generate longer term accumulation factors.

2.1. The Independent Log-normal (ILN) Model

We first consider the traditional log-normal equity return model. The log-normal model has a long and illustrious history, and has become “the workhorse of the financial asset pricing literature” (Campbell et al., 1997, p.16). The log-normal model assumes that log returns are independently and identically distributed (IID) normal variates with a constant mean and a constant variance.

It is well-known that the ILN model is closed under temporal aggregation (or scale invariance in the finance literature). If the monthly log return r_t as defined in equation (1) follows a normal distribution, that is

$$r_t \sim N(\mu, \sigma^2), \tag{4}$$

then the aggregated log return series for a m-month non-overlapping holding period also follows a normal distribution, i.e.,

$$R_T \sim N(m\mu, m\sigma^2). \tag{5}$$

The corresponding accumulation factor variable A_T as defined in equation (3) therefore follows the log-normal distribution.

Table 1. Performance of world stock markets, 2001-2005

		Index Value			Change(%)	
Country	Market	Jan 2001	Jan 2003	Jan 2005	2001-2003	2003-2005
Australia	All Ordinaries Stock Index	3292	2935	4107	-12.2%	28.5%
Canada	TSE 300 Stock Index	9322	6569	9204	-41.9%	28.6%
France	CAC 40 Stock Index	5998	2938	3913	-104.2%	24.9%
Hong Kong	Hang Seng Stock Index	16097	9259	13722	-73.9%	32.5%
Japan	Nikkei 225 Stock Index	13724	8372	11388	-63.9%	26.5%
United Kingdom	FTSE 100 Stock Index	6298	3567	4852	-76.6%	26.5%
United States	S&P 500 Stock Index	1367	856	1181	-59.7%	27.5%

2.2. The Independent Log-Stable (ILS) Model

Stable distributions are a class of probability laws that have intriguing theoretical and practical properties. The class is characterised by Lévy (1924) in his study of the sums of independent identically distributed variables. The application of stable laws to actuarial/financial modelling follows from the fact that stable distributions generalise the normal (Gaussian) distribution to accommodate heavy tails and skewness, which are frequently seen in investment data (e.g., see Finklestein, 1997; Embrechts et al., 1997; Nolan, 2003a).

There are a number of possible parameterisations of the class of stable distributions. Nolan (2003b) lists more than 10 different definitions of stable parameters. In this paper, we employ the "S0" parameterisation, which is better suited to numerical calculations than other representations (Nolan, 1998). Under this definition, the class of stable distributions is described by four parameters, which we call $(\alpha, \beta, \gamma, \delta)$. A random variable Y is S0$(\alpha, \beta, \gamma, \delta)$, if its characteristic function takes the form:

$$\Psi(t) = E[\exp(itY)]$$
$$= \begin{cases} \exp\Big(-\gamma^\alpha |t|^\alpha \Big[1 + i\beta \mathrm{sign}(t)\Big(\tan\frac{\pi\alpha}{2}\Big) \\ \qquad \Big(|\gamma t|^{1-\alpha} - 1\Big)\Big] + i\delta t\Big), \quad \alpha \neq 1, \\ \\ \exp\Big(-\gamma|t|\Big[1 + i\beta \mathrm{sign}(t)\Big(\frac{2}{\pi}\Big)\Big(\ln(\gamma|t|)\Big)\Big] \\ \qquad + i\delta t\Big), \qquad \alpha = 1. \end{cases} \tag{6}$$

It should be noted that Gaussian distributions are special cases of stable laws with $\alpha = 2$ and $\beta = 0$; more precisely, $N(\mu, \sigma^2) =$ S0$(2, 0, \sigma/\sqrt{2}, \mu)$.

The reason for terming this class of distributions *stable* is that they retain their main distributional characteristics under addition (Nolan, 2003b). This means that the stable model is closed under temporal aggregation. If the monthly log return r_t are independently and identically distributed as *stable*, that is

$$r_t \sim \mathrm{S0}(\alpha, \beta, \gamma, \delta), \tag{7}$$

then the aggregated log return series for a m-month non-overlapping holding period also follows a stable distribution, i.e.,

$$R_T \sim \mathrm{S0}(\alpha^*, \beta^*, \gamma^*, \delta^*). \tag{8}$$

The aggregated and disaggregated stable parameters are related by

$$\alpha^* = \alpha, \;\; \beta^* = \beta, \;\; \gamma^* = m^{\frac{1}{\alpha}}\,\gamma,$$

$$\delta^* = \begin{cases} m\delta + m^{\frac{1}{\alpha}}\beta\gamma\tan(\frac{\pi\alpha}{2}), & \alpha \neq 1, \\ m\delta + \frac{2}{\pi} m^{\frac{1}{\alpha}}\beta\gamma\ln(m^{\frac{1}{\alpha}}\gamma), & \alpha = 1. \end{cases} \tag{9}$$

The corresponding accumulation factor variable A_T therefore follows the log-stable distribution.

2.3. The Linear ARMA Model

Wilkie (1987, 1995) developed linear stochastic asset models for United Kingdom data. Wilkie's model is based on the orthodox Box and Jenkins (1976) ARMA (autoregressive moving average) modelling techniques. Suppose that time series Y_t has the stationary and invertible ARMA(p, q) representation

$$\phi(L)Y_t = \theta(L)a_t, \tag{10}$$

where L is the backshift operator such that $L^s Y_t = Y_{t-s}$, $\phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$, and $\theta(L) - \theta_1 L - \ldots - \theta_q L^q$; $\phi(L)$ and $\theta(L)$ have all of their roots outside of the unit circle, and a_t is Gaussian white noise with zero mean and constant variance $\sigma_a^2 < \infty$. Without a loss of generality, we assume that $E[Y_t] = 0$ (Should $E[Y_t] = \mu_Y \neq 0$ then, instead of working with $\{Y_t\}$, the mean-corrected process $\{\widetilde{Y}_t = Y_t - \mu_Y\}$ will be used).

In general, the class of linear ARMA processes is closed under temporal aggregation. If the disaggregated series follows an ARMA(p, q) model, then the temporal aggregated series follows an ARMA(c, d) process where the aggregated orders (c, d) could be the same as or different from the original orders (p, q). The parameters and the orders of the aggregated model can be derived through the relationship of autocovariances between the disaggregated and aggregated series (see Wei, 1990 and references therein).

The first-order autoregressive process is often employed to model the first-lag serial correlation that is observed in many stock return data. As an example, we illustrate the results discussed above using an AR(1) model. Assume that the monthly log return r_t follows an AR(1) process,

$$r_t = \phi r_{t-1} + a_t, \qquad a_t \sim \mathrm{N}(0, \sigma_a^2).$$

First, we derive the lag-s autocovariance function of the m-period aggregated log return variable, $\mathrm{Cov}[R_T, R_{T+s}]$

$$= \begin{cases} \Big[m + 2(m-1)\phi + 2(m-2)\phi^2 + \ldots \\ \qquad + 2\phi^{m-1}\Big]\frac{\sigma_a^2}{1-\phi^2}, \qquad \text{if } s = 0, \\ \\ \Big[1 + \phi + \phi^2 + \ldots + \phi^{m-1}\Big]^2 \times \\ \qquad \Big[\frac{\phi^{m(|s|-1)+1}}{1-\phi^2}\Big]\sigma_a^2, \\ \qquad\qquad\qquad \text{if } s = \pm 1, \pm 2, \ldots. \end{cases} \tag{11}$$

Equation (11) implies that R_T follows an ARMA(1,1) process, i.e.,

$$(1-\phi^* L)R_T = (1-\theta^* L)a_t^*, \qquad a_t^* \sim \mathrm{N}(0, \sigma_{a^*}^2),$$

with

$$\phi^* = \phi^m,$$

and $|\theta^*| < 1$ is the solution of the following quadratic equation

$$\frac{(\phi^m - \theta^*)(1-\phi^m\theta^*)}{1 - 2\phi^m\theta^* + \theta^{*2}} =$$

$$\frac{\phi(1+\phi+\phi^2+\ldots+\phi^{m-1})^2}{m + 2(m-1)\phi + 2(m-2)\phi^2 + \ldots + 2\phi^{m-1}}.$$

Finally, $\sigma_{a^*}^2$ can be computed from the Var$[R_T]$ equation in (11).

2.4. The GARCH Model

Time-varying volatility models have been popular since the early 1990s in financial research and applications, following the influential papers by Engle (1982) and Bollerslev (1986). Stochastic models of this type are known as generalised autoregressive conditional heteroscedastic (GARCH) models in the time series econometrics literature. GARCH processes are useful because these models are able to capture empirical regularities of asset returns such as thick tails of unconditional distributions, volatility clustering and negative correlation between lagged returns and conditional variance (Franses and Dijk, 2000 and Tsay, 2002).

Let $a_t = (r_t - \mu)$ be the mean-corrected log return. Then, a_t follows a GARCH(p, q) model if

$$\varepsilon_t = a_t/\sqrt{h_t}, \;\; h_t = \omega + \sum_{i=1}^{p} \alpha_i \, a_{t-j}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \tag{12}$$

where $\{\varepsilon_t\}$ is a sequence of IID random variables with mean zero and unit variance, $\omega > 0, \alpha_i \geq 0, \beta_j \geq 0$, and $\sum_{k=1}^{\max(p,q)} (\alpha_k + \beta_k) < 1$ with $\alpha_k = 0$ for $k > p$ and $\beta_k = 0$ for $k > q$.

Little is known about the impact of temporal aggregation on a general GARCH(p, q) process. For some lower order GARCH models, under certain conditions, Drost and Nijman (1993) show that they are closed under temporal aggregation. Fortunately, in most practical applications, lower order GARCH processes are adequate for modelling equity return data. In this section, we consider a GARCH(1,1) process and its innovation following a Student t distribution: i.e., ε_t in equation (12) has a marginal t distribution with mean zero, unit variance and degrees of freedom ν, and the conditional variance has the following representation

$$h_t = \omega + \beta\, h_{t-1} + \alpha\, a_{t-1}^2,$$

and the unconditional kurtosis of a_t is κ. Following Drost and Nijman (1993), we find that the aggregated return for a m-month non-overlapping period can be "weakly" approximated by a GARCH (1,1) process with the corresponding parameters:

$$\mu^* = m\,\mu, \; \omega^* = m\,\omega \left\{ \frac{1-(\alpha+\beta)^m}{1-(\alpha+\beta)} \right\},$$

$$\alpha^* = (\alpha+\beta)^m - \beta^*,$$

$$\kappa^* = 3 + \frac{(\kappa-3)}{m} + 6(\kappa-1)\times$$

$$\left\{ \frac{[m-1-m(\alpha+\beta)+(\alpha+\beta)^m][\alpha-\alpha\beta(\alpha+\beta)]}{m^2(1-\alpha-\beta)^2(1-2\alpha\beta-\beta^2)} \right\},$$

and $|\beta^*| < 1$ is the solution of the following quadratic equation

$$\frac{\beta^*}{1+\beta^{*2}} = \frac{\Theta[(\alpha+\beta)^m] - \Lambda}{\Theta[1+(\alpha+\beta)^{2m}] - 2\Lambda}, \quad \text{with}$$

$$\Theta = m(1-\beta)^2$$

$$+\left\{ \frac{2m(m-1)(1-\alpha-\beta)^2(1-2\alpha\beta-\beta^2)}{(\kappa-1)[1-(\alpha+\beta)^2]} \right\}$$

$$+4\left\{ \frac{[m-1-m(\alpha+\beta)+(\alpha+\beta)^m]\times D}{1-(\alpha+\beta)^2} \right\}, \quad \text{with}$$

$$D = [\alpha - \alpha\beta(\alpha+\beta)], \quad \text{and}$$

$$\Lambda = \frac{[\alpha-\alpha\beta(\alpha+\beta)][1-(\alpha+\beta)^{2m}]}{1-(\alpha+\beta)^2}.$$

The degrees of freedom of the marginal t distribution (ν^*) for the aggregated GARCH model can be derived via the aggregated kurtosis, κ^* (Bai et al., 2003).

2.5. The RSLN Model

In recent years, the use of regime switching log-normal (RSLN) processes for modelling maturity guarantees has been gaining popularity. Hardy (2001) proposes using Markov-type regime switching log-normal (RSLN) processes for modelling monthly equity returns. The RSLN model is defined as

$$r_t = \mu_{S_t} + \sigma_{S_t}\, \varepsilon_t \tag{13}$$

where $S_t = 1, 2, \ldots, k$ denotes the unobservable state indicator, which follows an ergodic k-state Markov process, and ε_t is a standard normal random variable that is IID over time. In most situations, $k = 2$ or 3 (i.e., two- or three-regime models) is sufficient for modelling monthly equity returns (Hardy, 2001). The stochastic transition probabilities that determine the evolution in S_t are given by

$$Pr\{S_{t+1} = j \mid S_t = i\} = p_{ij},$$

$$0 < p_{ij} < 1, \quad \sum_{j=1}^{k} p_{ij} = 1 \;\; \text{for all } \; i,$$

so that the states follow a homogenous Markov chain.

Research into the temporal aggregation of RSLN models is scanty and very much in its infancy. As a starting point to study the effect of temporal aggregation on RSLN processes, we examine the limiting behaviour (i.e., $m \to \infty$) of a simple two-regime RSLN model. Let

$$r_t = \begin{cases} \varepsilon_t^{(1)} \sim N(\mu_1, \sigma_1^2), \\ \varepsilon_t^{(2)} \sim N(\mu_2, \sigma_2^2), \end{cases} \tag{14}$$

with transition probability matrix

$$\mathcal{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \qquad 0 < p_{ij} < 1. \tag{15}$$

This implies that the vector of steady-state (ergodic) probabilities is

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{p_{21}}{p_{12}+p_{21}} \\ \frac{p_{12}}{p_{12}+p_{21}} \end{pmatrix}.$$

It is well-known that the limiting structure of time series aggregates from a covariance stationary process is white noise (e.g., see Wei, 1990, p. 412; Granger, 1990; Ohanissian et al., 2003). Timmermann (2000) gives the autocovariance function of the RSLN model in (14),

$$\gamma(s) = \pi_1\pi_2(\mu_1 - \mu_2)^2 \mathbf{vec}(\mathcal{P}^s)'\boldsymbol{\eta} \tag{16}$$

where **vec**$(\cdot)$ is the *vector of a matrix* operator (Graybill, 1983, p. 309), and $\boldsymbol{\eta} = (\pi_2, -\pi_2, -\pi_1, \pi_1)'$. Cox and Miller (1965, p. 82) derive that

$$\mathcal{P}^s = \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix} + (1 - p_{12} - p_{21})^s \begin{pmatrix} \pi_2 & -\pi_2 \\ -\pi_1 & \pi_1 \end{pmatrix}. \tag{17}$$

Combining (16) and (17), we have

$$\gamma(s) = \pi_1\pi_2(\mu_1 - \mu_2)^2(1 - p_{12} - p_{21})^s.$$

This autocovariance function is obviously geometrically bounded because $|(1 - p_{12} - p_{21})| < 1$, which is guaranteed by the restriction in (15) that all elements of $\mathcal{P}$ be positive. Hence, the limiting structure for the temporal aggregates of a two-regime RSLN model is white noise.

Deriving the effect of temporal aggregation on a more general (i.e., more than two regimes and finite values of m) RSLN model is not a trivial task. Research in this direction is in process.

3. APPLICATIONS

Quantile matching is the key of the Life Capital Adequacy Subcommittee (LCAS) calibration requirement in the United States (LCAS, 2002). In this application, we consider the monthly S&P 500 total return series from January 1945 to October 2002, which is the baseline series recommended by the Subcommittee. The main focus of this example is to numerically illustrate the quantile results using model aggregation formulae derived in the previous section. These results are then compared to the published calibration requirements. It is not our objective to recommend the "best" fitted model for the S&P 500 series.

From the five classes of models discussed in the last section, we find that the classes of ILS models and GARCH models are reasonably fitted to the data. First, we consider the class of independent stable distributions. Assume that the monthly S&P 500 log return follows an IID stable distribution, i.e.,

$$r_t \sim \mathrm{S0}(\alpha, \beta, \gamma, \delta),$$

see equations (6) to (9) in Section 2. Methods of computing the maximum likelihood estimation (MLE) of stable parameters are discussed by Nolan (2003b). Nolan (2003c) provides software for estimating and analysing stable models. The fitted stable parameters for the monthly series r_t are given in Table 2. The implied parameters for the aggregated stable models with different orders of aggregation m are calculated using equation (9). The results are listed in Table 2. The corresponding accumulation factor (A_T) for a holding period m, as defined in equation (3), follows an ILS model. The ILS model is simple and tractable. The cumulative distribution function (CDF) and the probability distribution function (PDF) of A_T are easy to be evaluated (Nolan, 2003c).

Financial asset return time series usually exhibit a characteristic known as volatility clustering, in which large changes tend to follow large changes, and small changes tend to follow small changes. Furthermore, probability distributions for asset returns often exhibit fatter tails than the standard normal distribution. The fat tail phenomenon is known as excess kurtosis in the finance literature. It is well-known that the GARCH model can accommodate these unique features (heavy tails and volatility clustering), which occur frequently in observed stock returns (Bollerslev et al., 1994).

The GARCH(1,1) model with marginal t distribution is fitted to the S&P 500 monthly total return series. The fitted GARCH parameters and their corresponding implied parameters for the aggregated models with different orders of aggregation m are given in Table 3. The implied parameters are computed using formulae discussed in Section 2.4.

Finally, lower quantiles for the 1-year accumulation factor using the aggregated ILS and GARCH models are obtained. Table 4 compares these quantiles to the calibration criteria derived from the S&P 500 empirical data by the AAA (LCAS, 2002 and Longley-Cook, 2003). The GARCH quantiles closely match to the AAA calibration points, but the ILS quantiles are too large as compared to the criteria.

4. CONCLUSION

This paper examines the effect of temporal aggregation on five classes of stochastic equity return models that are commonly used in actuarial practice. Analytical formulae for the linkages between some aggregated and disaggregated stochastic models are presented. If a model is closed under temporal aggregation, the parameters of the lower frequency model can be directly implied by the higher frequency (i.e., more data) model. This property is particularly useful when we need to study distributions and dynamics of longer term (for instance, more than 10 years) accumulation factors for equity-linked insurance products. The data on 10-year (or longer) accumulation factors is often very limited, even for the mature markets in Europe and North America.

Table 2. Stable parameters for the S&P 500 data

	Monthly	Aggregated Model		
Parameter	Model	$m = 12$	$m = 60$	$m = 120$
α	1.8678	1.8678	1.8678	1.8678
β	-0.7591	-0.7591	-0.7591	-0.7591
γ	0.0273	0.1032	0.2443	0.3540
δ	0.0128	0.1705	0.8088	1.5961

Table 3. GARCH parameters for the S&P 500 data

	Monthly	Aggregated Model		
Parameter	Model	$m = 12$	$m = 60$	$m = 120$
μ	0.01074	0.12885	0.64423	1.28845
ω	0.00014	0.01331	0.10738	0.21659
α	0.08296	0.00060	0.00007	0.00002
β	0.84070	0.38501	0.00846	0.00006
κ	4.92	4.18	3.52	3.29
ν	9.06	9.09	15.60	24.56

Table 4. Non-overlapping 1-year accumulation factor, S&P 500 data

	Empirical Range			Aggregated	Aggregated	AAA
Quantile	Minimum	Maximum	Mid	ILS	GARCH(1,1)	Calibration
1.72%	0.61	0.89	0.75	0.80	0.74	
2.50%	0.67	0.89	0.78	0.84	0.77	0.77
3.45%	0.73	0.89	0.81	0.87	0.81	
5.00%	0.79	0.90	0.85	0.90	0.84	0.84
5.17%	0.80	0.91	0.85	0.90	0.84	
6.90%	0.82	0.91	0.87	0.93	0.87	
8.62%	0.87	0.92	0.89	0.95	0.89	
10.00%	0.87	0.93	0.90	0.96	0.91	0.91
10.34%	0.87	0.93	0.90	0.97	0.91	

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