Dynamic Correlations in Symmetric Multivariate SV Models

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EXTENDED ABSTRACT

This paper proposes two types of stochastic correlation structures for Multivariate Stochastic Volatility (MSV) models, namely the constant correlation (CC) MSV and dynamic correlation (DC) MSV models, from which the stochastic covariance structures could be obtained easily. Both structures can be used for purposes of optimal portfolio and risk management, and for calculating Value-at-Risk (VaR) forecasts and optimal capital charges under the Basel Accord. The choice between the CC MSV and DC MSV models can be made using a deviance information criterion. A technique is developed to estimate the DC MSV model using the Markov Chain Monte Carlo (MCMC) procedure.

1. Introduction

Covariance and correlation structures are used routinely for optimal portfolio choice, risk management, obtaining Value-at-Risk (VaR) forecasts, and determining optimal capital charges under the Basel Accord. This issue does not yet seem to have been examined in the Multivariate Stochastic Volatility (MSV) literature.

For multivariate GARCH models, the most general expression is called the 'vec' model (see Engle and Kroner (1995)). The vec model parameterizes the vector of conditional covariance matrix of the returns vector, which is determined by its lags and the vector of outer products of the lagged returns vector. A serious issue with the vec model is that it has many parameters to be estimated, and will not guarantee positive definiteness of the conditional covariance matrix without further restrictions. Bollerslev et al. (1988) suggested the diagonal GARCH model, which restricts the off-diagonal elements of the parameters matrices to be zero, and also reduced the number of parameters drastically. Engle and Kroner (1995) proposed the Baba, Engle, Kraft and Kroner (or BEKK) specification that guaranteed the positive definiteness of the conditional covariance matrix. This is essential for obtaining sensible VaR forecasts. Bollerslev (1990) proposed the Constant Conditional Correlation (CCC) model, where the time-varying covariances are proportional to the conditional standard deviation derived from univariate GARCH processes. This specification also guarantees the positive definiteness of the conditional covariance matrix. Recently, Engle (2002) developed the Dynamic Conditional Correlation (DCC) model, which allows the conditional correlation matrix to vary parsimoniously over time. McAleer (2005) provides a comprehensive comparison of a wide range of univariate and multivariate, conditional and stochastic, financial volatility models.

This paper proposes two types of stochastic correlation structures for MSV models, namely the constant correlation (CC) MSV and dynamic correlation (DC) MSV models, from which the stochastic covariance structures could be obtained easily. Both structures can be used for purposes of optimal portfolio and risk management, for calculating VaR forecasts, and for determining optimal Basel Accord capital charges. The choice between the CC MSV and DC MSV models can be made using a deviance information criterion (see Berg et al. (2004)). A technique is developed to estimate the DC MSV model using the Markov Chain Monte Carlo (MCMC) procedure.

2 Constant and Dynamic Correlations

This section develops two types of correlation models for MSV models, namely the CC MSV and DC MSV models, from which the stochastic covariance structures may be obtained easily.

2.1 Constant Correlation MSV Models

Consider the constant correlation (CC) MSV model proposed by Harvey, Ruiz and Shephard

(1994). Let y_t be an $M \times 1$ vector of stochastic process, as follows:

$$y_t = D_t \varepsilon_t, \quad \varepsilon_t \sim N(0, \Gamma), \tag{1}$$

$$D_t = \mathrm{diag}\{\exp(h_t / 2)\}, \tag{2}$$

$$h_{t+1} = \mu + \phi \circ (h_t - \mu) + \eta_t, \quad \eta_t \sim N(0, \Sigma_\eta), \tag{3}$$

where the operator '$\circ$' denotes the Hadamard element-by-element product of two identically-sized matrices or vectors. Here, we use 'exp' as the operator for vectors to denote element-by-element exponentiation, and 'diag' for vectors to create a diagonal matrix. This model deals with two kinds of correlation matrices, namely the correlations between mean variables, Γ, and the covariance matrix of volatility, Σ_η. It should be noted that the model is different from the Constant Conditional Correlation (CCC) model of Bollerslev (1990) with respect to two major points, namely (i) the volatilities are stochastic, and (ii) the disturbances of the volatilities are correlated simultaneously.

In order to obtain the VaR forecasts for a given portfolio and to determine the optimal capital charges for a portfolio under the Basel Accord, it would be necessary to calculate the stochastic covariance matrix, Σ_t, from the correlation matrix, as $\Sigma_t = D_t \Gamma D_t$.

There are two ways in which to deal with the so-called 'leverage effects'. One approach is to extend the model (1)-(3) in order to assume negative correlations between returns and changes of volatilities as follows:

$$E(\varepsilon_t \eta_t') = \mathrm{diag}\{\kappa_1 \sigma_{\eta,11}^{1/2}, \ldots, \kappa_m \sigma_{\eta,mm}^{1/2}\},$$

where $\sigma_{\eta,ii}$ is the (i,i)th element of Σ_η, and κ_i is restricted to be negative. The model is a multivariate extension of Harvey and Shephard (1996), and has been analysed in Asai and McAleer (2004). The other approach is to incorporate y_t and $|y_t|$ in equation (3), as follows:

$$h_{t+1} = \mu + \phi \circ (h_t - \mu) + \lambda_1 \circ y_t + \lambda_2 \circ |y_t| + \eta_t, \quad \eta_t \sim N(0, \Sigma_\eta). \tag{4}$$

This model was proposed by Asai and McAleer (2004) to develop the multivariate extension of the SV model suggested by Danielsson (1994). In the Bayesian framework, Yu (2005) and Omori et al. (2004) estimated the univariate model of Harvey and Shephard (1996) using the MCMC technique.

As the purpose of the present paper is to consider the stochastic correlation MSV model, namely DC MSV, we do not consider asymmetric models, although such extensions are straightforward for the CC MSV model.

2.2 Dynamic Correlation MSV Models

This subsection develops a dynamic correlation (DC) MSV model, as an extension of the CC MSV model, which is based on the differences between the ε_{it}. Our approach is based on the Wishart distribution. It is known that the Wishart distribution of a sample variance covariance matrix computed from i.i.d. multivariate Gaussian observation (see Anderson (1984) and

Stuart and Ord (1994)). By considering the serially dependent Wishart Process, we have a process of positive definite matrices, under appropriate conditions. A standardization of the process leads to a serially dependent process of correlation matrices.

Let ε_t have a multivariate normal distribution, $N(0, \Gamma_t)$, conditional on the stochastic correlation matrix, Γ_t, where

$$\begin{aligned} &\Gamma_t = Q_t^{*-1} Q_t Q_t^{*-1}, \; Q_{t+1} = \Omega + \psi Q_t + \Xi_t, \\ &\Xi_t \sim W_m(\nu, \Lambda), \; Q_t^* = \left(\text{diagonal}\{Q_t\}\right)^{1/2}, \end{aligned} \tag{5}$$

and $W_m(\nu, \Lambda)$ denotes a Wishart distribution, and 'diagonal' creates a diagonal matrix by setting the off-diagonal elements of matrices to be zero. The DC MSV model guarantees the positive definiteness of Γ_t under the assumption that Q_t is positive definite, $0 < \psi < 1$, and that Ω and Λ are symmetric positive definite matrices. The second condition also implies that the time-varying stochastic correlations are mean reverting. Given Q_1 is positive definite, Q_t is guaranteed to be positive definite (the proof is obtained in a similar manner to that of the DCC model - see below for the case of $\nu = 1$). By the positive definiteness of Q_t, Γ_t will also be positive definite. With the above dynamic and stochastic structures, the DC MSV model for $y_t = D_t \varepsilon_t$ is defined by (2), (3) and (5).

As in the case of the CC MSV model, in order to obtain the VaR forecasts for a given portfolio and to determine the optimal capital charges for a portfolio under the Basel Accord for the DC MSV model, it is necessary to calculate the stochastic covariance matrix, Σ_t, from the stochastic correlation matrix, as $\Sigma_t = D_t \Gamma_t D_t$.

In order to investigate the properties of the DC MSV model, we consider the case of $\nu = 1$, for simplicity. In this case, Ξ_t can be expressed as $\Xi_t = \xi_t \xi_t'$, where $\xi_t : N(0, \Lambda)$. Then we have

$$Q_{t+1} = \Omega + \psi Q_t + \xi_t \xi_t', \tag{6}$$

which is analogous to the DCC model of Engle (2002), namely

$$Q_{t+1} = \Omega + \alpha Q_t + \beta \varepsilon_t \varepsilon_t'.$$

Comparing the last terms in the two equations given above, that of the DCC model is predetermined and can be observed by using estimated conditional volatility, while that of the DC MSV model is unobservable. Furthermore, letting $q_t = \text{vec}(Q_t)$, we have a VAR(1) process for q_t, as follows:

$$q_{t+1} = (\omega + \lambda) + \psi q_t + \text{vec}(\xi_t \xi_t' - \Lambda), \tag{7}$$

where $\omega = \text{vec}(\Omega)$ and $\lambda = \text{vec}(\Lambda)$. Since the expectation of the last term is zero, it is possible to obtain $E(q_t) = (1-\psi)^{-1}(\omega + \lambda)$, or $E(Q_t) = (1-\psi)^{-1}(\Omega + \Lambda)$. Similarly, it can be shown that

$$V(q_t) = (1-\psi)^{-2}\left(I_{m^2} + K_{mm}\right)(\Lambda \otimes \Lambda),$$

where K_{ij} is the commutation matrix. Noticing that $\Xi_t = \xi_t \xi_t'$, the other moments can be obtained through tedious calculation by using the moment of the multivariate normal distribution. When $\nu = 1$, it may be natural to assume that $Q_1 = (1-\psi)^{-1}(\Omega + \Xi_0)$, since its mean is equal to the unconditional mean, and as it is a positive definite matrix.

In the following, we consider $\nu = 1$ for convenience as (i) it is easy to interpret in this case, and (ii) an approximated Gaussian process of (7) based on ξ_t is used to develop the MCMC technique.

3 Bayesian MCMC: An Overview

In this section, we develop a technique to estimate the DC MSV model via Markov Chain Monte Carlo (MCMC) procedure (see, for example, Chib and Greenberg (1996)). For univariate SV models, there are two kinds of efficient and fast MCMC methods, namely (i) the integration sampler suggested by Kim et al. (1998) and Chib et al. (2002), and (ii) the multi-move sampler proposed by Shephard and Pitt (1997, 2004) and Watanabe and Omori (2004). For reasons explained in Subsection 3.2 below, we develop the block samplers which are an extension of Shephard and Pitt (1997, 2004).

Apart from MCMC, competing approaches include the likelihood approach based on importance sampling, such as Sandmann and Koopman (1998) and Liesenfeld and Richard (2003), and the reprojection method proposed by Gallant and Tauchen (1998). Asai and McAleer (2004) and Liesenfeld and Richard (2003) estimated MSV models by using the importance sampling procedures. Although we have to deal with the latent process of Q_t, which has Wishart disturbances, the importance sampling approach is inapplicable when the latent process is non-Gaussian. Furthermore, the reprojection method needs to be extended further for application to MSV models.

Let θ_1 denote the vector of parameters to be estimated in the latent process for the dynamic correlations, Q_t, and θ_2 the vector of parameters for the volatility processes, h_t. Given the prior density, $\pi(\theta_1, \theta_2)$, the aim of Bayesian inference is to obtain the parameter vector, (θ_1, θ_2), from the augmented posterior distribution, namely.

$$\pi(\theta_1, \theta_2, Q, h \mid y) \propto \pi(\theta_1, \theta_2) \times$$
$$\prod_{t=1}^{T} f(y_t \mid Q_t, h_t, \theta_1, \theta_2) \times$$
$$f(Q_t \mid Q_{t-1}, \theta_1)\, f(h_t \mid h_{t-1}, \theta_2).$$

In order to conduct inferences on the parameters, we produce a sample $\left(\theta_1^{(g)}, \theta_2^{(g)}, Q^{(g)}, h^{(g)}\right)$ from this density by the MCMC method. The produced draws $\left(\theta_1^{(g)}, \theta_2^{(g)}\right)$ are taken from the posterior density marginalized over (Q, h).

The proposed MCMC algorithm is based on the two blocks, (θ_1, Q) and (θ_2, h):

Algorithm

1. Initialize Q and (θ_1, θ_2).
2. Sample h and θ_2 from $\theta_2, h | y, Q$ through drawing:

(a) h from $h | y, \theta_2, Q$; (b) θ_2 from $\theta_2 \mid h$.

3. Sample Q and θ_1 from $\theta_1, Q | y, h$ through drawing:

(a) Q from $Q | y, \theta_1, h$; (b) θ_1 from $\theta_1 \mid Q$.

4. Go to Step 2.

The remainder of this section proposes the sampling method for each step mentioned above. Step 3a is most important. Our method is based on the Metropolis-Hastings (MH) algorithm (see Chib and Greenberg (1995)). Since the process of q_t is linear but non-Gaussian, we consider a linear approximation of the equation, and apply the simulation smoother proposed by de Jong and Shephard (1995) in order to generate candidates for the MH algorithm. Sampling all latent variables as one block will run into large very large rejection frequencies, while generating a single state at a time, which yields a highly autocorrelated sample, and needs a huge number of samples to conduct a statistical inference. We will take an intermediate approach, namely a block sampler such as Shephard and Pitt (1997) and Elerian et al. (2001).

Step 3b is based on the MH algorithm. We will use the Wishart distribution to generate candidates for Ω and Λ^{-1}, while we use the gamma distribution for ψ. In order to implement Step 2a, we extend the multi-move sampler proposed by Shephard and Pitt (1997, 2004) and Watanabe and Omori (2004). Although their approach was developed specifically for univariate SV models, the extension to MSV models is straightforward. As the model for h_t can be interpreted as a seemingly unrelated regression (SUR) model, in that SUR can be written as a regression model with parameter restrictions, Step 2b follows from the updates of a VAR(1) model with parameter restrictions.

5 Conclusion

As covariance and correlation structures are used routinely for optimal portfolio choice, risk management, obtaining Value-at-Risk (VaR) forecasts, and determining optimal capital charges under the Basel Accord, it is essential to model the covariances and correlations accurately. This issue does not yet seem to have been examined in the Multivariate Stochastic Volatility (MSV) literature.

This paper proposed two types of stochastic correlation structures for MSV models, namely the constant correlation (CC) MSV and dynamic correlation (DC) MSV models, from which the stochastic covariance structures could be obtained easily. The choice between the CC MSV and DC MSV models was shown to be based on a deviance information criterion. A technique was developed to estimate the DC MSV model using the Markov Chain Monte Carlo (MCMC) procedure.

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