Household Portfolio Behavior with Stochastic Interest Rates

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Abstract

The life cycle-permanent income (LPCI) hypothesis is one of the most important hypothesis describing how age-related patterns of consumer behavior affect consumption, saving and labor supply. The mathematical structure of the LPCI hypothesis consists of functional optimization. The model includes the performance function of lifetime utility, system equations and related constraints. In our model, we assume that a household allocates streams of consumption and investments in various financial assets so as to maximize the expected value of its lifetime utility. The introduction of this expected value is the main difference from our previous analysis in Maki and Aiyoshi (1997). From the start of employment until the time of retirement, a household considers the optimal combination of consumption and investment in a variety of financial assets. We assume interest rates of financial assets are discrete-type stochastic variables which have a probability corresponding to prevailing interest rates. The simulation suggests that the expected rate of return plays an important role in determining the period beginning of asset accumulation and reduction of consumption. Our finding is: there are many types of streams for consumption and asset accumulation paths reflecting different combinations of interest rates with related probabilities.

I. Introduction

The life cycle-permanent income (LPCI) hypothesis is one of the most important hypothesis describing how age-related patterns of consumer behavior affect consumption, saving and labor supply. The mathematical structure of the LPCI hypothesis consists of functional optimization. The model includes the performance function of lifetime utility, system equations and related constraints.

We employ a simulation method based on an algorithm of optimal control analysis using the penalty method to determine allocation of household consumption and financial assets over time.

In our model, we assume that a household allocates streams of consumption and investments in various financial assets so as to maximize the expected value of its lifetime utility. The introduction of this expected value is the main difference from our previous analysis in Maki and Aiyoshi (1997). From the start of employment until the time of retirement, a household considers the optimal combination of consumption and investment in a variety of financial assets. We assume interest rates of financial assets are discrete-type stochastic variables which have a probability corresponding to prevailing interest rates.

II. The Model

As a means to introduce stochastic variables in the model, stochastic movement of income streams and that of interest rates are considered. In the present model we introduce the stochastic variables for interest rates which have probabilities based on past trends. The expected portfolio model is estimated based on the following variables: t, time variable, $a(t)$ is the financial asset at t, $c(t)$ is the consumption at t, $w(t)$ is the income at t, δ is the discount rate, $r_j(t)$ is the interest rate of j-th asset, a_1 is the initial asset and a_N is the required financial assets at the time of retirement.

The maximization of the expected value of the lifetime utility function is formulated as:

$$\max_{c(.)} \quad E\left[\sum_{t=1}^{N}(1+\delta)^{-t}U(c(t))\right] \qquad (1)$$

We assume that there is no stochastic factor for the income stream, and m levels of interest rates are determined stochastically with the probability of interest rates of $r_1, \ldots, r_m$ as $P_t(r_1), P_t(r_2), \ldots$, and $P_t(r_m)$, respectively.

The constraint is

$$\sum_{i=1}^{m} P_t(r_i) = 1 \qquad (2)$$

Based on this assumption, let us consider the expected amount of assets for the time 2 (t = 2). The expected value of the asset, $a^*(2)$ are formulated as:

$$a^*(2) = \sum_{i=1}^{m} P_1(r_i)a_i(2)$$

$$= \sum_{i=1}^{m} P_1(r_i)(1+r_i)a(1)$$

$$+ w(1)\sum_{i=1}^{m} P_1(r_i) - c(1)\sum_{i=1}^{m} P_1(r_i)$$

$$= \left\{1 + \sum_{i=1}^{m} P_1(r_i)r_i\right\}a(1)$$

$$+ w(1) - c(1) \qquad (3)$$

The asset of the time 3 is:

$$a^*(3) = \sum_{i=1}^{m}\sum_{j=1}^{m}\{P_1(r_i)P_2(r_j)\}a_{ij}(3)$$

$$= \left\{1 + \sum_{j=1}^{m} P_2(r_j)r_j\right\}a^*(2)$$

$$+ w(2) - c(2) \qquad (4)$$

where a_{ij} indicates the accumulated assets in the third period reflecting interest rates of r_i during the first period and r_j during the second period.

The expected value of assets in the t-th period is:

$$a^*(t)$$

$$= \sum_{i(1)=1}^{m} \cdots \sum_{i(t-1)=1}^{m} P_1(r_{i(1)}) \ldots P_{t-1}(r_{i(t-1)}) \cdot$$

$$a_{i(1)\ldots i(t-1)}(t)$$

$$= \left\{1 + \sum_{j=1}^{m} P_{t-1}(r_j)r_j\right\} a^*(t-1)$$

$$+ w(t-1) - c(t-1) \qquad (5)$$

where $a_{i(1)\ldots i(t-1)}$ indicates the accumulated assets in the t-th period reflecting interest rates from the first to (t-1)-th period of $r_{i(1)}, \ldots$, and $r_{i(t-1)}$, respectively.

Extending the previous model developed by Maki and Aiyoshi (1997), we formulate three simulation models.

The simulation model is formulated so as to maximize the lifetime utility function as:

$$\max_{c(.)} \; -\sum_{t=1}^{N-1} \exp(-\delta t)(c(t)-c_1)(c(t)-c_2) \qquad (6)$$

The evaluation function is the same for all the following models. However, the restrictions of the evaluation function (6) are different from each other as follows:

(a) Model 1

Restrictions:

$$a^*(t+1) = \left\{1 + \sum_{j=1}^{m} P_t(r_j)r_j\right\}a^*(t)$$

$$+ w(t) - c(t) \qquad t=1,\dots,N-1$$

$$a(1) = a_1$$

$$0 \le c(t) \le 1/2(c_1+c_2) \qquad t = 1,\dots,N-1$$

$$a^*(t) \ge 0 \qquad t = 1,\dots,N$$

This model includes only one asset. The last inequality indicates liquidity constraints related to household assets.

(b) Model 2

Restrictions:

$$a_1^*(t+1) = \{1 + \sum_{j=1}^{m} P_t^1(r_j)r_j\}a_1^*(t) + \{1-\gamma(t)\}w(t) - c(t) \qquad t=1,\dots,N-1$$

$$a_2^*(t+1) = \{1+\sum_{j=1}^{m} P_t^2(r_j)r_j\}a_2^*(t) + \gamma(t)w(t) \qquad t = 1,\dots,N-1$$

$$a_1(1) = a_2(1) = 0$$

$$0 \le c(t) \le 1/2(c_1+c_2) \qquad t = 1,\dots,N$$

$$a_1^*(t), a_2^*(t) \ge 0 \qquad t = 1,\dots,N$$

$$0 \le \gamma(t) \le 1 \qquad t = 1,\dots,N$$

The first asset is used for both consumption and savings and the second asset is used for savings. The model includes forced savings because some amount, indicated in $\gamma(.)$, of current earnings accumulates to the assets. In this formulation the variables for maximization are not only $c(.)$ but also $\gamma(.)$.

(c) Model 3

Restrictions :

$$a_1^*(t+1) = \{1+\sum_{j=1}^{m} P_t^1(r_j)r_j\}a_1^*(t) + \{1-\gamma(t)\}\{w(t) - c(t)\} \qquad t=1,\dots,N-1$$

$$a_2^*(t+1) = \{1+\sum_{j=1}^{m} P_t^2(r_j)r_j\}a_2^*(t) + \gamma(t)\{w(t) - c(t)\} \qquad t=1,\dots,N-1$$

$$a_1(1) = a_2(1) = 0$$

$$0 \le c(t) \le 1/2(c_1+c_2) \qquad t = 1,\dots,N-1$$

$$a_1^*(t), a_2^*(t) \ge 0 \qquad t = 1,\dots,N$$

$$0 < \gamma(t) \le 1 \qquad t = 1,\dots,N$$

Current saving is divided into designated first and second financial assets at the rate of $1-\gamma(t)$ and $\gamma(t)$.

III. Numerical analysis of the Model

In solving the models, the original models are transformed by utilizing the penalty method to accommodate the terminal condition and inequality constraints. The problem including penalty terms is called the transformed problem against the original model. The transformed model 3 is:

$$\max_{c(.),\gamma(.)} - \sum_{t=1}^{N-1}[\exp(-\delta t)(c(t)-c_1)(c(t)-c_2)$$
$$- \epsilon(\min[a_1^*(t), 0])^2$$
$$- \epsilon(\min[a_2^*(t), 0])^2$$
$$- \epsilon(\min[c(t), 0])^2$$
$$- \epsilon(\max[c(t), 1/2(c_1 + c_2)])^2$$
$$- \epsilon(\min[\gamma(t), 0])^2$$

$$- \epsilon(\max[\gamma(t), 1])^2$$

$$- \epsilon(a_1^*(N)+a_2^*(N) - a_N)^2]$$

and the restrictions are:

$$a_1^*(t+1) = \{1 + \sum_{j=1}^{m} P_t^1(r_j)r_j\}a_1^*(t)$$

$$+ \{1 - \gamma(t)\}\{w(t) - c(t)\}$$

$$t=1,\dots,N-1$$

$$a_2^*(t+1) = \{1 + \sum_{j=1}^{m} P_t^2(r_j)r_j\}a_2^*(t)$$

$$+ \gamma(t)\{w(t) - c(t)\}$$

$$t=1,\dots,N-1$$

$$a_1(1) = a_2(1) = 0$$

The transformed problem is solved numerically by the gradient method. The optimal parameters are obtained by using the Hamiltonian equation:

$$H(a_1, a_2, c, \gamma, \lambda_1, \lambda_2, t)$$

$$= - \sum_{t=1}^{N-1} [\exp(-\delta t)(c(t)-c_1)(c(t)-c_2)$$

$$- \epsilon(\min[a_1^*(t), 0])^2$$

$$- \epsilon(\min[a_2^*(t), 0])^2$$

$$- \epsilon(\min[c(t), 0])^2$$

$$- \epsilon(\max[c(t), 1/2(c_1 + c_2)])^2$$

$$- \epsilon(\min[\gamma(t), 0])^2$$

$$- \epsilon(\max[\gamma(t), 1])^2$$

$$+ \lambda_1[\{1 + \sum_{j=1}^{m} P_t^1(r_j)r_j\}a_1^*(t)$$

$$+ \{1 - \gamma(t)\}\{w(t) - c(t)\}]$$

$$+ \lambda_2[\{1 + \sum_{j=1}^{m} P_t^2(r_j)r_j\}a_2^*(t)$$

$$+ \gamma(t)\{w(t) - c(t)\}]$$

The gradient of J(t) is:

$$\nabla_c J(t) = \partial H/\partial c$$

$$= -\exp(-\delta t)\{2c(t) - (c_1+c_2)\}$$

$$- \epsilon \min[2c(t), 0] - \epsilon\max[2c(t)$$

$$-(c_1+c_2), 0] - \lambda_1(t+1)\{1 + \gamma(t)\}$$

$$- \lambda_2(t+1)\gamma(t)$$

$$t=1,\dots,N-1$$

$$\nabla_\gamma J(t) = \partial H/\partial \gamma$$

$$= - \epsilon\min [2\gamma(t), 0]$$

$$- \epsilon \max[2\gamma(t)-2, 0]$$

$$- \{\lambda_1(t+1) - \lambda_2(t+1)\}\{w(t) - c(t)\}$$

And the adjoint equations are:

$$\lambda_1(t) = \lambda_1(t+1)\{1 + \sum_{j=1}^{m} P_t^1(r_j)r_j\}$$

$$- \epsilon\min[2a_1^*(t), 0]$$

$$t=1,\dots,N-1$$

$$\lambda_2(t) = \lambda_2(t+1)\{1+ \sum_{j=1}^{m} P_t^2(r_j)r_j\}$$

$$- \epsilon\min[2a_2^*(t), 0]$$

$$t=1,\dots,N-1$$

The transitivity condition is

$$\lambda_1(N) = \lambda_2(N)$$

$$= -2\epsilon(a_1^*(N)+a_2^*(N)-a_N)$$

IV. The results of the simulation

In simulation, constants are set in the following manner. (1) The total working period is divided into 20 sub-periods, namely N=20. (2) Annual income during this period is 6 million yen per period (w=6). (3) Initial assets, a_1, are zero at time 1 and, by retirement, the household accumulates assets of 10 million yen (a_N=10). (4) Parameters of the utility function are c_1=0 and c_2=60. This assumes that annual consumption does not exceed 30 million yen. (5) The time discount rate is 0.05. (6) For the interest rate, we introduce the probability concept:

Model 1

	Case 1	Case 2	Case 3
r_1=0.01 ;	p_1=0.20	p_1=0.60	p_1=0.20
r_2=0.03 ;	p_2=0.60	p_2=0.20	p_2=0.20
r_3=0.05 ;	p_3=0.20	p_3=0.20	p_3=0.60
expected returns:	0.03	0.022	0.038

Model 2

		Case 4	Case 5
1st asset			
r_1=0.01		p_1=0.60	p_1=0.20
r_2=0.03		p_2=0.20	p_2=0.20
r_3=0.05		p_3=0.20	p_3=0.60
expected returns:		0.022	0.038
2nd asset			
r_1= -0.05		p_1=0.20	p_1=0.30
r_2=0.03		p_2=0.50	p_2=0.50
r_3=0.11		p_3=0.30	p_3=0.20
expected returns:		0.038	0.022

Model 3

		Case 6	Case 7
1st asset			
r_1=0.01		p_1=0.20	p_1=0.20
r_2=0.03		p_2=0.60	p_1=0.20
r_3=0.05		p_3=0.20	p_3=0.60
expected returns:		0.03	0.038
2nd asset			
r_1=-0.05		p_1=0.30	p_1=0.20
r_2=0.03		p_2=0.50	p_2=0.60
r_3=0.11		p_3=0.20	p_3=0.20
expected returns:		0.022	0.03

In model 1, according to the difference of expected returns for cases 1,2 and 3, there are different streams for consumption and asset accumulation paths. In the optimal consumption profile, a sharp reduction of consumption is observed in case 2. The decrease in consumption begins after the 14th period and the speed of decline is the most rapid among the three. In case 2, the expected return is 2.2%, the lowest among the three. Tracing the asset accumulation path in case 2, the accumulation begins and consumption declines during the 15th period which is also the latest among the three. In case 3, the reduction of consumption begins in the 12th period and the rate of decline is less than that in case 2. Due to the earlier reduction in consumption, asset accumulation begins in the 13th period.

The simulation based on model 1 suggests that the expected rate of return plays an important role in determining the period beginning of asset accumulation and reduction of consumption.

In model 2, the expected rate of return for the designated 1st and 2nd assets are different from each other, but the values are the same. Because the related probabilities for r_1, r_2 and r_3 are different, the consumption and asset accumulation paths are asymmetric. In case 4, both assets 1 and 2 are utilized in order to satisfy the desired amount of consumption and savings, while in case 5 the first asset only is used for accumulating wealth for retirement. This causes a different optimal consumption path between cases 4 and 5.

In model 3, examining the asset accumulation process, only the 1st asset is used due to the higher expected returns.

V. Conclusion

We made a simulation model for household portfolio selection based on optimal control theory in the life cycle model with stochastic interest rates for assets. Our finding is: there are many types of streams for consumption and asset accumulation paths reflecting different combinations of interest rates with related probabilities.

References

Maki and Aiyoshi (1997), "Household portfolio behavior," *Mathematics and Computers in Simulation*, forthcoming.

Acknowledgement

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Figure 1. Optimal Consumption Stream (cases 1, 2 and 3)

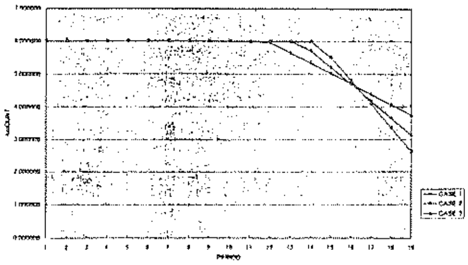

Figure 2. Optimal Asset Accumulation Path(cases 1, 2, and 3)

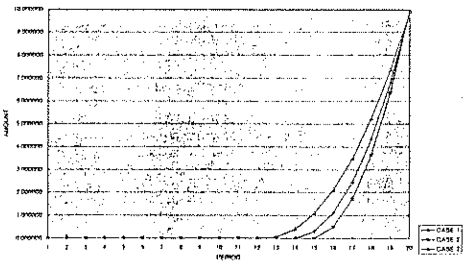

Figure 3. Optimal Consumption Path(case 4)

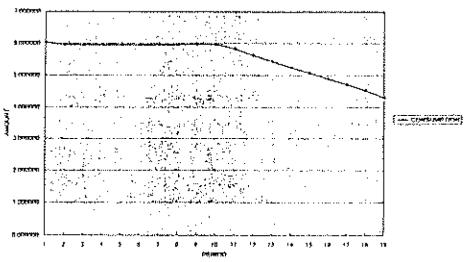

Figure 4. Optimal Accumulation Path for 1st and 2nd assets (case 4)

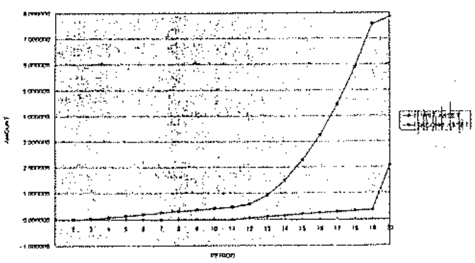

Figure 5. Optimal Consumption Path (case 5)

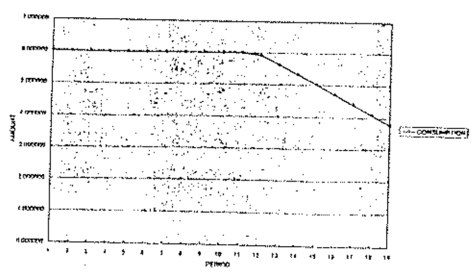

Figure 6. Optimal Asset Accumulation Path for the 1sst and 2nd assets (case 5)

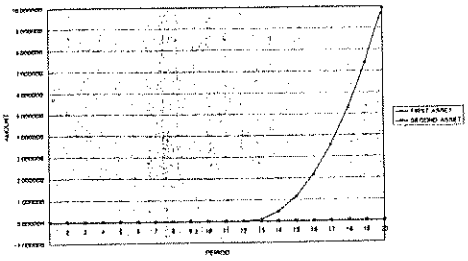

Figure 7. Optimal Consumption Path (case 6)

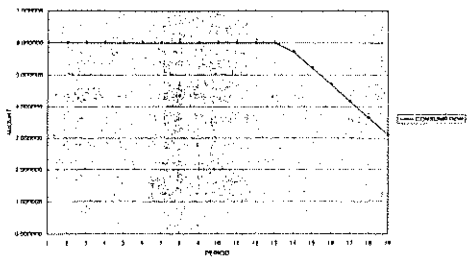

Figure 8. Optimal Asset Accumulation Path for the 1st and 2nd assets (case 6)

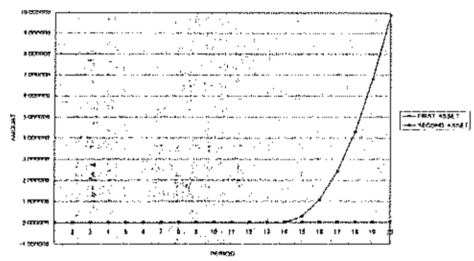

Figure 9. Optimal Consumption Path(case 7)

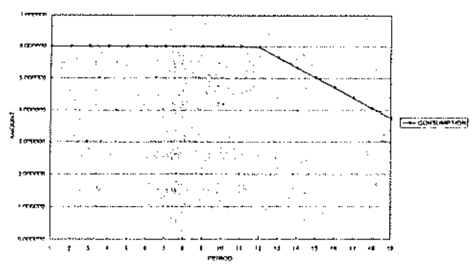

Figure 10. Optimal Asset Accumulation Path for the 1st and 2nd assets (case 7)