DYNAMIC CORRECTIVE TAXES WITH POLLUTION: A STOCHASTIC MODEL

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Abstract: The production of certain goods is associated with emission of pollution, and this causes both flow and stock externalities. Flow externalities are defined as the difference between social and private marginal costs. In addition pollution tends to aggregate, and the disutility associated with this is defined as the stock externality. We derive optimal corrective taxes in the presence of both externalities as explicit feedback control laws even when the decay of aggregated pollution is subject the a general stochastic process. Hence this represents a truly adaptive approach to regulation of the environment. The model applied is completely general in the state variable, pollution, and quadratic in the control variable. There are no assumptions about separability in the objective function which is to maximize social welfare defined as the sum of consumers' and producers' surplus adjusted for the externalities.

1 INTRODUCTION

The objective is to maximize social welfare defined as the sum of consumers' and producers' surplus corrected for externalities when the decay of the pollution causing externalities is subject to stochasticity. Flow externality is defined as the difference between social and private marginal costs whereas stock externality is defined as any disutility associated with the aggregated level of pollution. The corrective tax, which is the control variable in this problem, is defined as the difference between consumer and producer price of the product causing emissions. There is a fixed amount of emission associated with each unit produced.

2 THE STOCHASTIC MODEL

Let x denote production and a denote the aggregated level of pollution. To each unit produced there is a fixed amount of emission, δx, and the decay of pollution is a general function, $f(a)$. The time change in the aggregated level of pollution is then

$$da = [\delta x - f(a)]\, dt + \Omega(a) dw$$

where Ω is a general volatility function and the term dw is a standard Wiener process (independent and identically distributed) with variance dt and zero mean. Let the demand for x be given by

$$D(x) = p_0 - p_1 x$$

and private and social marginal costs are

$$\begin{aligned} MC^p &= c_{p0} + c_{p1} x, \\ MC^s &= c_{s0} + c_{s1} x. \end{aligned}$$

The parameters in the demand and cost functions can in principle be general functions in a. Private marginal costs represent the private supply curve which also is the producer price and the demand curve is the consumer price. Our control variable is a corrective unit tax denoted τ. As the corrective tax equals the difference between the consumer and producer price, production, x, in market equilibrium can be written as a linear function in τ:

$$x = x_0 - x_1 \tau$$

where

$$x_0 = \frac{p_0 - c_{p0}}{p_1 + c_{p1}}, \quad x_1 = \frac{1}{p_1 + c_{p1}}.$$

Note that $\tau = \tau_{\max} = x_0/x_1 \Rightarrow x = 0$. By rescaling a from physical to monetary units, A, through a monotone transformation $A = A(a)$ the dynamic equation can be rewritten

$$dA = [F(A) - \tau]\,dt + \sigma(A)dw \qquad (1)$$

where the derivative is chosen as $A' = 1/\delta x_1$, and

$$\begin{aligned} F(A) &= \left[\frac{x_0}{x_1} - \frac{f}{\delta x_1}\right] - \frac{1}{2}\frac{\Omega^2 x_1'}{\delta x_1^2} \equiv F_0(A) + F_1(A), \\ \sigma(A) &= \frac{\Omega(a)}{\delta x_1}. \end{aligned}$$

Note that F_1 can be written $F_1 = -\frac{1}{2}\sigma^2 \delta x_1'$.

We want to maximize social welfare defined as the sum of consumers' and producers' surplus corrected for externalities, which is the area between the demand curve and the social marginal cost curve up to production x. This implies that the social welfare function can be written as a quadratic function in τ :

$$W(\tau, A) = -\alpha(A) + \beta(A)\tau - \gamma(A)\tau^2$$

where α, β and γ are composite functions containing the parameters in the demand and cost functions.The maximization problem can then be written

$$\max_{\tau} E \int_0^T e^{-rt} W(\tau(t), A(t))dt \qquad (2)$$

subject to the dynamic constraint given by Eq.(1) and the appropriate transversality conditions. The operator E denotes expectation. The associated Bellman equation for the value function

$$V(t_0, A) = \max_{\tau} E \int_{t_0}^T e^{-r(s-t_0)} W(\tau(s), A(s))ds$$

is given by

$$\begin{aligned} & V_t + \max_{\tau} \left\{ e^{-r(t-t_0)} \left[-\alpha(A) + \beta(A)\tau - \gamma(A)\tau^2\right] + \right. \\ & \left. V_A \left[F(A) - \tau\right]\right\} + \frac{1}{2}\sigma^2(A) V_{AA} = 0. \qquad (3) \end{aligned}$$

Defining $W(x) = e^{r(t-t_0)}V$, the optimal control derived from (3) is given by $\tau = \frac{\beta - W'}{2\gamma}$. When inserted, this yields

$$\frac{1}{4\gamma}\left(W' - \mu\right)^2 + \frac{1}{2}\sigma^2 W'' + S = rW, \qquad (4)$$

where

$$\begin{aligned} S(A) &\equiv W(F(A), A) \\ &= -\alpha(A) + \beta(A)F(A) - \gamma(A)F(A)^2 \\ &= S_0 + \mu_0 F_1 - \gamma F_1^2 \equiv S_0 + S_1 + S_2, \\ \mu(x) &\equiv \frac{dW}{d\tau}\Big|_{\tau = F(A)} = \beta(A) - 2\gamma(A)F(A) \\ &= [\beta(A) - 2\gamma(A)F_0(A)] - 2\gamma(A)F_1(A) \\ &\equiv \mu_0 + \mu_1. \end{aligned}$$

We make a constant shift in the current value function by defining

$$\Lambda(A) \equiv W(A) - \frac{P}{r}$$

where P is a constant to be determined. The ordinary differential equation corresponding to (4), which will be used in the following, is then given by

$$\frac{1}{2}\sigma^2 \Lambda'' + \frac{1}{4\gamma}\left[\Lambda' - \mu\right]^2 - r\Lambda = P - S. \qquad (5)$$

2.1 The stochastic model with zero discounting

The differential equation corresponding to (5) for the optimization problem given by (2) and (1) is then

$$2\gamma\sigma^2 \Lambda'' + \left(\Lambda' - \mu\right)^2 = 4\gamma\left(P - S\right).$$

Defining a phase-space like coordinate, $\eta = \Lambda' - \mu = 2\gamma(F - \tau)$, the above equation can be rewritten

$$\eta^2 = 4\gamma\left(P - S\right) - 2\gamma\sigma^2\left(\mu' + \eta'\right). \qquad (6)$$

It is then assumed that the known, deterministic part dominates the stochastic part in magnitude in Eq. (6).

Let A be rescaled into z by a monotone transformation given by

$$2\gamma(A)\sigma^2(A)\frac{dz}{dA} = \varepsilon \Rightarrow 2\gamma\sigma^2\frac{d}{dA} \to \varepsilon\frac{d}{dz}$$

where ε is constant, $|\varepsilon| \ll 1$. Eq. (6) can again be rewritten

$$\eta^2(z) = Q(z) - \varepsilon\left[\mu'(z) + \eta'(z)\right] \qquad (7)$$

where $Q(z) \equiv 4\gamma(z)\left[P - S(z)\right]$. The ε-terms now represent the correction terms to the underlying deterministic structure. The deterministic analogue of this model is given when $z = A$ and $\varepsilon = 0$, implying $\eta = \mp\sqrt{Q}$, giving

$$\tau(A) = F_0(A) \pm \sqrt{\frac{P - S_0(A)}{\gamma(A)}} \qquad (8)$$

where P is a constant. With an infinite time horizon $P = \max S$, and this represents the separatrix solution. The solution with the plus sign is chosen in Eq. (8) when A is above the optimal steady state and vice versa.

In order to derive the optimal corrective tax as a feedback control law given by an analytical expression for the stochastic case, one needs to resort to perturbation methods, see e.g. Nayfeh (1973). The following perturbation scheme will be used here:

$$\begin{aligned} \eta(z) &= \sum_{k=0} \varepsilon^k \eta_k(z), \quad P = \sum_{k=0} \varepsilon^k P_k, \\ S(z) &= S_0 + \varepsilon\widehat{S}_1(z) + \varepsilon^2\widehat{S}_2(z), \\ \mu(z) &= \mu_0 + \varepsilon\widehat{\mu}_1(z), \quad F(z) = F_0 + \varepsilon\widehat{F}_1(z). \end{aligned}$$

This inserted into (6) yields simple algebraic expressions for the corrections to the control variable, τ, order by order without doing integration. Note that

$$(\eta^2)_m = 2\eta_0(z)\cdot\eta_m + \sum_{1\le k\le m-1} \eta_k\eta_{m-k}$$

where subscripts denote the order of the perturbation term. This yields the following results

$$\begin{aligned}
\eta_0^2 &= Q_0 = 4\gamma(P_0 - S_0), && (9)\\
2\eta_0\eta_1 &= Q_1 - \eta_0' - \mu_0' \\
&= 4\gamma(P_1 - S_1) - (\mu_0' + \eta_0'), \\
2\eta_0\eta_2 - \eta_1^2 &= Q_2 - (\mu_1' + \eta_1') \\
&= 4\gamma(P_2 - S_2) - (\mu_1' + \eta_1'), \\
2\eta_0\eta_m &= 4\gamma P_m - \eta_{m-1}' - \sum_{1\le k\le m-1} \eta_k\cdot\eta_{m-k}, \\
m &> 2,
\end{aligned}$$

A prerequisite for this perturbation scheme to be meaningful is that $P_0 - S_0 \ge 0$. If we let P_0 be the appropriate value in the deterministic case, then $\eta_0 = 0$ for the level of the pollution that defines the optimal equilibrium in the deterministic case. The constant terms P_1, P_2, P_3 and so on can then be found by setting the right hand sides of the equations in the perturbation scheme (9) equal to zero when $x = x_{0s} = \arg\max(S)$ and solving. This is the simplest modification of the deterministic case, as it leads to the same predefined optimum. The optimal control law can be written

$$\tau(A) = \tau_0(A) + F_1 + \frac{\left[S_1 + \frac{1}{2}\sigma^2(A)\, m_0'(A)\right]_{A_0}^{A}}{\gamma(A)\left[F_0(A) - \tau_0(A)\right]} + O(\varepsilon^2), \tag{10}$$

where τ_0 is the deterministic solution given by Eq. (8) and $m_0(A) = \frac{\partial W}{\partial \tau}(A;\tau_0)$. The function W is the social benefit function and $A_0 = \arg\max S$.

2.2 THE STOCHASTIC MODEL WITH DISCOUNTING

The purpose of this section is to find the feedback rule for the optimal corrective tax when discounting of the future is included through a constant discount rate.

By taking the derivative of (4) we get

$$\left[\frac{1}{4\gamma}(W' - \mu)^2 + \frac{1}{2}\sigma^2 W''\right]' = rW' - S'. \tag{11}$$

The problem is facilitated if we use the variable $\xi(A) = F(A) - \tau(A)$ instead of τ. Note that $\xi\, dt$ represents the expected change in the aggregated pollution level during dt. A plot of ξ against A is the well-known phase-plane in the determinstic analogue of the model ($\varepsilon \to 0$). We then have from the maximum principle ($\tau = \frac{\beta - W'}{2\gamma}$).

$$\xi(A) = \frac{W'(A) - \mu(A)}{2\gamma(A)}$$

and (11) can be rewritten

$$\left[\gamma\xi^2 + \sigma^2(\gamma\xi)'\right]' = 2r\gamma\xi + r\mu - \frac{1}{2}(\sigma^2\mu')' - S'.$$

An optimal ordering of the terms implies that as many corrections as possible appear as first-order corrections. Thus we avoid resorting to higher orders in order to encompass the corrections. The optimal ordering is given by

$$\begin{aligned}
\sigma^2 &\sim r \sim \varepsilon,\ \gamma \sim S_0 \sim \mu_0 \sim \varepsilon^0, \\
\frac{d}{dA} &\sim \varepsilon^0,\ \xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \\
\mu &= \mu_0 + \mu_1,\ S = S_0 + S_1 + S_2, \\
\mu_m &\sim S_m \sim \xi_m \sim \varepsilon^m.
\end{aligned}$$

With the ordering $r \sim \varepsilon^m$, $m > 1$, the solution resulting from the ordering above will still be valid. The optimal ordering implies

$$\begin{aligned}
(\gamma\xi_0^2)' &= -S_0', \\
&\left[2\gamma\xi_0\xi_1 + \sigma^2(\gamma\xi_0)'\right]' \\
&= 2r\gamma\xi_0 + r\mu_0 - S_1' - \frac{1}{2}(\sigma^2\mu_0')', \\
&\left[2\gamma\xi_0\xi_2 + \gamma\xi_1^2 + \sigma^2(\gamma\xi_1)'\right]' \\
&= 2r\gamma\xi_1 + r\mu_1 - S_2' - \frac{1}{2}(\sigma^2\mu_1')' \\
&\left[2\gamma\xi_0\xi_{M+1} + \gamma\sum_{k=1}^{M}\xi_k\xi_{M+1-k} + \sigma^2(\gamma\xi_M)'\right]' \\
&= 2r\gamma\xi_M,\quad M = 2,3,...
\end{aligned}$$

Direct integration yields

$$\begin{aligned}
\gamma\xi_0^2 &= P_0 - S_0, && (12)\\
&2\gamma\xi_0\xi_1 + \sigma^2(\gamma\xi_0)' \\
&= P_1 - S_1 + \int_{A_0}^{A}\left[2r\gamma\xi_0 + r\mu_0 - \frac{1}{2}(\sigma^2\mu_0')'\right] dA, \\
&2\gamma\xi_0\xi_2 + \gamma\xi_1^2 + \sigma^2(\gamma\xi_1)' \\
&= P_2 - S_2 + \int_{A_0}^{A}\left[2r\gamma\xi_1 + r\mu_1 - \frac{1}{2}(\sigma^2\mu_1')'\right] dA, \\
&2\gamma\xi_0\xi_{M+1} \\
&= P_{M+1} - \sigma^2(\gamma\xi_M)' - \gamma\sum_{k=1}^{M}\xi_k\xi_{M+1-k} \\
&+\quad 2r\int_{A_0}^{A}\gamma\xi_M\, dA,\quad M = 2,3,...
\end{aligned}$$

where P_0, P_1, $P_2, \ldots$ are constants of integration. The constants of integration have to be determined on the basis of certain additional requirements, e.g. barrier and regularity requirements and choice of the lower limit of integration ($A = A_0$).

Let us consider the case of an exogeneous upper limit on the level of pollution above which the corrective tax is so high that all emission is banned:

$$\begin{aligned} A &\geq \bar{A} : \tau \equiv \tau_{\max} \Rightarrow \xi \equiv F - \tau_{\max} \\ &= F_0 + F_1 - \frac{x_0}{x_1} = F_1 - \frac{f}{\delta x_1}, \\ A &= \bar{A} : \xi_0(\bar{A}) = -\frac{f}{\delta x_1}\bigg|_{A=\bar{A}}, \\ \xi_1(\bar{A}) &= F_1(\bar{A}), \ \xi_M(\bar{A}) = 0, \ M \geq 2. \end{aligned}$$

The tax, τ, is smooth at the transition $A = \bar{A}$. Thus we get the following when $\bar{A}$ is used as the lower limit of integration in (12).

$$\begin{aligned} P_0 &= \bar{S}_0 + \bar{\gamma}\bar{\xi}_0^2 = \bar{S}_0 + \left[\frac{\bar{\gamma}\bar{f}^2}{(\delta\,\bar{x}_1)^2}\right], \\ P_1 &= \bar{S}_1 + 2\bar{\gamma}\bar{\xi}_0\bar{\xi}_1 + \bar{\sigma}^2\left(\bar{\gamma}\,\bar{\xi}_0\right)', \end{aligned}$$

where the bar indicates that the value $\bar{A}$ is inserted. The terms ξ_0, ξ_1, $\xi_2, \ldots$ are now completely determined and hence the optimal tax $\tau_0 = F_0 - \xi_0$, $\tau_1 = F_1 - \xi_1$, $\tau_M = -\xi_M$ for $M \geq 2$. The solution given by (12) is uniformly valid whenever $P_0 > S(A)$. In the opposite case there are intervals in which the solution of (12) becomes complex. In that case it is necessary to determine a singular perturbation scheme that handles these intervals, but this leads to technicalities that we avoid here. For a discussion of this case see Sandal and Steinshamn (1997).

The explicit feedback control law for the optimal corrective tax to second order can now be written

$$\tau(A) = \tau_0(A) + \tau_1(A) + O(\varepsilon^2) \tag{13}$$

for $A < \bar{A}$ and $\tau = \tau_{\max}$ for $A \geq \bar{A}$ where

$$\begin{aligned} \tau_0(A) &= F_0(A) \pm \sqrt{\frac{P_0 - S_0(A)}{\gamma(A)}}, \\ \tau_1(A) &= F_1(A) + \frac{\bar{\gamma}\bar{\xi}_0\bar{F}_1}{\gamma(A)\xi_0(A)} \\ &\quad + \frac{1}{2\gamma(A)\xi_0(A)}\left[S_1(A) + \frac{1}{2}\sigma^2(A)m_0'\right]_{\bar{A}}^{A} \\ &\quad + \frac{r}{2\gamma(A)\xi_0(A)}\int_{\bar{A}}^{A} m_0(z)\,dz, \\ m_0(A) &= \mu_0(A) + 2\gamma(A)\xi_0(A) = \beta(A) - 2\gamma(A)\tau_0(A). \end{aligned}$$

The feedback rules given by (10) and (13) are now operational functions that can be used by the authorities to determine the optimal corrective tax given that they have the relevant economic and technical information, that is; estimates of the demand and cost functions and the decay of pollution.

The expression in (13) can also be applied to investigate the qualitative effects of stochasticity upon the optimal corrective tax. In particular, when the slopes of the demand and cost curves are independent of the aggregated level of pollution such that $x_1' = 0$, τ_1 can be rewritten

$$\begin{aligned} \tau_1(A) &= \frac{1}{4\gamma(A)\xi_0(A)}\left[\sigma^2(A)m_0'\right]_{\bar{A}}^{A} \\ &\quad + \frac{r}{2\gamma(A)\xi_0(A)}\int_{\bar{A}}^{A} m_0(z)\,dz, \end{aligned}$$

It is seen that the volatility term appears only in the first term in square brackets, and therefore this is the term of interest here. Furthermore, by defining $\Phi = -\sigma^2 m_0'$ this term can be interpreted as the *cost of uncertainty*. The variance, σ^2, is measured as unit of mass squared (e.g. kg.^{2}) and m_0' is measured as value per unit of mass squared (e.g. \$/kg.^{2}). Therefore, Φ is a measure of value (or cost), e.g. \$, where $-m_0'$ can be interpreted as the cost per unit of variance. If $\Phi(A) < \Phi(\bar{A})$ for some $A < \bar{A}$, that is, the cost of uncertainty is lower for a lower pollution level, then it is optimal to increase the corrective tax (reduce emissions). This will probably be the usual case. It is, however, also possible to think of cases where $\Phi(A) > \Phi(\bar{A})$ due to the functional form of σ, and in this case the presence of stochasticity will call for increased emissions along an optimal path which is an interesting and quite counterintuitive case.

References

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