Estimating the Hurst Parameter in Fractional ARIMA(p, d, q) Models via the Quasi-likelihood Method

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Abstract

This paper is concerned with R/S analysis given a fractional ARIMA(p,d,q) model with finite variance where the aim is to estimate the intensity of long-range dependence of the particular series. This is done through what is commonly referred to as the Hurst parameter (denoted by H). H is a measure of self-similarity of a given time series. The goal of this paper is to examine the effectiveness of applying the method of asymptotic quasi-likelihood to R/S analysis instead of the conventional method of least squares.

1. Introduction

The Hurst parameter (H) is a measure of the intensity of self-similarity of a particular time series. Fractional autoregressive integrated moving average or fractional ARIMA(p,d,q) processes (with $0 < d < 0.5$, since the process is not stationary if $d \geq 0.5$) are examples of asymptotically second-order self-similar processes with self-similarity parameter $H = d + 0.5$ (providing the process under consideration has finite variance) or $H = d + 1/\alpha$ (if the process being analysed possesses infinite variance). Taqqu and Teverovsky (1996) examined fractional ARIMA(p,d,q) models with both finite and infinite variance structures and found that the resulting estimates are not unduly influenced when either of the variance structures are considered. Our attention, in this paper, is limited to finite variance structures.

The alternative to modelling long-range dependence via self-similar processes is via time series methods which would involve more parameters as the sample size increases thus making the analysis and interpretation of the results even more complicated.

In Section 2 we outline the rescaled adjusted range (R/S) procedure and discuss Hurst's empirical power law relation. We also define the range and the standard deviation in Hurst's rescaled adjusted range statistic, discuss the effectiveness of R/S analysis for the fractional ARIMA(p,d,q) model and examine the effect of a well-controlled short-range dependence structure on R/S analysis (i.e. the order of p and/or q is not equal to 0). Simulations are performed to show what effect such a short-range dependence structure has on the accuracy of the resulting estimates of H. The asymptotic quasi-likelihood procedure is outlined in Section 3 along with an example of its application. We compare the estimate of H via R/S analysis using the asymptotic quasi-likelihood method to that obtained where the method of least squares is employed and compare these results to those obtained by Taqqu and Teverovsky. Finally, numerous simulations are performed in Section 4 and the resulting estimates analysed via R/S analysis using both the traditional method (least squares) and the method of asymptotic quasi-likelihood. All simulations involve 8192 data values using the Durbin-Levinson algorithm in S-plus.

2. The R/S Method

The rescaled range (R/S) method was first introduced by Hurst (1965). In this method the range $R(t,m)$ is defined as

$$R(t,m) = \max_{1 \leq t \leq m} Y(t,m) - \min_{1 \leq t \leq m} Y(t,m),$$

where t is the discrete integer-valued time and m the time-span considered. The standard deviation, denoted by $S(t,m)$, is defined as

$$S(t,m) = (\frac{1}{m}\sum_{t=1}^{m}(Y(t,m) - \bar{Y}(m))^2)^{.5}.$$

The use of this dimensionless R/S ratio allows observed ranges of various phenomena to be compared over long time periods. Hurst found the following power law empirical relation between the quotient of the range $R(t,m)$ and the standard deviation, $S(t,m)$;

$$E[R(m)/S(m)] = cm^H, \quad \text{as } m \to \infty.$$

where H denotes the Hurst parameter $(0 < H < 1)$, and c is a finite positive constant that does not depend on the time span m.

Taking logarithms (base 10) of both sides of the rescaled range gives

$$\log[R(m)/S(m)] = C + H\log(m) + \epsilon(m). \qquad (1)$$

The logarithm of the rescaled range, is then plotted as a function of the time-scale index, $\log(m)$. This is known as the rescaled adjusted range plot (also called the pox diagram) of R/S. Providing the parameter H in relation (1) is well defined a typical rescaled adjusted range plot commences with a transient region representing the nature of short-range dependence in the sample (in this transient region the quantity R/S grows faster than $m^{.5}$ for small or moderate m), but eventually settles down as m increases and fluctuates in a straight "street" of a certain asymptotic slope. The estimated slope, $\hat{H}$, is typically obtained by using the least squares linear regression of $\log[R(m)/S(m)]$ on $\log(m)$.

Mandelbrot and Wallis (1969) have shown that the division of R by S leads to robustness against extreme deviations from normality, including the infinite variance syndrome. It is particularly robust with respect to heavy-tailed distributions. The biggest drawback however, is the loss of efficiency under Gaussian models than is the case with maximum likelihood estimators, and thus this method does not necessarily minimise the bias.

The modified R/S statistic, introduced by Lo (1991), corrects Hurst's classical R/S, allowing for the effects of possible short-term dependence. The resulting statistic is found to be invariant over the general class of short memory processes but deviates for long-memory processes.

We now focus our attention on the power of R/S analysis on the fractional ARIMA(p,d,q) model. Under the scheme of least squares the R/S method can be affected by a variety of factors, namely;

1. the range of d,
2. the order of the autoregressive and/or moving average components, and
3. the fact that the $\{\epsilon(m)\}$ in (1) may possess non-constant variance and are thus not independent.

Estimates of the Hurst exponent H via R/S analysis are found to be biased towards 0.72. More specifically, in using the empirical Hurst law the estimate of H when the true value of H is less than 0.72 tends to be overestimated and the estimate of H when the true value of H is greater than 0.72 tends to be underestimated.

The second point has been addressed by Taqqu and Teverovsky. They applied the R/S method to data simulated from a fractional ARIMA(p,d,q) model and found that this estimator does not work as well when the order of either p or q is not zero. However, if the model under consideration exhibits a well-controlled short-range dependence structure (e.g. Fractional Gaussian ARIMA$(0,d,0)$) R/S analysis always leads to a very accurate estimate of the parameter d (and H). Using R/S analysis in situations where short-range dependence is also present leads to biases in the final estimate of the parameter d (and H). Another interesting result that Taqqu and Teverovsky found in analysing a process with a short-term dependence structure is that, if the parameter(s) chosen for p and/or q are negative, there will be significantly less induced bias in the estimate than would be the case if the parameter(s) are positive. R/S analysis is biased in this case even though the estimator is still efficient.

The third point is an important reason for introducing the asymptotic quasi-likelihood method to the R/S procedure. The usual method of applying least squares linear regression to the data transformed via R/S analysis would not provide an accurate estimate of H when the residuals possess non-constant variance. The asymptotic quasi-likelihood estimate, in such circumstances, would appear to be effective.

Note that fractional ARIMA(p,d,q) processes (with $0 < d < 0.5$) exhibit long-range dependence where the parameter d determines the level of long-range dependence whilst short-range dependence is modelled through the parameters p and q. The effectiveness of several estimators used by Taqqu and Teverovsky to estimate d decreases when there is an additional short-term dependence structure (i.e. when either the order of p and/or q is not equal to zero). The results for the R/S analysis are dependent on the number of subintervals and the minimum and maximum lags chosen.

We now simulate data from several fractional autoregressive moving average processes. Due to the amount of time it takes for simulations to be performed forty simulations will be carried out for each different model discussed. Furthermore we use Bodruzzaman et al's (1991) method of applying Hurst's Rescaled-Range (R/S) method. The window is defined as the segment of the particular time series, the beginning of which is not allowed to move but the size of the window is doubled every time the R/S ratio is calculated. This is in contrast to the usual application of the R/S method where, for each window size m, there are N/m different R/S values, the mean of which is the statistic analysed. The estimated slope is the estimate of the Hurst parameter. By applying this method eleven values of R/S ($R(m)/S(m)$ where $m = 2^{n+1}$ and $n = 1,2,...11$) are obtained from the original 8192 observations. We commence with 4 observations (i.e. $n = 1,2,...,11$ but also compare the results via R/S analysis when the initial window size is 8 ($n = 2, 3, ..., 11$) and 16 ($n = 3, 4, ...11$) respectively. This is done because, as mentioned before, the usual rescaled adjusted range plot commences with a transient region in which the quantity R/S grows faster than $m^{.5}$ for small m. Therefore small values of m should be discarded when calculating the slope so as not to unduly influence the resulting estimate of H.

Upon obtaining the eleven R/S values the method of least squares is applied to this transformed data (i.e. the slope of the line in the graph of $\log(R/S)$ versus $\log(m)$ is estimated where m is the window size). If the model under consideration has a well controlled short-range dependence structure (e.g. Fractional Gaussian ARIMA$(0, d, 0)$) the R/S method will always provide a very accurate estimate of the parameter d. In this paper the emphasis is on models with additional short-range dependence components.

Example 1: We now wish to demonstrate the effect of different initial window lengths. In this example data is simulated from the fractional ARIMA(1,.22,0) process

$$(1 - 0.8B)(1 - B)^{.22}Y_t = \epsilon_t.$$

where ϵ_t is white noise. Note that the true value of H equals 0.72. Forty data sets are simulated from the above model. Taking the initial window size to be 4, 8 and 16 respectively the average of the forty estimates of H are given in Table 1. We see that the estimate becomes more accurate when at least the first observation ($n = 1$, when the data is transformed via R/S analysis) is discarded before applying the method of least squares. Generally the initial window length, m, (or lag) is taken to be about 10.

Method	mean($\hat{H}$)	stand.error($\hat{H}$)
LS(4)	0.803	0.005
LS(8)	0.778	0.006
LS(16)	0.750	0.007

Table 1: Least squares for three different initial window sizes.

Example 2: We now simulate forty sets of data from the fractional ARIMA(2,.3,0) process

$$(1 - 0.2B - 0.6B^2)(1 - B)^{.3}Y_t = \epsilon_t.$$

and the output is given in Table 2. The estimates are improved when the initial window length is increased although the final estimate is still not as accurate as hoped.

As can be seen increasing the initial window length might lead to improved estimates of H. This leads to the question of how to determine the initial window length, the answer to which is not very clear. In the following section the asymptotic quasi-likelihood method will be introduced to estimate H. Via this method the accuracy of the estimate is improved and the determination of initial window length is avoided.

Method	mean($\hat{H}$)	stand.error($\hat{H}$)
LS(4)	0.872	0.006
LS(8)	0.863	0.007
LS(16)	0.848	0.008

Table 2: Least squares for three different initial window sizes.

3. The Asymptotic Quasi-likelihood Method

Assume that the observed process $\{X_n\}$ satisfies the model

$$X_n = f_n(\theta) + M_n(\theta). \qquad (2)$$

where $n = 1, 2, ...T$, f_n is a predictable process, θ is an unknown parameter from an open parameter space Θ, $\mathcal{F}_n$ denotes a standard filtration generated from X_s, $s \leq n$ and M_n is an error process such that

$$E(M_n|\mathcal{F}_{n-1}) = 0. \qquad (3)$$
$$E(M_n^2|\mathcal{F}_{n-1}) < \infty.$$

Equation (3) implies that M_i and M_j are uncorrelated, $i \neq j$. The case where M_i are mutually independent with mean 0 is a special case of (3).

According to the quasi-likelihood method (see Godambe and Heyde, 1987) a quasi-score estimating function can be determined based on model (2) and has the expression

$$G_N^*(\theta) = \sum_{n=1}^{N} \frac{\dot{f}_n(\theta)}{E(M_n^2|\mathcal{F}_{n-1})} M_n, \qquad (4)$$

where $\dot{f}_n$ represents the derivative of f_n with respect to the unknown parameter θ.

The quasi-likelihood estimate of θ is obtained by solving the quasi-score normal equation $G_N^*(\theta) = 0$. When $f_n(\theta)$ is a linear function of θ, the quasi-likelihood estimate always provides a good estimate of θ without knowing the distribution of M_n as long as $E(M_n^2|\mathcal{F}_{n-1})$ is known. But in practice it is very difficult to accurately determine $E(M_n^2|\mathcal{F}_{n-1})$ and thus, the expression for the quasi-score estimating function. Therefore, a possible approach of the asymptotic quasi-likelihood method was discussed by Lin (1995) and Mvoi et al (1997) and an inference procedure was given. The procedure is as follows; we accept the true model is (2). If, for given X_n, we can determine a predictable process g_n such that $E(X_n^2 - g_t|\mathcal{F}_{n-1})$ is small enough for all n, then

$$E(M_n^2|\mathcal{F}_{n-1}) \approx g_n - f_n^2(\theta),$$

and the asymptotic quasi-score estimating function

$$\tilde{G}_N^*(\theta) = \sum_{n=1}^{N} \frac{\dot{f}_n(\theta) M_n}{g_n - f_n^2(\theta)}$$

is obtained. The solution of the asymptotic quasi-score normal equation $\tilde{G}_N^*(\theta) = 0$, obtained via the twostage method, is called the asymptotic quasi-likelihood estimate. Mvoi et al (1997) have proved that, under certain conditions, the asymptotic quasi-likelihood estimate is a good estimate of the true parameter. In particular, when $f_n(\theta)$ is a linear function of θ the asymptotic quasi-likelihood estimate is consistent as sample size is increasing.

Example 3: Our analysis now turns to the simulating of data from the fractional ARIMA(2,.3,0) process as in Example 2. It is shown how to apply the asymptotic quasi-likelihood method to the data and obtain accurate estimates of H. 8192 data values were simulated from this model and by applying R/S analysis to the data we transform the 8192 data values to eleven data points. We consider model (2) where n=1,2,..,11 and $X_n = \log[R(m)/S(m)]$ (where $m = 2^{n+1}$ and $f_n(\theta) = C + H\log n$. Based on this procedure of asymptotic quasi-likelihood three possible g_n's are determined based on this sample of eleven data points X_n. The predictable processes are listed below:

$$\begin{aligned} g_1 &= -0.742 - 0.312X_{n-1}^2 + 0.850(\log n)^2, \\ g_2 &= -1.426 - 0.477X_{n-1}^2 - 0.427X_{n-2}^2 \\ &\quad +1.172(\log n)^2, \\ g_3 &= -1.714 - 0.536X_{n-1}^2 - 0.486X_{n-2}^2 \\ &\quad -0.094X_{n-3}^2 + 1.282(\log n)^2. \end{aligned}$$

According to the criterion discussed by Biondini and Lin (1997) for selecting g_n it can be seen from Figure 1 that g_{1n} is not as good at approaching X_n^2 as the other two predictable processes. It can be seen that there is very little difference between g_{2n} and g_{3n}. Turning our attention to the $\{X_n^2 - g_n\}$, they can be accepted as stationary for each of the three g_n's. From Table 3 it is seen that the most accurate asymptotic quasi-likelihood estimate occurs when the third predictable process is used followed by g_{2n}.

Method	LS(4)	LS(8)	LS(16)
H	0.928	0.938	0.903
Method	AQL(g_1)	AQL(g_2)	AQL(g_3)
H	0.975	0.870	0.835

Table 3: Least squares (for three different initial window sizes) and asymptotic quasi-likelihood estimates (for three possible predictable processes) for Example 3.

This example shows the fact that the possibility of improving the estimate of H via the asymptotic quasi-likelihood method by means of choosing a better g_n to approach the quantity y_n^2 without serious reservations concerning the initial window length.

4. Comparison of Methods using Simulations of Fractional ARIMA(p, d, q)

In this section we will further compare the accuracy of the estimate of H via the least squares method and the method of quasi-likelihood. We will also discuss what situations the AQL method will be much more

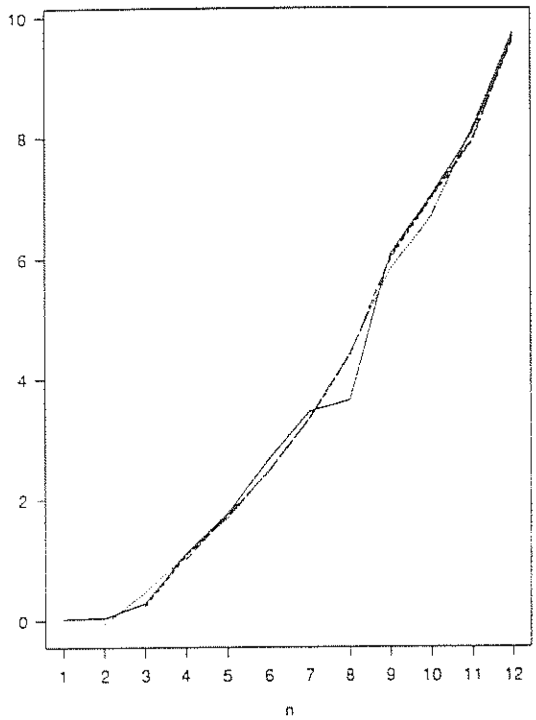

Figure 1: X_n^2 (hard line) and three possible g_n's (dotted lines) for Example 1. g_{1n} starts at the first lag, g_{2n} starts at the second lag and g_{3n} starts at the third lag.

accurate than the method of least squares.

Example 4: Forty simulations are performed of the following model:

$$(1 - 0.2B - 0.6B^2)(1 - B)^{.3}Y_t = \epsilon_t,$$

From Table 4 it may be seen that when taking the mean value of the forty simulations we find that the method of least squares yields a value of 0.872 with a standard error of 0.018 and the method of asymptotic quasi-likelihood with a predictable process of the form $g_n = \theta_0 + \theta_1 X_{n-1}^2 + \theta_2 X_{n-2}^2 + \theta_3 X_{n-3}^2 + c(\log n)^2$ yields a mean value of 0.818 with a standard error of 0.010.

The asymptotic quasi-likelihood is by far the most accurate method in this example. It is seen that the mean value is within two standard errors of the true value (when the predictable process is of the form $g_n = \theta_0 + \theta_1 X_{n-1}^2 + \theta_2 X_{n-2}^2 + \theta_3 X_{n-3}^2 + c(\log n)^2$) whereas this is not the case when the least squares method is applied.

Even when the initial window size in increased from 4 to 8 the least squares estimates do not improve much at all (the mean of the estimates is now 0.863 with a standard error of 0.010). Increasing the initial window length to 16 the estimate becomes 0.848 with a standard error of 0.012. In this instance even relatively large initial window sizes do not significantly improve

Method	mean($\hat{H}$)	stand.error($\hat{H}$)
LS(4)	0.872	0.018
LS(8)	0.863	0.010
LS(16)	0.848	0.012
AQL(g_3)	0.818	0.010

Table 4: Least squares (for three different initial window sizes) and asymptotic quasi-likelihood estimates for Example 4.

the estimates.

In the following we consider various cases for different d in the ARIMA(p,d,q) model and the resulting estimates of H via both the method of least squares and the method of asymptotic quasi-likelihood are compared. Thirty simulations are performed for each model varying the values of d from 0 to 0.4 in increments of 0.1. The order of the parameters p and q may either be 0 or 1. If the order of both p and q are equal to 1 then the coefficients of the model are either $\phi_1 = 0.3$ and $\theta_1 = 0.7$ or $\phi_1 = -0.3$ and $\theta_1 = -0.7$ respectively. Otherwise $\phi_1 = 0.5$ or $\theta_1 = 0.5$. Once again we compare the mean of the least squares estimates to the mean of the asymptotic quasi-likelihood estimates using a predictable process of the form $g_n = \theta_0 + \theta_1 X_{n-1}^2 + \theta_2 X_{n-2}^2 + \theta_3 X_{n-3}^2 + c(\log n)^2$. The results are reported in tables 5, 6, 7 and 8. The initial window length for all these simulations was 8.

When $p = 1$ and $q = 0$ the asymptotic quasi-likelihood estimates are very accurate. The least squares method only yields accurate estimates when $d = 0.3$, otherwise this method overestimates the value of d when the true value is less than 0.3 and underestimates the true value when $d = 0.4$.

The algorithm used by Taqqu and Teverovsky leads to least squares estimates which are more biased than those obtained using the Durbin-Levinson algorithm at smaller values of d. Some of the bias may come from the fact that Taqqu and Teverovsky commence with an initial window length of 5 compared to 8 in our analysis. However, the R/S procedure is strictly adhered to by these authors whereas we use a simplified method to calculate the R/S ratio where there is only one window and the window length varies.

When $p = 0$ and $q = 1$ there is the reverse trend in the resulting estimates, the true value is always underestimated. The most accurate estimate occur around $d = 0.3$. The biases obtained by Taqqu and Teverovsky are very large compared to our results. They obtain mean biases (via least squares) of -0.113, -0.122 and -0.141 when the value of d varies from 0.2 to 0.3 and finally to 0.4. The mean biases we obtain via least squares are -0.074, -0.013 and -0.048 respectively. The biases obtained via the method of asymptotic are usually less than those obtained via least squares.

When $p = 1$ and $q = 1$ (with $\phi_1 = 0.3$ and $\theta_1 = 0.7$) the estimate of d is always less than the true value. The biases are once again less than those obtained by Taqqu and Teverovsky. For all d, the least squares estimate is further from the true value than the asymptotic quasi-likelihood estimate. When $d = 0.3$ the bias via the asymptotic quasi-likelihood method is -0.054 which compares favourably to the bias of -0.157 obtained by Taqqu and Teverovsky. The mean square errors are also much smaller in our simulations than those by Taqqu and Teverovsky.

When $p = 1$ and $q = 1$ (with $\phi_1 = -0.3$ and $\theta_1 = -0.7$ d is in fact overestimated when $d = 0$, 0.1 and 0.2 and the asymptotic quasi-likelihood estimate is less biased than the method of least squares at each value of d. Compared to the case when both coefficients were positive there is much less bias induced in this instance. The bias obtained is very close to 0 via both methods in this example although when the coefficients were positive (i.e. $\phi_1 = 0.3$ and $\theta_1 = 0.7$) the biases obtained were 0.090 using the least squares method and 0.054 when the method of asymptotic quasi-likelihood was used.

5. Conclusion

Our method of estimating d in a fractional ARIMA(p,d,q) model (i.e. R/S analysis with the final estimate of d coming from the application of the asymptotic quasi-likelihood method) will only affect the results if the variances are not equal (i.e. the bias with the application of our method does not come from unequal variances). This method seems to be very effective when there exists a short-range dependence structure, and is much more effective than when the method of least squares is applied. The asymptotic quasi-likelihood method clearly outperforms the least squares method when we consider the ARIMA$(1,d,0)$ model. When considering the ARIMA$(0,d,1)$ model the estimates of H via both methods are comparable for high values of d but the method of asymptotic quasi-likelihood is clearly much more accurate when $d = 0.1$ and $d = 0.2$.

When the ARIMA$(1,d,1)$ model (with positive coefficients) is considered the method of asymptotic quasi-likelihood is much more effective than least squares and only in the case where $d = 0$ are both methods comparable. However if the ARIMA$(1,d,1)$ model (with negative coefficients) is considered the asymptotic quasi-likelihood method is much more accurate than the method of least squares for low values of d. For $d = 0.2$, $d = 0.3$ and $d = 0.4$ there appears to be little difference. The estimates in general via both methods are much closer to the true value than is the case when the coefficients are positive.

According to our other simulations (not shown in this paper) we have seen that when considering higher order pure autoregressive models the asymptotic quasi-likelihood method far outperforms the method of least squares when applied to R/S analysis.

As we have seen, the asymptotic quasi-likelihood method will only affect the results if the variances are not equal. The bias, in this instance, is not a result of unequal variances.

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	d=0		d=0.1		d=0.2		d=0.3		d=0.4	
	AQL	LS	AQL	LS	AQL	LS	AQL	LS	AQL	LS
Bias	0.029	0.086	0.021	0.062	0.029	0.041	0.000	0.001	-0.087	-0.085
$\hat{\sigma}$	0.064	0.036	0.067	0.045	0.084	0.038	0.077	0.049	0.066	0.052
$\sqrt{MSE}$	0.069	0.093	0.069	0.076	0.088	0.056	0.075	0.049	0.108	0.100

Table 5: Least squares and asymptotic quasi-likelihood estimates for the fractional ARIMA(1, d, 0) model.

	d=0		d=0.1		d=0.2		d=0.3		d=0.4	
	AQL	LS	AQL	LS	AQL	LS	AQL	LS	AQL	LS
Bias	-0.038	-0.046	-0.040	-0.058	-0.050	-0.074	-0.022	-0.013	-0.051	-0.048
$\hat{\sigma}$	0.062	0.046	0.063	0.039	0.082	0.041	0.074	0.038	0.065	0.037
$\sqrt{MSE}$	0.071	0.061	0.067	0.070	0.095	0.085	0.076	0.040	0.082	0.061

Table 6: Least squares and asymptotic quasi-likelihood estimates for the fractional ARIMA(0, d, 1) model.

	d=0		d=0.1		d=0.2		d=0.3		d=0.4	
	AQL	LS	AQL	LS	AQL	LS	AQL	LS	AQL	LS
Bias	-0.052	-0.072	-0.059	-0.082	-0.059	-0.105	-0.054	-0.090	-0.094	-0.126
$\hat{\sigma}$	0.050	0.034	0.056	0.032	0.068	0.037	0.083	0.038	0.088	0.056
$\sqrt{MSE}$	0.072	0.080	0.082	0.088	0.090	0.111	0.098	0.098	0.128	0.137

Table 7: Least squares and asymptotic quasi-likelihood estimates for the fractional ARIMA(1, d, 1) model (with positive coefficients).

	d=0		d=0.1		d=0.2		d=0.3		d=0.4	
	AQL	LS	AQL	LS	AQL	LS	AQL	LS	AQL	LS
Bias	0.042	0.057	-0.004	0.028	0.001	0.014	-0.003	-0.008	-0.049	-0.055
$\hat{\sigma}$	0.057	0.036	0.070	0.037	0.075	0.036	0.072	0.035	0.064	0.039
$\sqrt{MSE}$	0.068	0.065	0.069	0.046	0.074	0.038	0.071	0.036	0.079	0.067

Table 8: Least squares and asymptotic quasi-likelihood estimates for the fractional ARIMA(1, d, 1) model (with negative coefficients).