ARIMA Approach to the Unit Root Analysis of Macro Economic Time Series

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Abstract In this paper, The extended Nelson-Plosser data on the historical US macro-economic time series is re-analyzed from the view of the Dickey-Fuller type auto-regressive (AR) unit root test as well as from the view of the recently developed moving average (MA) unit root test. Hatanaka and Koto analyzed the extended Nelson-Plosser data based on a simplified encompassing principle. An example of the test proposed by Hatanaka and Koto is illustrated when a time series is doubted whether it is stationary ARMA or non-stationary ARIMA about trend. Our search for ARMA regression always includes the MA error term. This is because long lags in the AR process often caused over-differencing phenomena, and it was preferred to estimate ARMA processes with short AR lags by including MA error. The ARMA regressions whose MA coefficient is estimated to be unity is avoided to use in the AR unit root test. After arriving at proper ARMA regressions, the adjusted likelihood ratio test is calculated. On the other hand, our search for ARIMA regression is carried out by examining the MA error term. The ARIMA regression whose MA coefficient is estimated to be unity is avoided to use in the MA unit root test. After arriving at proper ARIMA regressions, the MA unit root test is calculated. The p-values of the AR and MA unit root tests are compared to reach at conclusions. Various lag length are examined and overall judgments are drown.

1. Introduction

Hatanaka and Koto (1994) analyzed the extended Nelson-Plosser data on the historical US macro-economic time series based on a simplified encompassing principle. An example of the test proposed by Hatanaka and Koto is illustrated when it is doubted whether a time series is stationary or non-stationary about trend. The former process is represented by a transformed ARMA process about a trend such as M(0, 1): $\Delta X = \alpha + \beta t + \phi X_{-1} + \gamma_1 \Delta X_{-1} + \cdot + \gamma_p \Delta X_{-p} + u$, and the latter is represented by ARIMA process with a drift such as M(1, 1): $\Delta X = \alpha + \gamma_1 \Delta X_{-1} + \cdot + \gamma_p \Delta X_{-p} + u$ where $u = \varepsilon - \theta \varepsilon_{-1}$. (The first argument in M is the order of the integrated process, and the second argument is the power of the deterministic trend in level X.) The latter process includes a random walk and follows from the former under the null of the AR unit root test. Alternatively when θ is unity, the latter reduces to the former but AR lag is shorter by one, and MA error is removed. A time series which includes a random walk is called difference stationary (DS), and that which does not is called trend stationary (TS) throughout this paper following Nelson and Plosser (1982) and Hatanaka and Koto (1995).

Our search for a proper stationary regression includes the MA error term always. This is because the MA coefficient is often estimated to be unity when the AR part has long lags, and it was preferred to estimate ARMA processes with short AR lags by including MA error. The ARMA regressions whose MA coefficient is estimated to be unity is avoided when the AR unit root test is applied. After arriving at proper ARMA regressions, the adjusted likelihood ratio denoted Φ_3, is used for testing the AR unit root as an alternative to the Dickey-Fuller (DF) Φ_3 statistic. (See Sargan and Bhargava (1983) and Tanaka and Satchel (1989). The m.l.e. has positive probability of estimating the unit MA coefficient when the true is not unity.) On the other hand, our search for ARIMA regression is carried out by examining the MA error term. The ARIMA regression whose MA coefficient is estimated to be unity is avoided when the MA unit root test is applied. The MA unit root test is calculated after arriving at proper ARIMA regressions. The process is DS if the MA unit root test rejects the stationarity. The process can be either DS or TS if the null cannot be rejected. The p-values of the AR and MA unit root tests are compared to reach conclusions. Various lag length are examined and overall judgments are drown. (See Saikkonen and Luukkonen (1993), Kwiatkovski et. al. (1992), Tanaka (1990), Koto and Hatanaka (1993) which has extensive tables, and the Appendix A in Morimune and Zhao(1995).)

Details of the test which includes shifts in constant term, trend term and / or polynomial in trend is found in Hatanaka and Koto (1995) and in Koto and Hatanaka(1994). In this paper, the order of the deterministic polynomial in the ARMA processes is congruent with that in the ARIMA processes, and neither examined are constant shift, trend shift nor the higher order trend polynomials. Higher order integration is neither considered. The selection procedure used by this paper tried to be more informative, look at estimated equations more closely than Hatanaka and Koto and also looked for simplicity in the resulting processes. It will be found that our classifications of the time series do not differ much from Hatanaka and Kotos' classifications. This means that the more complicated procedures which are caused by the structural shift and / or polynomial trend can be avoided in the unit root testing procedure.

2 Specification search in ARMA regression

First, the null hypothesis of AR unit root is tested against the alternative hypothesis of stationarity by the augmented DF (ADF) tests in the transformed ARMA equation M(0, 1) such as

$\Delta X = \alpha + \beta t + \phi X_{-1} + \gamma_1 \Delta X_{-1} + \cdot + \gamma_p \Delta X_{-p} + u$,

the error term is $u = \varepsilon - \theta \varepsilon_{-1}$ where the t-value on θ is used against the standard normal distribution for testing the significance of θ coefficient, and ε_t is a white noise. The first order MA lag is always used in our analyses. When serial correlation is detected by the Lagrange multiplier test which was often the case, MA is doubted more than AR in the error term. This is because the specified equations have AR part already, and, therefore, AR in the residuals is considered less likely than MA. It is noted that the AR and MA in the error term are the locally equivalent alternatives. See Godfrey (1988). See Said and Dickey (1984, 1985) for the DF tests in the ARMA models and their small sample properties. The null hypothesis of the DF test can be either $\beta = \phi = 0$ or $\alpha = \beta = \phi = 0$ when a trend is included in the regression. The original notations such as $\tau_{\alpha\tau}$, $\tau_{\beta\tau}$, τ_τ, Φ_1, Φ_2, and Φ_3, and $T\hat{\phi}$ for testing significance of ϕ coefficient are used throughout the paper. The modified LR test Φ_{3*} is carried out for testing the null M(0, 1) against the alternative M(1, 1). See the Appendix A for a proof.

The lag order of the ARMA process was selected basically but not mechanically by the *forward solution* method which applies the t test sequentially starting from the highest lag order which was chosen prior to the test. The null distribution of the t ratio on the differenced lag variable is asymptotically n(0, 1), and the size of the test is set to one percent at each lagged differenced variable and the MA term. Since the MA part includes only one lag, the likelihood function with respect to θ was examined in each estimation, and a regression which has a bi-modal likelihood function or a spike at minus one was avoided in ARMA regressions. When the MA coefficient is estimated unity, it is interpreted that the shorter AR lag is desirable. (See Plosser and Schwarz (1977), and Sargan and Bhargava (1983)). The Durbin and Watson (DW) value is shown at each regression since it gives diagnostic information on the degree of serial correlation among residuals, in particular, when the time series is stationary.

On the other hand, ARIMA process about a drift M(1, 1) is

$$\Delta X = \alpha + \gamma_1 \Delta X_{-1} + \cdot + \gamma_p \Delta X_{-p} + u$$

where $u = \varepsilon - \theta \varepsilon_{-1}$ and $\beta = \phi = 0$. This process includes a random walk. This process is over differenced when θ is unity. If this happens, M(1, 1) reduces to M(0, 1) but AR lag is shorter by one and the MA error is removed. That means over-differencing is doubted when the MA coefficient is estimated unity in the ARIMA models. It was avoided to use the ARIMA regression the MA coefficient of which is estimated to be unity. For proper regressions, the MA unit root test (η_τ) by Saikkonen and Luukkonen (SL) is applied where the null is M(0, 1) and the alternative is M(1, 1). See Appendix B for the test.

This test is achieved by calculating the residuals $\hat{u} = X - \hat{\gamma}_1 X_{-1} - \cdots - \hat{\gamma}_p X_{-p}$ where $\hat{\gamma}$ s' are the maximum likelihood estimator in M(1, 1) with the MA error, and calculating

$$\eta_\tau = \frac{\hat{u}^{*\prime} D' D \hat{u}^*}{T\hat{u}^{*\prime}\hat{u}^*}$$

where $\hat{u}^*$ is the demeaned and detrended residual vector, and D is the partial sum operator matrix. When the null is rejected by the MA unit root test, the process is at least I(1), then the DF test is applied to calculated residuals. If this rejects the null hypothesis of unit root, X is at most I(1) and cannot be I(2). However, caution must be exercised since the common factor between the AR and MA part should be avoided. See J. MacKinnon (1991).

Along Hatanaka and Kotos' procedure, the P-values implied by Φ_{3^*} and η_τ are compared. If the P-value of Φ_{3^*} is smaller than that of η_τ, M(1, 1) is favored to M(0, 1) Similarly, if the P-value of η_τ is smaller than that of Φ_{3^*}, M(0, 1) is favored to M(1, 1). Various lag length are examined and overall judgments are drown.

The non-parametric AR as well as MA unit root tests are calculated, and the MA unit root test is found to be dependent upon the size of the Bartlett window. As it can be seen from the Table 2, the test statistic decreases in value as the size of the Bartlett window increases. Phillips-Perron type non-parametric AR test is not affected much by the size of the Bartlett window, and may well replace the DF type tests. However, it does not take the MA error directly into account.

Using M(0, 1) and M(1, 1), some other tests can be explained. Test M(0, 0) against M(1, 0) is achieved by a modified likelihood ratio test Φ_{1^*} where the former does not include trend and the latter does not include drift in M(0, 1) and in M(1, 1), respectively. Test M(1, 0) against M(0, 0) is by a MA unit root test η_μ where $\hat{u}^*$ in η_τ is only demeaned but not detrended. These tests were used in analyzing the real rate of interest and the unemployment rate. Estimation was mostly performed by Microfit, and the MA unit root test was calculated by Gauss.

3 Empirical results

3.1 Real GNP, 1909-1988 in log.

The plot of real GNP seems to include a linear trend, then the ADF test is performed. The selected ARMA regression is

(3.1) $\Delta X = 0.51 + 0.0054t - 0.17X_{-1} + 0.47\Delta X_{-1}$
(5.7) (6.1) (-6.1) (2.6)

the MA error is $\varepsilon - 0.08\varepsilon_{-1}$ where t-value is -0.35, log L =122.1, and DW=2.0. The τ_τ test is significant (1%) but the $T\hat{\phi}$ test is insignificant (10%). The MA coefficient is estimated to be unity when p is 2 through 6. Phillips-Perron tests are insignificant in all cases (5%). The ARMA process is properly estimated only when p is 0 and 1. The implied ARIMA regression by the null is estimated as

(3.2) $\Delta X = 0.023 + 0.22\Delta X_{-1}$,
(2.3) (0.89)

the MA error is $\varepsilon + 0.14\varepsilon_{-1}$ where the t-value is 0.59, log L =116.4, and DW =2.0. Then Φ_{3^*} =6.1 which is barely insignificant (5%). Various ARIMA regressions were estimated and again, the estimated MA coefficient was unity when p is 2 through 5. The SL test value resulted from (3.2) is 0.46 which is significant (1%) and the null of stationarity is rejected. However, the Φ_{3^*} test and the unit MA coefficient in various ARIMA specifications drove us to classify real GNP into TS. Hatanaka and Koto uses the intercept shift and also favors TS to DS.

3.2 Nominal GNP, 1909-1988 in log.

The MA coefficient in ARMA regressions when p is 2 through 6 is estimated unity. The selected ARMA regression is

(3.3) $\Delta X = 0.46 + 0.0044t - 0.065X_{-1} + 0.37\Delta X_{-1}$,
(2.0) (2.2) (-2.0) (1.8)

the MA error is $\varepsilon + 0.13\varepsilon_{-1}$ where t-value is 0.57, log L = 90.2, and DW = 2.0. The τ_τ test and $T\hat{\phi}$ test are insignificant (10%). Phillips-Perron tests are insignificant in all cases (5%). The ARIMA regression is estimated as

(3.4) $\Delta X = 0.044 + 0.31\Delta X_{-1}$,
(2.5) (1.4)

the MA error is $\varepsilon + 0.18\varepsilon_{-1}$ where t-value is 0.81, log L = 87.9, and DW = 2.0. Then Φ_{3^*} is 2.4 and is insignificant (10%). The SL test calculated from (3.4) is η_τ =1.16 which is significant (1%). The SL test is significant when p is 0 through 6, neither the MA coefficient is estimated unity. The nominal GNP is classified into DS. Hatanaka and Koto uses the intercept shift in nominal GNP, and prefers TS to DS mainly because real GNP is TS.

3.3 **Money stock**, 1989-1988 in log.

The selected auto-regression is

$$(3.5)\quad \underset{(-1.5)}{\Delta X = -0.04} + \underset{(3.4)}{0.0058t} - \underset{(-3.4)}{0.094X_{-1}} + \underset{(1.5)}{0.29\Delta X_{-1}} + \underset{(1.2)}{0.20\Delta X_{-2}} + \underset{(0.71)}{0.074\Delta X_{-3}},$$

the MA error is $\varepsilon + 0.5\varepsilon_{-1}$ where t-value is 2.8, log L = 167.7, and DW = 1.9. The ARMA estimation when p is 0 through 3 is similar in the sense that ϕ value as well as t-values are stable. τ_τ is significant except for the case when p is 0, but $T\hat{\phi}$ is insignificant (10%). The trend term is found marginally significant (1%) by the $\tau_{\beta\tau}$ test, then the τ_τ is tested against n(0, 1) and is found significant. Phillips-Perron tests are insignificant in all cases (5%). The implied ARIMA regression is estimated as

$$(3.6)\quad \Delta X = \underset{(3.2)}{.038} + \underset{(1.2)}{.24\Delta X_{-1}} + \underset{(1.0)}{.17\Delta X_{-2}} - \underset{(-0.13)}{.014\Delta X_{-3}}$$

the MA error is $\varepsilon + 0.53\varepsilon_{-1}$ where t-value is 3.1, log L = 163.9, and DW = 1.9. Then Φ_{3*} is 3.9 which is insignificant(10%). Since Φ_{3*} is more reliable than individual t-test because the former seems to avoids the multi-collinearity between trend and the lagged level, the unit root may be included in the AR regression. The SL test η_τ resulted from (3.6) is 0.6, and is significant (1%). Since the η_τ are significant when p is 0 through 4 (1%), money stock is classified as DS. Hatanaka and Koto uses an intercept shift and a dummy variable for 1932, and found this series is DS.

3.4 **Real wage**, 1900-1988 in log.

The selected auto-regression is

$$(3.7)\quad \Delta X = \underset{(2.8)}{0.18} + \underset{(2.5)}{0.0013t} - \underset{(-2.7)}{0.075X_{-1}} + \underset{(0.51)}{0.16\Delta X_{-1}},$$

the MA error is $\varepsilon + 0.12\varepsilon_{-1}$ where t-value is 0.36, log L = 170.3, and DW = 2.0. $T\hat{\phi}$ and τ_τ are both insignificant (10%). The same result holds when p is 0, 4, and 5. The estimated MA coefficient is unity in other cases. Phillips-Perron test is insignificant in all cases (5%). The ARIMA regression is

$$(3.8)\quad \Delta X = \underset{(2.0)}{0.014} + \underset{(0.075)}{0.025\Delta X_{-1}},$$

MA error is $\varepsilon + 0.22\varepsilon_{-1}$ where t-value is 0.65, log L = 168.9, and DW = 2.0. The Φ_{3*} = 1.4 which is insignificant (10%), and real wage is DS. The SL test is 0.8 which is significant (1%). This result holds the same when p is 0 through 6, and real wage is DS overall. Hatanaka and Koto takes the intercept shift into account, and classified real wage into DS.

3.5 **Real rate of interest**, 1900-1988

Trend is insignificant in real rate of interest, and the selected regression is

$$(3.9)\quad \Delta X = \underset{(1.6)}{1.2} - \underset{(-3.5)}{.67X_{-1}} - \underset{(-0.55)}{.099\Delta X_{-1}} - \underset{(-0.10)}{.013\Delta X_{-2}},$$

the MA error is $\varepsilon + 0.59\varepsilon_{-1}$ where t-value is 2.4, log L = -240.4 and DW = 2.0 The $T\hat{\phi}$ and τ_μ tests are significant (1%). Similar result holds when p is 0 through 2, and the MA coefficient is estimated unity when p is 3, 5, and 6. Phillips-Perron test is significant in all cases (1%). The ARIMA regression is

$$(3.10)\quad \Delta X = \underset{(7.2)}{0.76\Delta X_{-1}} - \underset{(-2.2)}{0.23\Delta X_{-2}},$$

the MA error is $\varepsilon - \varepsilon_{-1}$, log L = -243.9, and DW = 1.2, and Φ_{1*} =3.6 which is insignificant (10%), and the null of non-stationarity cannot be rejected. Since the MA coefficient is estimated unity when p is 1 through 6 which is interpreted as evidence for TS. Real rate of interest is TS overall. When k is zero, the SL test η_μ is 0.17 which is insignificant (10%). This also supports TS. Based on ARIMA estimation, real rate of interest is classified into TS. Hatanaka and Koto used a dummy variable to explain an outliner in 1921, and concluded that the real rate of interest is TS.

3.6 **Unemployment rate**, 1890-1988

Trend is insignificant in the unemployment, and the selected ARMA regression is

$$(3.11)\ \Delta X = .53 - .31X_{-1} + .31\Delta X_{-1} - .18\Delta X_{-2} + .22\Delta X_{-3}$$
$$(3.1)(-3.3)\quad (1.7)\quad (-1.8)\quad (2.2)$$

the MA error is $\varepsilon + 0.11\varepsilon_{-1}$ where t-value is 0.44, log L = -37.1, and DW = 2.0. The $T\hat{\phi}$ and τ_μ tests are significant (1% and 2.5%, respectively). Results are similar when p is 0 through 3. The MA coefficient is unity when p is 4 through 6. Phillips-Perron test is significant in all cases (at least at 5%). The corresponding ARIMA regression is

$$(3.12)\ \Delta X = 0.030\Delta X_{-1} - 0.33\Delta X_{-2} + 0.081\Delta X_{-3}\ ,$$
$$(0.14)\quad (-3.5)\quad (0.77)$$

the MA error is $\varepsilon + 0.23\varepsilon_{-1}$ where t-value is 0.98, log L = -44.4, and DW = 2.0, and Φ_1, which is 7.9 is significant (1%). Then unemployment rate is TS. The SL tests η_μ is 0.25 which is insignificant (5%). Hatanaka and Koto gave the same classification.

4 Conclusion

1: Dickey-Fuller Φ_3, test is natural to use since it is calculated from the maximized values of likelihood function associated with ARMA and ARIMA regressions. This test may avoid multi-collinearity between trend and the lagged difference which is often evident in the macro time series. The t-tests such as $T\hat{\phi}$ and τ_τ are convenient but partial and may suffer from multi-collinearity. (It is noted that the null hypothesis of these t-tests are $\beta = \phi = 0$, not $\phi = 0$ or $\beta = 0$ separately, and the power of the joint test Φ_3, seems naturally greater than that of the partial test.) Phillips and Perron type non-parametric AR unit root test seems useful.

2: The MA coefficient in ARMA processes indicates whether the estimated regression has over-differencing phenomena or not. It is comfortable to use shorter AR lags when the MA coefficient is unity. This does not contradict to what Sargan and Bhargava (1983) analyzed on the maximum likelihood estimator of the MA coefficient since it is not proved wrong to use shorter AR lags.

3: The small sample properties are not studied but Saikkonen and Luukkonens' MA unit root test seems to over-reject the null of stationarity. Kwiatkovsky *et. al.*'s non-parametric test seems to under-reject the null of stationarity. Caution should be exercised when these MA tests are used. The Kwiatkovsky *et. al.*'s non-parametric test depends heavily on the window size, and there is no criterion in the selection of the window size. The test value gets smaller as the window size becomes larger. Selection between DS and TS should not solely rely on these tests, but should also be consulted with the estimated MA coefficient.

4: Our selection between DS and TS does not use structural shift in intercept nor in trend, dummy variables to explain sudden changes, nor polynomial trend which were included in Hatanaka and Koto's analysis. Neither do we use trend of different orders in TS and DS. Trend in ARMA is consistent with that in ARIMA. It was tried to look at the estimated regressions more closely than Hatanaka and Koto, and our classification is similar to their results.

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Appendix A Using standard notations, define

$$A=\begin{pmatrix} 1 & \int_0^1 B(r)dr & 1/2 \\ \int_0^1 B(r)dr & \int_0^1 B(r)^2 dr & \int_0^1 rB(r)dr \\ 1/2 & \int_0^1 r\, B(r)dr & 1/3 \end{pmatrix},$$

$$b=\begin{pmatrix} B(1) \\ \int_0^1 B(r)dB(r) \\ \int_0^1 r\, dB(r) \end{pmatrix}.$$

The likelihood ratio test statistic $2\log L(\hat{\theta})$ - $2\log L(\hat{\theta}_0)$ is asymptotically equal to $b' A^{-1} b$ $-b_1^2$ which is the same as that of the Φ_3 by Dickey and Fuller. Details may be requested from the author. The Φ_{3*} (Φ_{2*}) is calculated by exponential transformation of { twice the log likelihood ratio, divide by two (three) } minus one.

Appendix B Setting all pre-historical values of X and u to zero, the DGP of M(1, 1) is $X_1 - \mu = \varepsilon_1$, $\alpha(B)(\Delta X_t - \beta) = (1 - \theta B)\varepsilon_t$, $t = 2,T$. The process for the level is $\alpha(B)(X_t - \mu - (t-1)\beta) = (1 - \theta B)\sum_{t=1,T}\varepsilon_t$ where $\theta \neq 1$. When $\theta = 1$, the auto-regression is M(0, 1). The likelihood ratio test is not consistent in the MA unit root test since the SSR from estimation of the null model is $O_p(T)$ under the alternative. This is seen by the following example where the alternative model is assumed to be $(1-\alpha B)\Delta X = (1-\theta B)\varepsilon$, $t = 1,\cdots,T$. The null model is $(1-\alpha B)X = \varepsilon$. The SSR is $O_p(T)$ under the null obviously. Further, as it was proved by Phillips (1988), the OLS estimator $\hat{\alpha}$ of α under the null converges to 1 under the alternative hypothesis. Then the residual $(1-\hat{\alpha} B)X$ is approximately ΔX which is stationary under the alternative hypothesis. This implies that the null SSR is $O_p(T)$ under the alternative, and the likelihood ratio does not diverge. In short, $\hat{\alpha}$ miraculously converges to unity, and the residual converge to the difference.

Table 1 Parametric Test

series	T	p	ARMA test: τ_β	ϕ	τ_τ (τ_μ)	θ	t_θ	log LF	$\Phi 3$ ($\Phi 1$)	ADF test	ARIMA test: θ	t_θ	log LF	η_μ (η_τ)	MA test
Real GNP	80	0	3.3	-0.19	-3.3	0.36	3.8	122.1	4.3	insig	0.31	3.3	118	0.54	DS
1909-1988	ln	1	6.1	-0.17	-6.1	-0.1	-0.3	122.1	6.1	TS	0.14	0.59	116.4	0.46	DS
		2				-1		122.7	5.1		-1		117.9		
		3				-1		120.8	5.1		-1		116		
		4				-1		119.1	4.4		-1		114.9		
		5				-1		117.4	4.8		-1		112.9		
		6				-1		117	6.1		0.81	8.8	111.3	0.51	DS
Nominal GNP	80	0	1.7	-0.06	-1.5	0.4	4.7	90.5	1.8	insig	0.44	4.67	88.7	1.24	DS
1909-1988	ln	1	2.2	-0.07	-2	0.1	0.6	90.2	2.4	insig	0.18	0.81	87.9	1.16	DS
		2				-1		88.7	1.7		0.88	11.3	87.0	1.14	DS
		3				-1		90.7	5.3		-0.1	-0.23	85.7	1.17	DS
		4				-1		89.4	4.2		0.15	0.36	85.4	1.22	DS
		5				-1		90.7	6.0		-0.24	-0.73	85.1	0.99	DS
		6				-1		92.7	8.7		0.83	8.54	84.8	1.19	DS
Money stock	100	0	1.7	-0.05	-1.6	0.6	8.8	164.9	1.3	insig	0.6	8.7	163.6	0.60	DS
1889-1988	ln	1	-4.0	-0.07	-4.0	0.1	0.7	169.4	3.4	insig	0.27	1.6	166.1	0.59	DS
		2	3.1	-0.09	-3.1	0.6	4	168.8	3.6	insig	0.59	3.6	165.3	0.59	DS
		3	3.4	-0.09	-3.4	0.5	2.8	167.7	3.9	insig	0.53	3.1	163.9	0.60	DS
		4	4.36	-0.77	-4.5	0.17	0.69	168.2	3.7	insig	0.17	0.66	164.6	0.62	DS
		5				-1		167.2	3.8		-1		163.5		
		6				1		166.3	4.3		1		162.2		
Real wages	89	0	1.8	-0.07	-2.1	0.3	2.7	172.2	1.4	insig	0.24	2.3	170.8	0.80	DS
1900-1988	ln	1	2.7	-0.07	-2.6	0.2	0.8	170.3	1.4	insig	0.22	0.65	168.9	0.80	DS
		2				-1		168.7	1.8		-0.58	-1.4	166.9	0.65	DS
		3				-1		166.7	2.0		-0.36	-0.5	164.7	0.78	DS
		4	1.2	-0.1	-1.4	0.8	6.5	164.9	1.2	insig	0.8	6.6	163.7	0.84	DS
		5	1.3	-0.1	-1.5	0.76	5.5	162.7	1.1	insig	0.76	5.4	161.6	0.86	DS
		6				-1		161.3	1.6		0.74	5.0	159.7	0.87	DS
Real rate	89	0	n.a.	-0.69	-4.4	0.51	3.3	-245.3	n.a.		n.a.	n.a.	n.a.	0.17	insig
of interest		1	n.a.	-0.69	-4.9	0.62	3.4	-242.7	6.4	TS	-1		-248.7		
1900-1988		2	n.a.	-0.67	-3.48	0.59	2.4	-240.4	3.6	insig	-1		-243.9		
		3	n.a.			-1		-237.2	3.9		-1		-240.9		
		4	n.a.	-0.47	-2.11	0.24	0.65	-234.3	4.1	insig	-1		-238.2		
		5	n.a.			-1		-229.2	5.5		-1		-234.4		
		6	n.a.			-1		-226.9	5.0		-1		-231.6		
Unemploy	99	0	n.a.	-0.5	-4.4	0.6	4.8	-47	n.a.		n.a.	n.a.	n.a.	0.19	insig
-ment rate		1	n.a.	-0.45	-4.2	0.75	8.7	-44	8.9	TS	0.77	9.9	-52.2	0.21	insig
1890-1988		2	n.a.	-0.41	-3.6	0.64	5.4	-41.5	7.2	TS	0.61	4.9	-48.2	0.24	insig
		3	n.a.	-0.31	-3.3	0.11	0.4	-37.1	7.9	TS	0.23	0.98	-44.4	0.25	insig
		4	n.a.			-1		-35.5	4.3		-1		-39.6		
		5	n.a.			-1		-34.7	5.2		-1		-39.6		
		6	n.a.			-1		-33.6	5.9		-1		-39.2		

Table 2 Non-Parametric Test

	size of Bartlett widow		0	1	2	3	4	5	6
Real GNP	Phillip	$Z\alpha$	-10.1	-13.3	-14.6	-14.6	-13.9	-13.0	-12.3
1909-1988	&Perron	$Z\tau$	-2.5	-2.7	-2.9	-2.9	-2.8	-2.7	-2.7
	Kwiatkovsky		0.54	0.29	0.21	0.17	0.15	0.13	0.12
Nominal GNP	Phillip	$Z\alpha$	-3.0	-4.4	-5.1	-5.3	-5.2	-5.0	-5.0
1909-1988	&Perron	$Z\tau$	-1.2	-1.4	-1.6	-1.6	-1.6	-1.5	-1.5
	Kwiatkovsky		1.24	0.64	0.44	0.34	0.28	0.24	0.22
Money stock	Phillip	$Z\alpha$	-2.7	-5.1	-6.8	-7.9	-8.7	-9.1	-9.5
1889-1988	&Perron	$Z\tau$	-1.0	-1.4	-1.7	-1.9	-2.0	-2.0	-2.1
	Kwiatkovsky		0.60	0.31	0.21	0.17	0.14	0.12	0.11
Real wages	Phillip	$Z\alpha$	-3.5	-5.1	-5.5	-5.6	-5.5	-5.3	-5.1
1900-1988	&Perron	$Z\tau$	-1.0	-1.3	-1.4	-1.4	-1.3	-1.3	-1.3
	Kwiatkovsky		0.80	0.42	0.29	0.23	0.20	0.17	0.15
Real rate of interest	Phillip	$Z\alpha$	-33.9	-37.8	-35.8	-34.3	-31.9	-31.0	-31.1
1900-1988	&Perron	$Z\tau$	-4.6	-4.8	-4.7	-4.6	-4.5	-4.4	-4.4
	Kwiatkovsky		0.17	0.11	0.09	0.08	0.07	0.07	0.07
Unemploy -ment rate	Phillip	$Z\alpha$	-19.8	-22.9	-21.6	-21.6	-21.7	-21.1	-20.8
1890-1988	&Perron	$Z\tau$	-3.2	-3.5	-3.4	-3.4	-3.4	-3.3	-3.3
	Kwiatkovsky		0.19	0.11	0.08	0.07	0.06	0.06	0.05

Critical Value

	5%	1%
τ	-2	-2.6
τ_μ	-2.9	-3.5
τ_τ	-3.5	-4.4
η_μ	0.46	0.74
η_τ	0.15	0.22
$\Phi 1$	4.6	6.4
$\Phi 3$	6.3	8.3
ϕ_μ	-14.1	-20.7
ϕ_τ	-21.8	-29.5

Trend is not included in the tests associated with real rate of interest and unemployment rate.

Concentrated log likelihood on θ : Real GNP case