A Technique for the Sensitivity Analysis of Functions in Relation to Decision-making Objectives

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EXTENDED ABSTRACT

Sensitivity analysis is usually focussed on parameters and is a relatively well-developed field compared with function sensitivity. But how sensitive are model conclusions to the choice of functions used in the right hand side of differential equation models? Most work in this area has been scenario-based where alternative functions are tested. In this paper, we examine the sensitivity of a model to changes in the shape of the functions. We do this in an automated way without the need to specify alternative functional forms.

The question then is how much can we change a function that defines the dynamics of the system without producing a significant change in some performance measure? If the changes in functions need to be large in some sense, to cause significant changes in a performance measure, then there is less need to focus attention on getting the model functions correct. A method of approach to this type of analysis is presented and illustrated on an ecosystem model. Testing the proposed method on a simple model demonstrates that quite large changes can be made to functions before reaching a critical value in the decision criterion. This insight is as useful as the corresponding knowledge of the effect of uncertainty in parameter values.

1. INTRODUCTION

It has long been recognised that, “the simple obtaining of solutions for the equations of the mathematical model of a dynamic system – or even a set of solutions – is no longer sufficient” (Tomovic 1963). Further, in 1968, Quade expressed the view that: “A good system study will include sensitivity tests on the assumptions in order to find out which ones really affect the outcome and to what extent. This enables the analyst to determine where further investigation of assumptions is needed”. Along these lines Forrester (1969) demonstrated that the question of sensitivity is important from a policy viewpoint only when parameter changes would render a proposed policy ineffective. At that time and subsequently most sensitivity analyses were only performed on parameters and initial values. This approach can be found, for example, in Barnes and Yeaple (1968), Thornton and Lessem (1976), and Vermeulen and De Jongh (1976 and 1977).

A notable early exception to straight parameter sensitivity analysis (SA) was the practical approach adopted by Ford and Gardiner (1979). They convened a workshop of public and private leaders where the group was presented with model forecasts and asked to decide on a policy. Changes were then made to the model and the group was presented with the new forecasts. Based on those forecasts, the group was again asked to vote on the policy. If the policy decision was unchanged the model could be regarded as insensitive to the changes. So it might well be that some change in a parameter value or function causes a very large change in a state variable but if this does not alter the decision of the policy-making body then in practical terms the model is insensitive.

The main purpose of a SA of a model used in decision support should be to determine the extent to which the decisions or policies based on model results are robust with respect to the uncertainty in the model. Walker *et al* (2003) recently noted the increasing requirement to articulate uncertainty, when working at the interface of science and management, in model-based decision support. They recognise two extremes as a feature of the nature of uncertainty: ‘Epistemic uncertainty’, which is due to the imperfection of our knowledge and may be reduced by more research or data, and ‘variability uncertainty’, which is due to the inherent variability in a system.

Epistemic uncertainty includes uncertainty in parameter values, model inputs, and the functions in a model. As noted, most analyses of models do include a SA of the parameters and a scenario analysis of the inputs. Alternative functions within a model, however, are only sometimes tested and in an *ad hoc* manner. Little has been done to perform a SA of *functions* in the automated way that is done with parameters. This paper takes some tentative steps towards addressing this problem.

2. METHOD

Consider the following system of difference equations:

$$x_i(t+\Delta t) = x_i(t) + f_i(\underline{x}, t, \underline{\alpha})\Delta t,\ i = 1,2,\ldots,n, \tag{1}$$

where $\underline{x}$ is the state vector and $\underline{\alpha}$ a vector of parameters.

A basic parameter sensitivity analysis involves changing the values of the parameters in the functions $f_i(\underline{x}, t, \underline{\alpha})$ by a small amount, one at a time, and observing the change it produces in the output. Although changing the parameters in the functions $f_i(\underline{x}, t, \underline{\alpha})$ does change the shape of the functions it does so in very restrictive ways. Clearly other changes to the shape of the functions are possible. This is an important consideration if there is uncertainty about the appropriateness of the functional form chosen for the model. The functions contain the information about the dynamics of the model, with different functional forms corresponding to different choices of dynamics for the system. Changes in the functions then correspond to changes in the dynamics of the model.

A possible pragmatic approach for this more general form of function SA is to multiply each function or rate by a parameter with a nominal value of one. These parameters can then be perturbed as is done in parameter SA. This should yield some indication as to which rates are the most sensitive. This method was tried with some success by Lawrie and Hearne (2007). One of the shortcomings of this approach, however, is that no information is obtained on the sensitivity of the output to changes in the shape of the functions. The simplest approach towards this end, going beyond the method mentioned above, is to multiply each function by the following function which comprises a product of triangular-shaped functions:

$$H(\underline{x}, \underline{p}, \underline{m}) = \prod_{i=1}^{n} (1 + h_i(x_i, p_i, m_i)), \tag{2}$$

where

$$h_i(x_i, p_i, m_i) = m_i(x_i - c)/(p_i - c)$$
$$\text{and } c = \begin{cases} a_i \text{ if } x \le p_i, \\ b_i \text{ if } x > p_i. \end{cases} \tag{3}$$

Note that by choosing m_i to be negative we can invert the triangular shaped function h_i. Reasonable choices of a_i and b_i are the respective minimum and maximum values of the corresponding state variable x_i over the solution interval $[a_i, b_i]$.

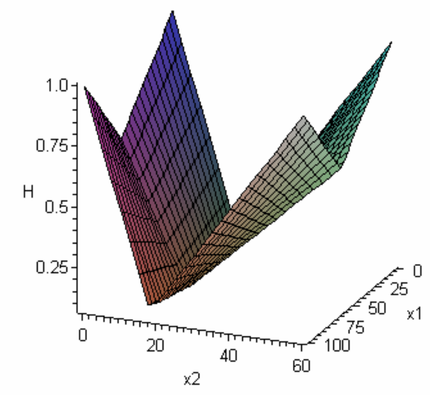

Figure 1. An example of the function $H(\underline{x}, \underline{p}, \underline{m})$ in two dimensions, where both m_1 and m_2 are negative.

The function $H(\underline{x}, \underline{p}, \underline{m})$ deviates from the constant function 1, where the greatest change occurs at $\underline{x} = \underline{p}$, with magnitude determined by $\underline{m}$.

The SA now proceeds by investigating changes in each of the functions $f_i(\underline{x}, t, \underline{\alpha})$ obtained by multiplying each in turn by a function $H(\underline{x}, \underline{p}, \underline{m})$. The SA tests the sensitivity of some objective measure chosen by the user. In this project, we are interested in a decision-making objective. In this context further development of our idea is best achieved with an illustrative example.

3. ILLUSTRATIVE EXAMPLE

An agricultural product X will be ready for harvesting in T (=12) months time. A pest species Y consumes X at a certain rate depending on the density of X. The damage caused by Y is unacceptable and two means of controlling the pest have been proposed: (1) biological control through the introduction of a parasitoid Z and (2) chemical control. The first method is much cheaper and also more desirable from environmental considerations but there is more confidence in the efficacy of chemical control. To facilitate making a decision, a model of the system with Z has been formulated. The aim of the model is to answer the following question:

Will the introduction of population Z ensure that the biomass of X achieves a minimum level at harvest time T? In particular will the 10th percentile of X be above a threshold value V (=60)?

Let x_1, x_2, and x_3 denote the population levels of X, Y, and Z, respectively. The model is given by the system of equations (1) with the following RHS functions:

$$f_1 = r_1 x_1 (1 - x_1 / k) - \frac{\alpha_1 x_2 x_1}{(x_1 + \alpha_2)},$$
$$f_2 = \frac{r_2 x_2 x_1}{(x_1 + \alpha_2)} - \frac{\beta_1 x_3 x_2}{(x_2 + \beta_2)},$$
$$f_3 = \frac{r_3 x_3 x_2}{(x_2 + \beta_2)} - \gamma x_3. \tag{4}$$

Initial and parameter values are

$$x_1(0) = 40,\ x_2(0) = 10,\ x_3(0) = 6,$$
$$r_1 = 0.8,\ r_2 = 0.4,\ r_3 = 0.25,$$
$$\alpha_1 = 0.4,\ \alpha_2 = 20,\ \beta_1 = 0.2,\ \beta_2 = 4,$$
$$\gamma = 0.001,\ k = 100(1 + N(0,10)), \tag{5}$$

where *N(0,10)* is a normally distributed random number with mean 0 and standard deviation 10.

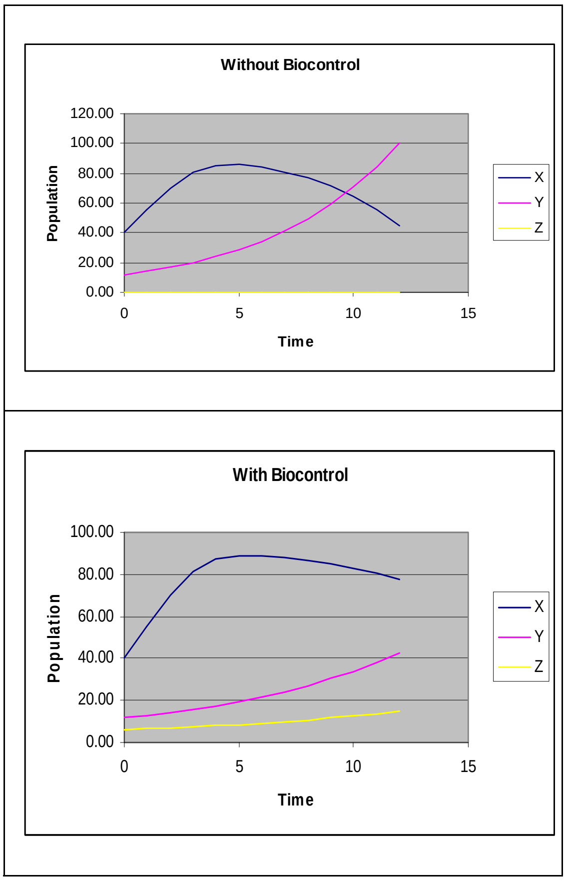

Figure 2. Deterministic solution: The system behaviour without bio-control shows population X decreasing below the acceptable threshold. The introduction of population Z reduces Y enabling X to maintain a level well above the threshold at harvest.

That the introduction of Z is effective can be seen by comparing the two graphs with and without population Z in Figure 2. These are solutions of the deterministic model with k held constant at 100. Further analysis was undertaken by performing 500 simulations of the stochastic model. These solutions indicated that at the final time T, X would have a mean of approximately 78 and a 10th percentile of 66 (>V). This suggests that the decision can be made tentatively to go for option (1), biological control.

Normally at this point SA of parameters and initial values would be undertaken and possibly some experimenting with alternative model formulations. As the purpose of this project is to go beyond that, we assume that all parameter and initial values are perfectly known. The question then remains whether the functions $f_i(\underline{x}, t, \underline{\alpha})$

of the model are correct. In particular, we are interested in the following:

By how much can the functions be distorted while still ensuring that the decision criterion is satisfied? The decision criterion being that the 10^{th} percentile of population X lies above the threshold V at the final time and hence that the first option for control will be the preferred one?

Function	$\lvert \underline{m} \rvert$	Relative Insensitivity	
f_1	0.97	6.06	
f_2	0.16	1.00	
f_3	2.30	14.38	
Function	Peak Vector		
	m_1	m_2	m_3
f_1	-0.35	-0.9	---
f_2	0.11	0.11	0.04
f_3	---	-1.5	-1.75
Function	Critical Point		
	p_1	p_2	p_3
f_1	58	18	---
f_2	56	28	8
f_3	---	18	10

Table 1: Results from the application of the proposed method to the illustrative model. The third column, Relative Insensitivity, is indicative of the relative magnitude of change that can be made to a function before the critical value of the decision criterion is reached. The last three rows contain the point where the function is most sensitive to function changes.

If the criterion is satisfied, despite large changes to the functions, then one might conclude that the decision is insensitive to the choice of model functions. We now formulate the mathematical problem to answer this question.

Formulation of the function sensitivity problem

Problem P1

Consider, one at a time, a change to each function $f_i(\underline{x},t,\underline{\alpha})$, where the changed function is given by $H(\underline{x},\underline{p},\underline{m})\,f_i(\underline{x},t,\underline{\alpha})$ (see equation (2) for definition of $H(\underline{x},\underline{p},\underline{m})$). If the functions $f_i(\underline{x},t,\underline{\alpha})$ are independent of x_i then $h_i(x_i,p_i,m_i)$ are set to zero. For the i^{th} equation this means:

Find $(\underline{p}^*,\underline{m}^*)$, the solution to the constrained minimization problem:

$$\min_{\underline{p},\underline{m}} \underline{m}^2$$

constrained by the condition that the 10^{th} percentile of $x_1(T) \le V$.

This is the smallest change of $f_i(\underline{x},t,\underline{\alpha})$ which no longer ensures that the harvest has minimum biomass greater than 60 units.

Effectively P1 means that we find the position in state space where the model is most sensitive to changes in the functions $f_i(\underline{x},t,\underline{\alpha})$. Moreover, we can determine if increasing the function or decreasing the function at that point produces the greatest change in the output measure – through the sign of $\underline{m}$. This means that regardless of position or direction (increase or decrease) any smaller change in the function will ensure that the final level of population X is acceptable, and hence robust to the decision.

4. RESULTS

Problem P1 is solved for the three cases corresponding to each of the three RHS functions. The results are shown in Table 1. Note that, looking at the values of $\lvert \underline{m} \rvert$, one can see that the function f_2 is the most sensitive. Compared to function f_2 the other two functions are relatively insensitive. In fact functions f_3 and f_1 can endure changes which are 14.38 and 6.06 times larger, respectively, than that of f_2. This relative measure is given as the "Relative Insensitivity" in Table 1.

The position of the peak changes the shape of the function. This in turn generally influences the results. The original function $f_1(\underline{x},t,\underline{\alpha})$ is shown in Figure 3. The modified function $H(\underline{x},\underline{p}^*,\underline{m}^*)\,f_1(\underline{x},t,\underline{\alpha})$, where the $(\underline{p}^*,\underline{m}^*)$ values for function f_1 are given in Table 1, is shown in Figure 4.

To further show that the result depends on the shape of the function we change the location of the peak of $H(\underline{x},\underline{p},\underline{m})$. To do this, we replace the function $H(\underline{x},\underline{p}^*,\underline{m}^*)$ with the function

$H(\underline{x}, \underline{p}^{\#}, \underline{m}^{*})$, where $\underline{p}^{\#}$ is the new location for the peak. Choosing the function f_1, for example and using $\underline{p}^{\#} = (65{,}30)$ we get the 10^{th} percentile for population X greater than 68, compared with 60 in the previous solution.

For the purposes of comparison we also apply an approach similar to that taken by Lawrie and Hearne (2007). In this approach, each RHS function is multiplied by a constant parameter of nominal value one. The constants are then perturbed by 1%, one at a time, and the simulations are repeated.

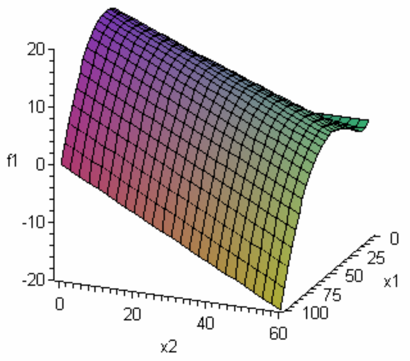

Figure 3. This figure is the original function $f_1(\underline{x}, t, \underline{\alpha})$ given in Equation 4 above.

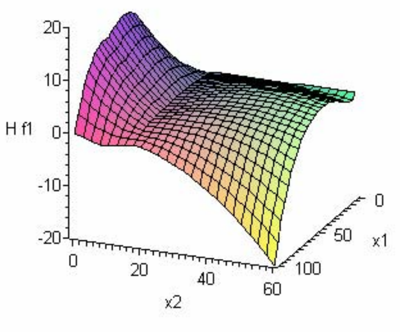

Figure 4. This figure shows the modified function $H(\underline{x}, \underline{p}^{*}, \underline{m}^{*})\, f_1(\underline{x}, t, \underline{\alpha})$.

This is equivalent to setting $h_i(x_i, p_i, m_i) = k_i$ for some small constants k_i, or alternatively setting $H(\underline{x}, \underline{p}, \underline{m}) = 1 + K$ for a small constant K.

For perturbations to functions f_1, f_2, and f_3 the changes in the magnitude of the 10^{th} percentile of X are 2.36%, 1.89% and 1.5% respectively. This suggests that f_1 is most sensitive to change followed by f_2 and then f_3. This contrasts with the results given in Table 1.

5. CONCLUSION

Decision and policy making can be determined or influenced by the output of a model. In this context the techniques needed to explore the relationship between a model's output and uncertainties in its parameter and initial values are well-developed. However, techniques for the analysis of uncertainty in the functions used in a model is less developed. It is often dealt with by changing the functions used in the model on a trial and error basis, or as a scenario based analysis. This is both difficult and time-consuming when dealing with large complex models. In this paper, we investigate the effects of uncertainty in the functions of a model through an automated process.

The proposed method is tested in our simple model. For this model quite large changes can be made to functions before reaching a critical value in the decision criterion. This insight is as useful as the corresponding knowledge of the effect of uncertainty in parameter values.

Research is in progress to investigate the procedure further by extending the process to individual terms in a RHS function. This may in turn yield useful information about all the relationships within a model. The intention is then to test the whole procedure on a large complex model.

6. ACKNOWLEDGEMENTS

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