Solving a Non-Linear Model: The Importance of Model Specification for Deriving a Suitable Solution

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EXTENDED ABSTRACT

Saddle-path instabilities frequently arise in dynamic macroeconomic models with forward-looking expectations. Macroeconomic models with saddle-path instabilities can be solved using shooting algorithms, such as forward-shooting and reverse-shooting. In practice, such macroeconomic models, when derived from optimizing behavior, are also likely to contain non-linearities

In this paper, we investigate the interaction between the use of shooting algorithms and the precise specification of non-linear dynamic macroeconomic models.

First, we consider the basic Cagan (1956) Model augmented by perfect foresight for price expectations. This gives us the well-known model by Sargent and Wallace (1973). We consider both a linear specification of the model (expressed in logarithms) and a non-linear specification (expressed in levels).

For the linear model, p denotes the price level (expressed in logarithms) and $\bar{m}$ denotes the exogenously fixed money supply (expressed in logarithms). For the non-linear model, P denotes the price level and $\bar{M}$ denotes the exogenously fixed money supply.

Examination of the linear specification of the model, shows that there are essentially three types of solution path:
(a) the model can jump to the equilibrium given by $p = \bar{m}$; or
(b) p can diverge to $-\infty$; or
(c) p can diverge to $+\infty$.

This corresponds to the following types of solution path for the non-linear specification of the model:
(a) the model can jump to the equilibrium given by $P = \bar{M}$; or
(b) P can converge to the equilibrium given by $P = 0$; or
(c) P can diverge to $+\infty$.

Thus, for the vast majority of initial conditions, the variable P will either diverge; or converge to the stable equilibrium given by $P = 0$. It is highly improbable that a randomly chosen initial condition will lead the model to the equilibrium given by $P = \bar{M}$.

It is essentially this property of the solution to the non-linear model that drives the results. The non-linear model has two equilibria. Given a random choice of initial condition, the variable P will either diverge; or converge to the stable equilibrium given by $P = 0$. With probability 1, it will not locate the equilibrium given by $P = \bar{M}$. Thus, the forward-shooting algorithm will almost certainly conclude that a path converging to $P = 0$ is the appropriate solution. This, of course, is the incorrect solution.

Because the reverse-shooting algorithm requires the specification of the appropriate equilibrium steady-state, it does not suffer from the problem that it will choose the wrong equilibrium. By specifying the appropriate equilibrium, that is, $P = \bar{M}$, it is possible to ensure that the correct path is chosen by the reverse-shooting algorithm.

We then use a more complex macroeconomics framework to demonstrate how similar issues can arise in larger models.

INTRODUCTION

Saddle-path instabilities frequently arise in dynamic macroeconomic models with forward-looking expectations. Macroeconomic models with saddle-path instabilities can be solved using shooting algorithms, such as forward-shooting and reverse-shooting. In practice, such macroeconomic models, when derived from optimizing behavior, are also likely to contain non-linearities

In this paper, we investigate the interaction between the use of shooting algorithms and the precise specification of non-linear dynamic macroeconomic models. We show that, with some model specifications, it is possible that a forward-shooting algorithm will choose an inappropriate solution path.

This paper extends issues previously discussed by Herbert, Stemp and Griffiths (2005) and Stemp and Herbert (2007).

2. THE EXTENDED CAGAN MODEL

The linear model (expressed in logarithms)

First, we consider the basic Cagan (1956) Model augmented by perfect foresight for price expectations. This gives us the well-known model by Sargent and Wallace (1973), represented by the following equation:

$$\bar{m} - p = -\alpha \dot{p} \tag{1}$$

where α is positive, all variables are functions of time, and lower-case letters denote logarithms:
p = price level (expressed in logarithms); and
$\bar{m}$ = nominal money stock (expressed in logarithms), assumed to be constant.

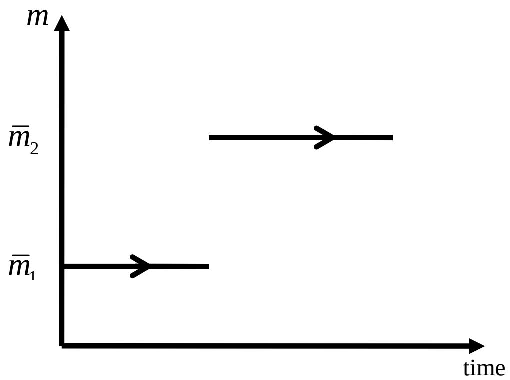

Figure 1
Shock to Money Supply

This is a linear model and has a unique equilibrium given by: $p = \bar{m}$.

Starting from an initial condition given by $p(0) = p_0$, the dynamics of p are described by the following equation:

$$p = \bar{m} + \left[p_0 - \bar{m}\right]e^{+\left(\frac{1}{\alpha}\right)t} \tag{2}$$

Whenever $p_0 \neq \bar{m}$, then p diverges. Assuming that p evolves continuously, then following an increase in $\bar{m}$ from $\bar{m}_1$ to $\bar{m}_2$ as shown in Figure 1, the dynamics of p can be described by Figure 2.

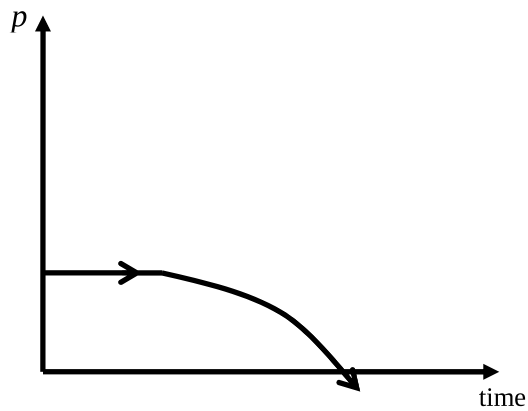

Figure 2
Price Dynamics: Continuous Adjustment from Initial Condition

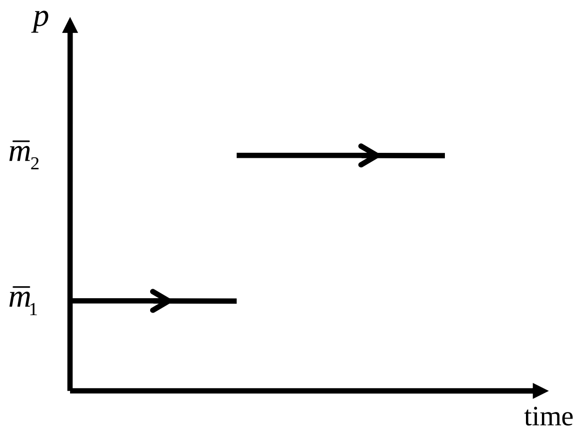

Figure 3
Price Dynamics: Initial Jump in Prices

Sargent and Wallace's (1973) seminal contribution was to suggest that one way we can handle the perverse result of instability is to allow prices to jump. Assume initial equilibrium where $p(0) = \bar{m}_1$. Then let $\bar{m}$ increase from $\bar{m}_1$ to $\bar{m}_2$.

Then if p must evolve continuously, p will decline exponentially. If p jumps to $\bar{m}_2$ when money supply jumps, then this gives a stable solution. This gives the result for the time-path of prices as described in Figure 3.

This use of an initial jump in an endogenous variable following an initial shock to the economy has become the standard approach to solving dynamic models with at least some variables having expectations that satisfy a rational expectations assumption. Blanchard and Kahn (1980) have extended this approach to show that, following an initial shock, an economy will evolve towards a stable equilibrium if there are as many "jump variables" as there are unstable eigenvalues.

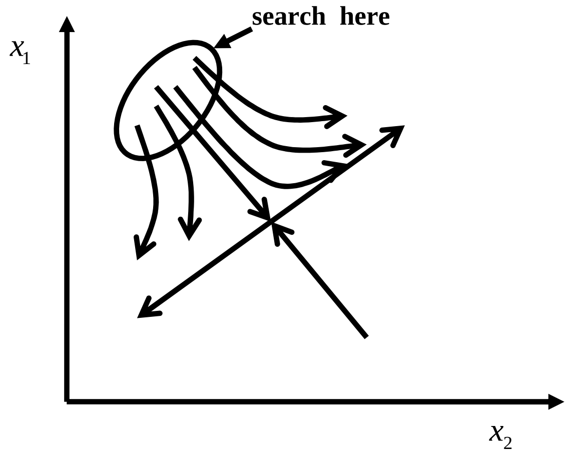

Figure 4
Forward-Shooting in Two-Dimensional Model

The non-linear model (expressed in levels)

The above model specification can be rewritten in levels as follows:

$$\dot{P} = P.\log\left[\frac{P}{\bar{M}}\right]^{\frac{1}{\alpha_2}} \qquad (3)$$

The variables are as for the model above except that upper case letters denote levels, so that:
P = price level; and
$\bar{M}$ = nominal money stock, assumed to be constant.

This specification of the model has two equilibria:

$$P = 0 \qquad (4)$$

and

$$P = \bar{M}\ . \qquad (5)$$

Examination of the linear specification of the model, shows that there are essentially three types of solution path:
(a) the model can jump to the equilibrium given by $p = \bar{m}$; or
(b) p can diverge to $-\infty$; or
(c) p can diverge to $+\infty$.

This corresponds to the following types of solution path for the non-linear specification of the model:
(a) the model can jump to the equilibrium given by $P = \bar{M}$; or
(b) P can converge to the equilibrium given by $P = 0$; or
(c) P can diverge to $+\infty$.

Thus, for the vast majority of initial conditions, the variable P will either diverge; or converge to the stable equilibrium given by $P = 0$. It is highly improbable that a randomly chosen initial condition will lead the model to the equilibrium given by $P = \bar{M}$.

3. SOLVING THE MODELS USING FORWARD-SHOOTING

The forward-shooting approach

In the case of the standard two-dimensional model, forward-shooting requires searching over a grid of initial points (see Figure 4).

Forward-shooting for higher dimensional models requires searching over a larger grid (equal to the entire space) with the dimension of grid equal to the sum of stable and unstable eigenvalues (see Figure 5).

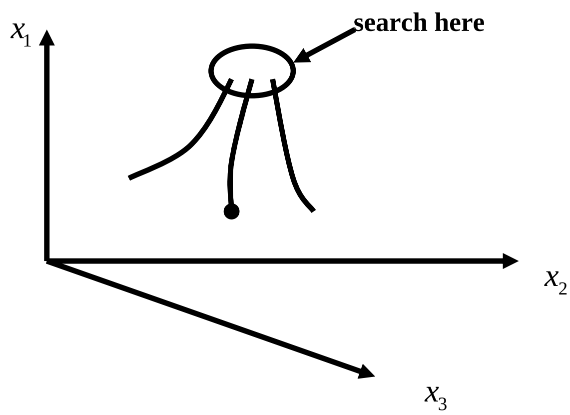

Figure 5
Forward-Shooting in Higher-Dimensional Models

In all cases, the success of the solution can be assessed by evaluating whether or not the chosen solution gives a time-path for each variable that goes from the chosen initial condition to (a small neighbourhood of) the steady-state.

Model solutions

The linear model has a unique steady-state equilibrium, and all other initial conditions other than the one associated with this steady-state, will lead to a divergence in *p*. The forward-shooting approach will reject all divergent solutions. Thus, a systematic search by the forward-shooting algorithm will eventually locate the unique jump and dynamic path that converges towards the unique equilibrium, given by $p = \bar{m}$. This is the solution considered the correct one by the theoretical literature.

The non-linear model has two equilibria. Given a random choice of initial condition, the variable P will either diverge; or converge to the stable equilibrium given by $P = 0$. With probability 1, it will not locate the equilibrium given by $P = \bar{M}$. Thus, the forward-shooting algorithm will almost certainly conclude that a path converging to $P = 0$ is the appropriate solution. This, of course, is the incorrect solution.

4. SOLVING THE MODELS USING REVERSE-SHOOTING

The reverse-shooting approach

In the case of the standard two-dimensional model, reverse-shooting involves just one search in reverse time starting from the neighbourhood of the steady-state. The model dynamics throw the dynamic solution onto the stable arm of the saddle-path. See Figure 6.

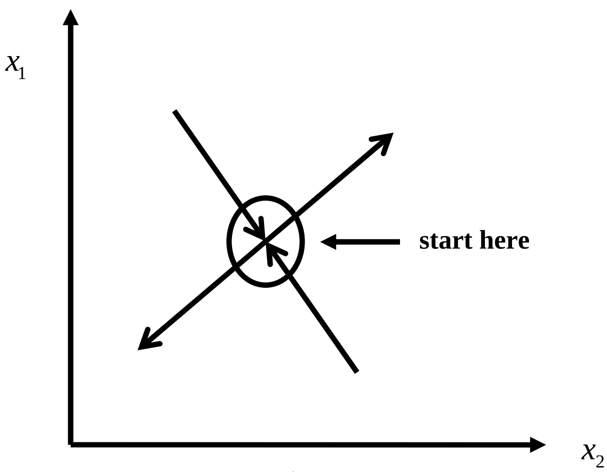

Figure 6
Reverse-Shooting in Two-Dimensional Model

Reverse-shooting for higher dimensional models with more than two stable eigenvalues requires searching over a grid (the stable manifold) with the dimension of grid equal to the number of stable eigenvalues (see Figure 7).

Model solutions

As for forward-shooting, the unique equilibrium of the linear model ensure that the correct equilibrium and associated dynamic path will be chosen.

Because the reverse-shooting algorithm requires the specification of the appropriate equilibrium steady-state, it does not suffer from the problem that it will choose the wrong equilibrium. By specifying the appropriate equilibrium, that is, $P = \bar{M}$, it is possible to ensure that the correct path is chosen by the reverse-shooting algorithm.

Furthermore, the other equilibrium, that is, $P = 0$, will be associated with a stable eigenvalue. Hence, it will be associated with an infinite number of different solution paths, all equally plausible, but unable to be distinguished by the reverse-shooting algorithm.

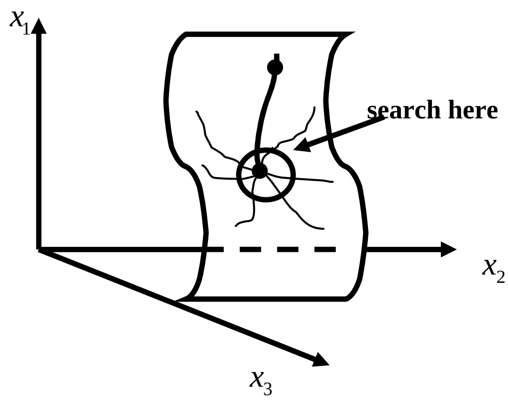

Figure 7
Reverse-Shooting in Higher-Dimensional Models

5. MORE COMPLICATED MODELS

The linear model (expressed in logarithms)

The following model has previously been considered by Stemp and Herbert (2006) and is an extension of that described by Turnovsky (2000). It can be derived from the model of Section 2 by adding a labour market defined by both employment and wages and by introducing sluggish adjustment for both these variables. The model is given by the following set of equations:

$$\bar{m} - p = \alpha_1 y - \alpha_2 \dot{p} \quad (6)$$

$$y = \beta + (1-\gamma)n, 0 < \gamma < 1 \quad (7)$$

$$\dot{n} = \theta(\delta - \gamma n - w + p) \quad (8)$$

$$\dot{w} = \eta(n - \bar{n}) \quad (9)$$

where Greek symbols denote parameters with a positive value, all variables are functions of time and lower-case letters denote logarithms:
y = output (expressed in logarithms);
n = employment (expressed in logarithms);
$\bar{n}$ = full employment (expressed in logarithms);
p = price level (expressed in logarithms);
$\bar{m}$ = nominal money stock (expressed in logarithms), assumed to be constant; and
w = wage rate (expressed in logarithms).

Since this model is linear, it has a unique equilibrium which satisfies the following equations:

$$\bar{m} - p = \alpha_1 y \quad (10)$$

$$y = \beta + (1-\gamma)n \quad (11)$$

$$\delta - \gamma n = w - p \quad (12)$$

$$n = \bar{n} \quad (13)$$

The non-linear model (expressed in levels)

An equivalent model specification can be rewritten in levels as follows:

$$\dot{P} = P.\log\left[\frac{P.A^{\alpha_1} N^{(1-\gamma)\alpha_1}}{\bar{M}}\right]^{\frac{1}{\alpha_2}} \quad (14)$$

$$\dot{N} = N.\log\left[\frac{P.(1-\gamma)AN^{-\gamma}}{W}\right]^{\theta} \quad (15)$$

$$\dot{W} = W.\log\left(\frac{N}{\bar{N}}\right)^{\eta} \quad (16)$$

The variables are as for the model above except that upper case letters denote levels, so that:
Y = output;
N = employment;
$\bar{N}$ = full employment;
P = price level;
$\bar{M}$ = nominal money stock, assumed to be constant; and
W = wage rate.

As for the extended Cagan model, this non-linear specification of the model has multiple equilibria. There is an equilibrium that corresponds to the equilibrium for the linear specification of the model. But there is another equilibrium as well. For example, there is an additional equilibrium given by:

$$P = 0 \quad (17)$$

$$N = \bar{N} \quad (18)$$

$$W = 0 \quad (19)$$

We will demonstrate below that this leads to similar solution issues as arose in the case of the extended Cagan model.

6. SOLVING THESE MODELS USING SHOOTING METHODS

Using forward-shooting

It is possible to derive analytic solutions for the linear version of the complicated model but this model is too complex to derive an analytic solution for the non-linear version. Accordingly, in order to derive comparable solutions for these models it is necessary to calibrate the underlying parameters.

In Stemp and Herbert (2006), we focused on the case when the eigenvalues of the linear model were complex-valued so that the model exhibited cyclic dynamic behavior. We discussed in considerable detail the forward-shooting and reverse-shooting algorithms and the properties of corresponding solutions to the linear model.

In that paper, we showed that the most crucial aspect in ensuring similar solutions, using the two shooting algorithms, was making sure that the step-size of the shooting algorithms was sufficiently small. This previous paper also established that a unique and meaningful solution was derived for the linear model using each algorithm.

In the case of the non-linear model, however, one runs into similar problems to those encountered earlier in the paper when using the forward-shooting algorithm to solve the non-linear version of the extended Cagan model. Specifically, the forward-shooting algorithm chooses a time-path that converges towards the equilibrium given by equations (17-19). One such solution is demonstrated in Figure 8. This would not be considered as the conventional solution to the underlying problem described by the model. It can best be described as the “wrong” solution.

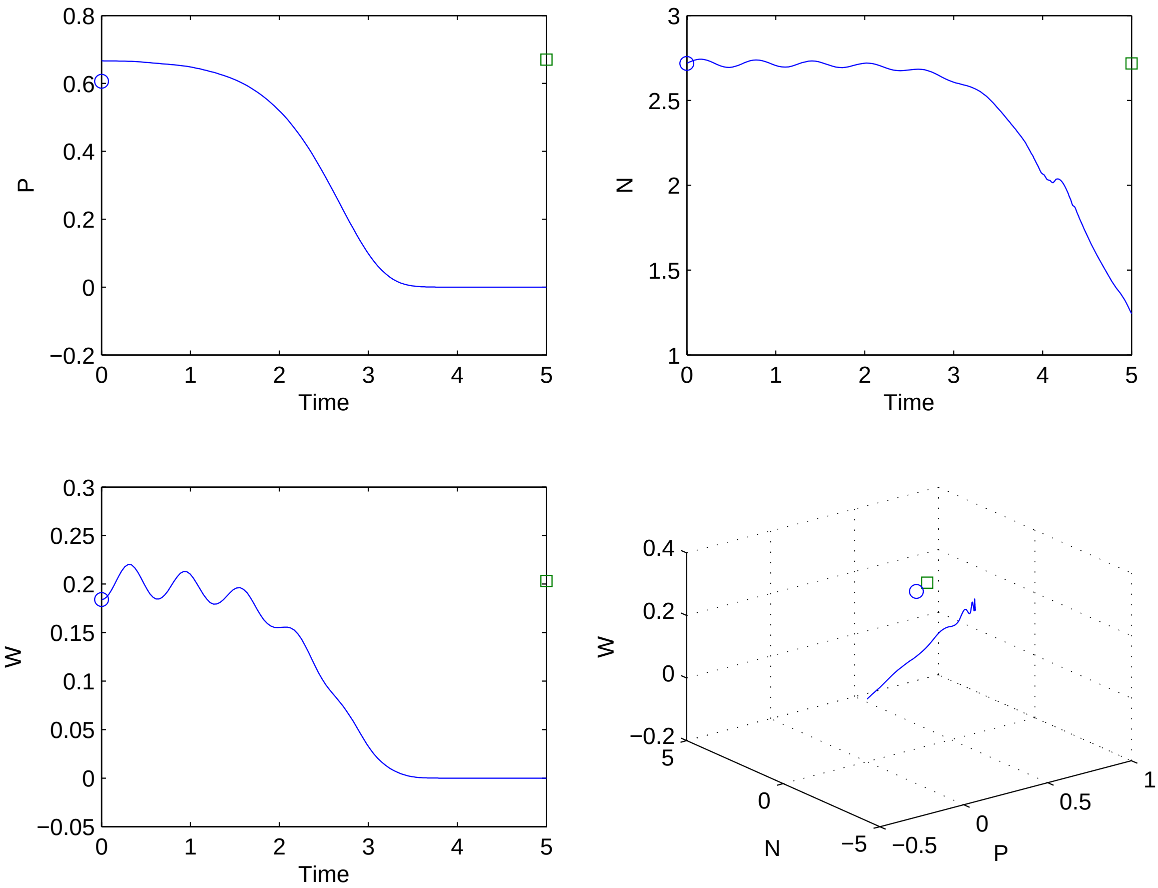

Figure 8
Forward-Shooting: Solution of Non-Linear Model

Using reverse-shooting

As in the previously considered model, the reverse-shooting algorithm works fine for both the linear and non-linear model, choosing the appropriate solution in each case. For the linear model this arises because there is a unique equilibrium.

In the case of the non-linear model, the correct solution is derived provided the desired appropriate initial equilibrium is chosen.

Figure 9 shows the dynamic solution path derived for both linear and non-linear models, with the solutions to the linear model having been transformed so as to be comparable with solutions for the non-linear model. The solutions are identical in both cases and converge to a meaningful (non-zero) equilibrium.

7. CONCLUSION

In this paper, we have considered two macroeconomic models with alternative linear and non-linear specifications. One version of each model, expressed in levels, is non-linear and has at least two steady-state equilibria. One of these equilibria has an economically-meaningful interpretation, while the other does not have a sensible economic interpretation. A second version of each model, expressed in logarithms, is linear and has a unique steady-state equilibrium, which corresponds to the economically-meaningful equilibrium of the non-linear version of the model.

The dynamic solution of each model has a combination of stable and unstable eigenvalues so that any dynamic solution requires the calculation of appropriate "jumps" in endogenous variables. Attempts to solve these models, using forward-shooting and reverse-shooting algorithms, show that the forward-shooting algorithm chooses the "wrong" solution for the non-linear model, but the "right" solution for the linear model. The reverse-shooting algorithm chooses the "right" solution in both cases. We have demonstrated how this result is driven by particular properties of each model.

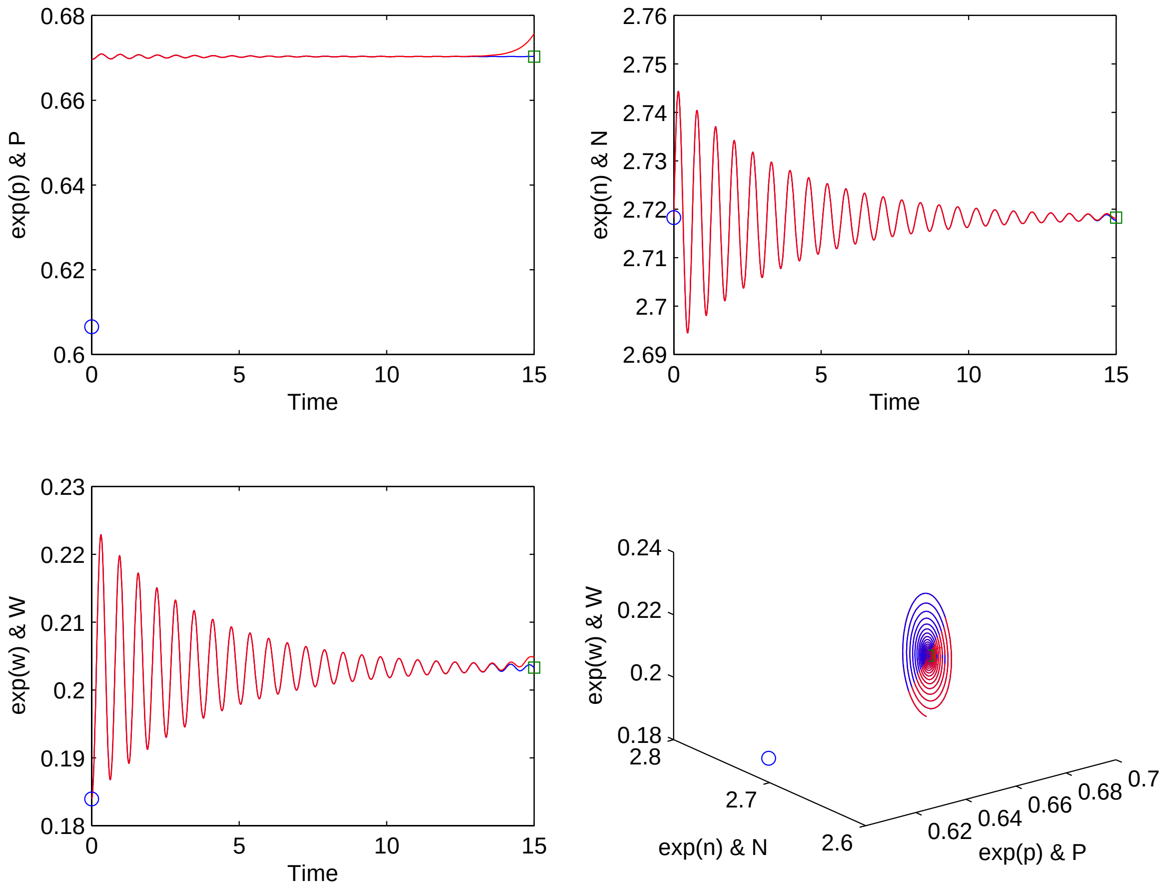

Figure 9
Reverse-Shooting: Solution of Both Linear and Non-Linear Models

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