A simulation of the non-cooperative Nash equilibrium model

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Abstract: The present analysis is an application of the continuous time replicator dynamic to a market equilibrium model. Let us consider that there are two automobile corporations that make two types of automobiles, namely deluxe and ordinary cars. Each corporation has its own production limits for these two types of automobiles, and each produces two types of automobiles so as to maximize profits, calculated to include conjectural variations between the two firms. Let us define x_i^p as the quantity of automobiles for corporation p (p=1,2) and type i (i=1,2). The non-cooperative Nash equilibrium solution is obtained after assuming the profit maximization behavior for each corporation under the conditions of normalized constraints as : $x_1^1+x_2^1=1$ and $x_1^2+x_2^2=1$ and non-negative constraints, $x_i^p \geq 0$ (p,i=1,2). To get the Nash equilibrium point, the profit function of corporation p is specified as : $E^p(\mathbf{x}^1,\mathbf{x}^2)=\sum f_i^p(x_i^p)-\sum\sum\theta_{ij}^p x_i^p x_j^q$ (p=1,2,i,j=1,2(i≠j,p≠q)), where θ_{ij}^p is the conjectural variations. The replicator dynamic for corporation 1 is specified as : $dx_1^1(t)/dt=x_1^1(t)x_2^1(t)\{df_1^1(x_1^1(t)/dx_1^1-df_2^1(x_2^1(t))/dx_2^1-\sum(\theta_{1j}^1-\theta_{2j}^1)x_j^2\}$ and $dx_2^1(t)/dt= x_1^1(t)x_2^1(t)$ $\{df_2^1(x_2^1(t)\ /dx_2^1-df_1^1(x_1^1(t))/dx_1^1-\sum(\theta_{2j}^1-\theta_{1j}^1)x_j^2\}$. Changes in the values of θ_{ij}^p and the parameters included in the profit functions make possible many alternative production mixes between deluxe and ordinary cars for the two corporations.

Keywords: *non-cooperative Nash equilibrium, replicator dynamic, conjectural variation*

1. Introduction

There are various types of markets ranging from competitive markets to monopoly markets. In contemporary industrial society, oligopoly markets prevail in manufacturing industries such as automobiles, electric appliances, PCs, etc. One of the key problems facing producers is to decide on how much to produce and how to allocate production capacity among a mix of goods.

The Nash equilibrium model is a useful tool for clarifying the structure of oligopoly markets. Here, we will propose a simple model of the Nash equilibrium and use a simulation method to derive an optimal solution for production decisions by rival firms.

Section 2 explains the model in general, and section 3 denotes how to derive the optimal production solution using numerical methods. The present analysis uses the numerical method of the continuous time replicator dynamic that is used for the game-theoretic problem on ecology, group genetics and evolutionary economics. Section 4 proposes a simulation model and the results are reported in section 5. Finally section 6 presents some conclusions.

2. Non-cooperative Nash equilibrium model and resource allocation

A game of P-players is considered in general with profit function E^p, decision variables $\mathbf{x}^p \in R^n$. All profit functions E^p(p=1,2,…, P) are assumed to be the function of $\mathbf{x}^1,\ldots,\mathbf{x}^P$. The constraints of $\mathbf{x}^p$ are independent of other players and are included in S^P. The problem of a non-cooperative game is formulated as:

$$\text{Max}_{\,xp}\ E^p(\mathbf{x}^1,\ldots,\mathbf{x}^P) \text{ subject to } \mathbf{x}^p \in S^P \tag{1}$$

For simplicity, X is defined as

$$\mathbf{X}=(\mathbf{x}^{1T},\ldots,\mathbf{x}^{PT})^T \tag{2}$$

where T is used for transposition. As a rational solution for a non-cooperative game, the Nash equilibrium is proposed. The Nash equilibrium solution, $\mathbf{X}^*=(\mathbf{x}^{*1T},\ldots,\mathbf{x}^{*PT})^T$, is obtained by the following P-simultaneous equations including maximizing operation as:

$$E^p(\mathbf{x}^{*1},\ldots,\mathbf{x}^{*p-1},\mathbf{x}^{*p}, \mathbf{x}^{*p+1}\ldots,\mathbf{x}^{*P})$$
$$= \max_{xp} E^p(\mathbf{x}^{*1},\ldots,\mathbf{x}^{*p-1},\mathbf{x}^{p}, \mathbf{x}^{*p+1}\ldots,\mathbf{x}^{*P})$$
$$\text{subject to } \mathbf{x}^p\in S^P \ (p=1,\ldots,P) \tag{3}$$

where

$$S^P=\{\mathbf{x}^p \mid \sum x_i^p = 1,\ x_i^p \geq 0,\ i=1,\ldots,n\} \tag{4}$$

3. Dynamics to solve Nash equilibrium in terms of resource allocation

A player, under the condition of fixing the values of decision variables for other players, considers gradient vector with respect to own decision variables

$$\nabla_{xp}E^p(\mathbf{X}) = (\partial E^p(\mathbf{X})/\partial x_1^p,\ldots,\partial E^p(\mathbf{X})/\partial x_n^p)^T \tag{5}$$

and the gradient system by the steepest ascent method for changing in x^p as:

$$d\mathbf{x}^p(t)/dt =\nabla_{xp}E^p(\mathbf{X}(t)),\ p=1,\ldots,P \tag{6}$$

The trajectory of dynamics surely does not satisfy the constraint set of (4). To avoid such a situation by reducing the velocity of the gradient when reaching the border of the non-negative constraint of (4), equation (6) is modified as

$$d\mathbf{x}^p(t)/dt \quad =Q_A^{M(\mathbf{x}p)}M(\mathbf{x}^p)^{-1}\nabla_{xp}E^p(X(t)),\quad p=1,\ldots,P \tag{7}$$

where $M(\mathbf{x}^p)$ is the variable metric matrix

$$M(\mathbf{x}^p)=\mathrm{diag}[(1/x_i^p)], \tag{8}$$

and $Q_A^{M(\mathbf{x}p)}$ is the variable metric projection matrix

$$Q_A^{M(\mathbf{x}p)} =I-M(\mathbf{x}^p)^{-1}A^T(AM(\mathbf{x}^p)^{-1}A^T)^{-1}A$$
$$=I-\mathrm{diag}[x_i^p][1,\ldots,1]^T[\sum x_i^p]^{-1}[1,\ldots,1] \tag{9}$$

onto the hyper-plane by normalized equality, where A= [1,…,1]

$$[1,\ldots,1]\mathbf{x}^p = 1 \tag{10}$$

The elements of (7) is described as

$$dx_i^p(t)/dt$$
$$=\{x_i^p(t)(1-x_i^p(t))(\partial E^p(\mathbf{X})(t)/\partial x_i^p)-\sum_{j\neq i}x_i^p(t)x_j^p(t)(\partial E^p(\mathbf{X}(t))/\partial x_j^p)\} \tag{11}$$

that is called the replicator dynamic with interference (cf. Fisher (1930), Schuster and Sigmund (1983), Sigmund (1984)).

4. Application of the replicator dynamic to non-cooperative Nash equilibrium for resource allocation

For simplicity, we consider a two-person (P=2) game with two products (n=2). Even in the simplest model, there is no loss of generality in the model described in section 2. As a concrete example, there are two automobile companies such as Toyota and Honda, and both of them produce two types of automobiles, namely ordinary and luxury cars. The decision variables are $\mathbf{x}^1=(x_1^1,x_2^1)^T$ and $\mathbf{x}^2=(x_1^2,x_2^2)^T$ where suffix indicates product and superfix indicates firm. The profit functions of each firm are

$$E^1(\mathbf{x}^1,\mathbf{x}^2)=\sum f_i^1(x_i^1)-\sum\sum\theta_{ij}^1x_i^1x_j^2 \tag{12a}$$
$$E^2(\mathbf{x}^1,\mathbf{x}^2)=\sum f_i^2(x_i^2)-\sum\sum\theta_{ij}^2x_i^2x_j^2 \tag{12b}$$

where f_i^p is the gain from product i of firm p, and θ_{ij}^p is a loss parameter suffered by the product of x_j^q ($q\neq p$) when p produces x_i^p and q produces x_j^q. In economics of firms, gain is the corporate profit and loss is various kinds of conjectural costs. The constraints are:

$$x_1^1+x_2^1=1 \tag{13a}$$
$$x_1^2+x_2^2=1 \tag{13b}$$

The dynamic for firm 1 of (11) becomes

$$dx_1^1(t)/dt = x_1^1(t)x_2^1(t)\{df_1^1(x_1^1(t)/dx_1^1 - df_2^1(x_2^1(t))/dx_2^1 - \sum(\theta_{1j}^1 - \theta_{2j}^1)x_j^2\} \quad (14a)$$

$$dx_2^1(t)/dt = x_1^1(t)x_2^1(t)\{df_2^1(x_2^1(t)/dx_2^1 - df_1^1(x_1^1(t))/dx_1^1 - \sum(\theta_{2j}^1 - \theta_{1j}^1)x_j^2\} \quad (14b)$$

$$dx_1^2(t)/dt = x_1^2(t)x_2^2(t)\{df_1^2(x_1^2(t)/dx_1^2 - df_2^2(x_2^2(t))/dx_2^2 - \sum(\theta_{1j}^2 - \theta_{2j}^2)x_j^1\} \quad (14c)$$

$$dx_2^2(t)/dt = x_1^2(t)x_2^2(t)\{df_2^2(x_2^2(t)/dx_2^2 - df_1^2(x_1^2(t))/dx_1^2 - \sum(\theta_{2j}^2 - \theta_{1j}^2)x_j^1\} \quad (14d)$$

5. Simulation

For firm 1, the gain from products 1 and 2 are indicated respectively as:

$$f_1^1(x_1^1) = -2(x_1^1 - 1.5)^2 + 3$$

$$f_2^1(x_2^1) = -0.5(x_2^1 - 2.5)^2 + 3,$$

The difference in the f_i^p functions is due to that of production technology for products and firms. The profit function, $E^1(\mathbf{x}^1,\mathbf{x}^2)$, for firm 1 is specified as:

$$E^1(\mathbf{x}^1,\mathbf{x}^2) = f_1^1(x_1^1) + f_2^1(x_2^1) - (\theta_1^1 x_1^1 x_1^2 + \theta_2^1 x_2^1 x_2^2)$$

where $\theta_1^1 = \theta_{11}^1$, $\theta_2^1 = \theta_{22}^1$ and θ_{12}^1 and θ_{21}^1 are assumed to be zero. For firm 2, the gain functions for products 1 and 2 are, respectively:

$$f_1^2(x_1^2) = -2(x_1^2 - 1.7)^2 + 3$$

$$f_2^2(x_2^2) = -0.5(x_2^2 - 2.7)^2 + 3,$$

The profit function, $E^2(\mathbf{x}^1,\mathbf{x}^2)$, for firm 2 is specified as:

$$E^2(\mathbf{x}^1,\mathbf{x}^2) = f_1^2(x_1^1) + f_2^2(x_2^2) - (\theta_1^2 x_1^2 x_1^1 + \theta_2^2 x_2^2 x_2^1)$$

where $\theta_1^2 = \theta_{11}^2$, $\theta_2^2 = \theta_{22}^2$ and θ_{12}^2 and θ_{21}^2 are assumed to be zero.

In the simulation, we will change the four values of θ_1^1, θ_1^2, θ_2^1 and θ_2^2, and get the Nash equilibrium solution for the decision variables of x_1^1, x_1^2, x_2^1 and x_2^2. It is difficult to show this eight-dimensional space in a graph, so we show a three-dimensional figure that moves x_1^1, θ_1^1 and θ_2^1 after fixing θ_1^2 =1and θ_2^2=1. Figure 1 indicates the change in the Nash equilibrium value for x_1^1.

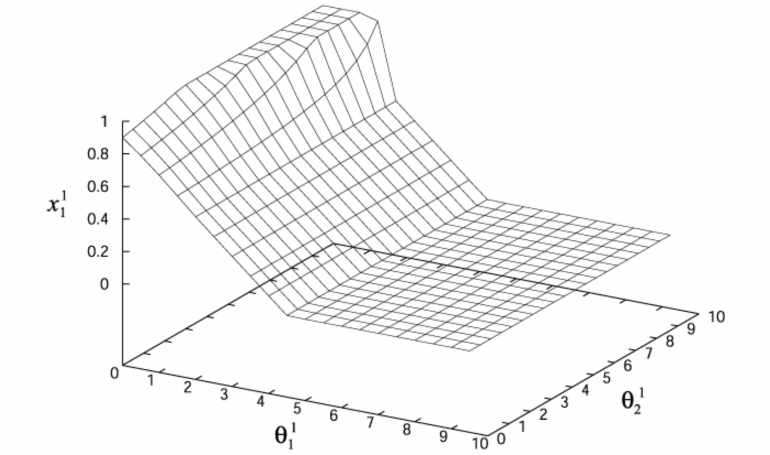

Figure1 : Equilibrium of x_1^1 in response to $\left(\theta_1^1, \theta_2^1\right)\left(\theta_1^2 = 1.0, \theta_2^2 = 1.0\right)$

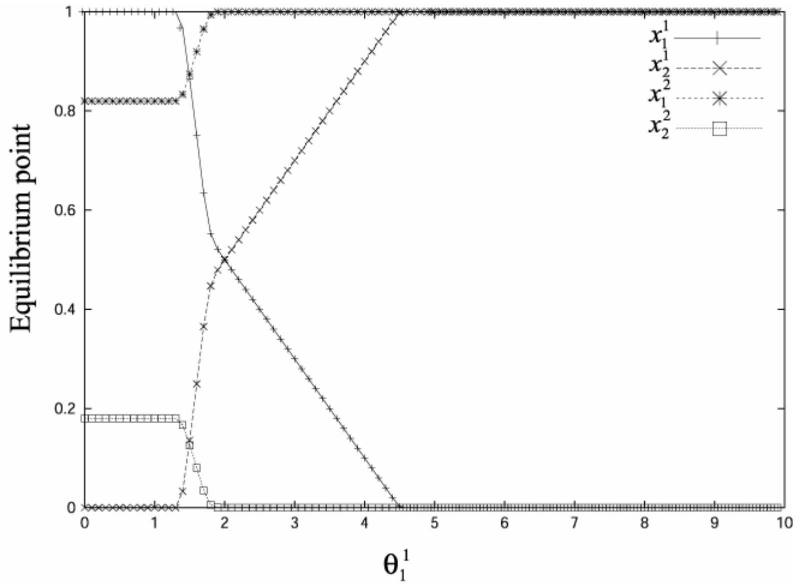

Figure2 : Changes in equilibrium point varying

$\theta_1^1 \left(\theta_2^1 = 9.0, \theta_1^2 = 1.0, \theta_2^2 = 1.0\right)$

Figure 2 indicates the Nash equilibrium solution for x_1^1, x_1^2, x_2^1 and x_2^2 by changing one parameter, θ_1^1. From figure 2, we understand the changes of the product mix for firms 1 and 2 due to changes in the parameter of θ_1^1. When θ_1^1 is between 0 and 1.5, firm 1 only produces product 1, while firm 2 divides production between the two products, with 82% of total production for product 1 and 18% for product 2 . When θ_1^1 is between 1.5 and 2, firm 1 produces both products, and firm 2 increases the share of product 2. At the point where θ_1^1 is 2, the share of production of products 1 and 2 for firm 1 is equal (50%), while firm 2 produces only product 1. According to the changing value of θ_1^1, the share of resource allocation changes drastically. This means that in an oligopoly market without cooperation, anticipating the production decisions of rival firms plays an important role in making production decisions.

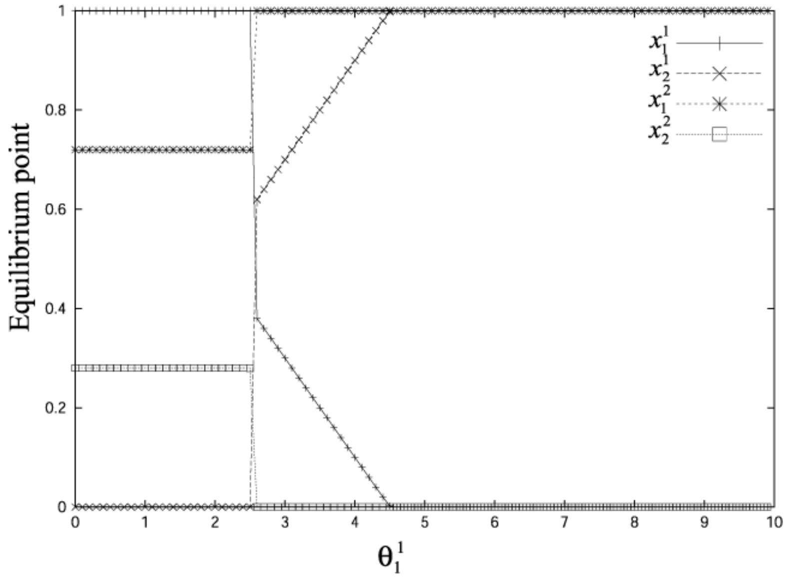

Figure3 : Changes in equilibrium point varying

$\theta_1^1 \left(\theta_2^1 = 8.5, \theta_1^2 = 1.5, \theta_2^2 = 1.0\right)$

Figure 3 indicates a big discrepancy at θ_1^1=2.5. Before the point of 2.5, firm 1 produces only product 1, while firm 2 produces both products, with 70% devoted to production of product 1 and 30% for product 2. However, after the point where θ_1^1=2.5, firm 2 specializes in production of product 1. On the other hand, firm 1 produces both products until the point where θ_1^1=4.5. After θ_1^1=4.5, firm 1 produces only product 2. According to changes in θ_1^1, the product-mix for firms 1 and 2 are changed drastically.

Figure 4 indicates changes in the production share for products 1 and 2 by the two firms less drastic changes than in previous example. When θ_1^2 is between 2 and 4, the two firms produce both products at the same time.

6. Conclusion

Using the Nash equilibrium simulation model, we can generate various kinds of optimal paths for changing the conjecture between two firms. In the simulation the share of products produced varies according to changes in conjecture. To test the validity of the Nash equilibrium model, we need to construct an empirical model using existing data for oligopoly markets by estimating profit functions. The conjectural factor is calculated by the gap between observed data and estimated values.

7. Acknowledgement

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8. References

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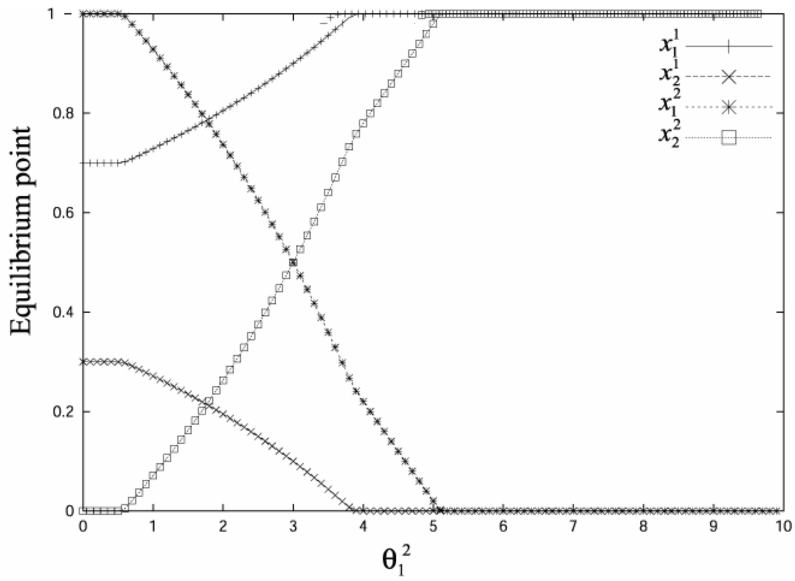

Figure4 : Changes in equilibrium point varying

$\theta_1^2 \left(\theta_1^1 = 1.0, \theta_2^1 = 1.0, \theta_2^2 = 1.0\right)$