New Approximation For Tide-Induced Water Table Fluctuations At A Sloping Beach

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Abstract: Tidal motions of the water table adjacent to a sloping beach are investigated theoretically. It is shown that a previous analytical solution by Nielsen (1990) only contained part of the first-order solution. A new approximation is introduced to provide a model more consistent with the special case of a vertical beach. Comparisons between the Nielsen (1990) and the present solutions indicate that the former solution is inadequate for a beach with small slopes. Numerical examples demonstrate the significant influence of higher-order components and the beach slope.

Keywords: *Groundwater; Sloping beach; Perturbation approximation; Tidal oscillations*

1. INTRODUCTION

The groundwater table adjacent to a sandy beach is of great interest in the management of coastal aquifers. In particular, an accurate prediction of dynamic groundwater hydraulics in coastal zones is important to erosion (Grant, 1948), saltwater intrusion (Dagan and Zeitoun, 1998) and biological activity (Pollock and Hummon, 1971, McArdle and McLachan, 1991). Most studies of coastal aquifers are based on the Boussinesq equation, (Dagan, 1967, Bear, 1972, Parlange et al., 1984, Barry et al., 1996, Li et al., 2000).

Most previous investigations have been limited to the case of a vertical beach. Only a few researchers considered the case of a sloping beach. Nielsen (1990) and Li et al. (2000) used analytic solutions of the Boussinesq equation to examine the effects of a moving shoreline on tide-induced water table fluctuations. In Nielsen's solution only part of higher-order components are included, the details will be discussed later. Li et al. (1997) further considered capillary effects on tidal fluctuations through numerical simulations, while some field studies have been reported in Raubenheimer et al. (1999).

In this paper, a new analytical solution will be derived. The new approximation is applicable to a large range of beach slopes. Based on the new solution, the effect of high-order components and different beach slopes on tidal fluctuation in coastal aquifers will be investigated.

2. BOUNDARY VALUE PROBLEM

2.1 Problem set-up

The flow is taken to be homogeneous, isothermal and incompressible in a rigid porous medium. The configuration of the tidal forced dynamic groundwater flow is shown in Figure 1.

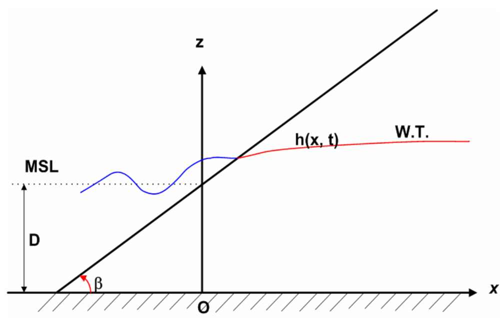

Figure 1. Definition sketch.

In Figure 1, *h(x,t)* is the total water table height and *D* is the average height of the water table at the coastline. Hence the condition that the water table height at the boundary of the ocean and coast (i.e., $x = x_0(t)$) is equal to the specified tidal variation becomes

$$h(x_0(t),t) = D(1+\alpha\cos\omega t) \tag{1}$$

Here the aquifer thickness (D), amplitude parameter (α) and the frequency (ω) are prescribed constants. $\alpha=A/D$ is a dimensionless *amplitude parameter*, representing the ratio of the maximum tidal variation, A, to the average height of the water table, D. Note that we have neglected the seepage face. If β is the slope of the beach, the horizontal extent of the tidal variation is

$$x_0(t) = A\cot\beta\cos\omega t\ . \tag{2}$$

The standard linear solution has a *decay length*, L, defined by (Nielsen, 1990)

$$L = \sqrt{2KD/n_e\omega}\ . \tag{3}$$

L is the length scale for significant variations in the x direction. K and n_e are the hydraulic conductivity and the porosity of the soil respectively. Here we consider shallow water flows and hence define the *shallow water parameter*, ε, as

$$\varepsilon = \frac{D}{L} = \sqrt{\frac{Dn_e\omega}{2K}}\ . \tag{4}$$

This represents the ratio of the water table height to the linear decay length. Note that ε is entirely controlled by the material constants and the prescribed boundary condition (1). For shallow water flows, $\varepsilon \ll 1$. Thus in this problem there are three independent parameters defined by the material and the boundary condition: the shallow water parameter, ε, the amplitude parameter α and the beach slope β.We construct solutions valid for small ε and α and a large range of β.

Following Barry *et al.* (1996), the fluid potential (ϕ) satisfies Laplace's equation since we assume the fluid to be incompressible (Bear, 1972), i.e.

$$\phi_{xx} + \phi_{zz} = 0 \tag{5}$$

where $\phi = z + p/\rho g$. The fluid potential in (5) is to be solved subject to the following boundary conditions,

$$\phi_z(x,0;t) = 0\ , \tag{6a}$$
$$\phi(x,h;t) = h(x,t) \tag{6b}$$
$$\phi_x(\infty,z;t) = 0 \tag{6c}$$
$$n_e\phi_t = K[\phi_x^2 + \phi_z^2] - K\phi_z \tag{6d}$$

and (1).

2.2 Non-dimensional parameters

Before constructing the solution, it is convenient to rewrite the governing equations in a dimensionless form. Thus, we define the new non-dimensional variables

$$X = \frac{x}{L},\ H = \frac{h}{D},\ T = \omega t,\ \Phi = \frac{\phi}{D}\ . \tag{7}$$

Since $x_0(t)$ is a time-dependent boundary, we also introduce a transformation so that in the new coordinates the boundary is fixed (Li et al., 2000):

$$X_1 = X - X_0(T),\ T_1 = T\ , \tag{8}$$

where $X_0(T) = \alpha\varepsilon S\cos T$ and $S = \cot(\beta)$.

Then

$$\frac{\partial f}{\partial T} = \frac{\partial f}{\partial T_1} + \frac{\partial f}{\partial X_1}\frac{\partial X_1}{\partial T} = \frac{\partial f}{\partial T_1} + \alpha\varepsilon S\sin(T)\frac{\partial f}{\partial X_1}. \tag{9}$$

where f is a dependent variable.

2.3 Shallow water expansions

Following Dagan (1967), the standard non-linear kinematic boundary condition is expanded in powers of ε:

$$H = \sum_{n=0}^{\infty}\varepsilon^n H_n \text{ and } \Phi = \sum_{n=0}^{\infty}\varepsilon^n\Phi_n\ , \tag{10}$$

resulting in the following equations up to the second-order:

$$O(1): 2H_{0T_1} = (H_0H_{0X_1})_{X_1}\ , \tag{11a}$$
$$O(\varepsilon): 2[H_{1T_1} + \alpha\sin(T_1)SH_{0X_1}] = (H_0H_1)_{X_1X_1}\ , \tag{11b}$$
$$O(\varepsilon^2): 2[H_{2T_1} + \alpha\sin(T_1)SH_{1X_1}] = \tfrac{1}{2}(H_1^2)_{X_1X_1} + (H_0H_2)_{X_1X_1} + \frac{1}{3}(H_0^3H_{0X_1X_1})_{X_1X_1}\ . \tag{11c}$$

Note that (11a) is the well-known Boussinesq equation. The subscripts X_1 and T_1 denote derivatives with respect to X_1 and T_1.

These are to be solved subject to (1) and the condition that, far from the coastline, the water table fluctuations should vanish:

$$H_{nX_1} = 0 \quad as\ x \to \infty \tag{12}$$

The appropriate boundary conditions for (11) at each order are then

$$H_0(0,T_1) = 1 + \alpha\cos(T_1),$$
$$H_1(0,T_1) = \cdots = 0. \tag{13}$$

ANALYTICAL SOLUTIONS

3.1 Previous solution-Nielsen (1990)

Nielsen's (1990) solution is re-organised in a non-dimensional form here:

$$H_{Nielsen} = 1 + \alpha e^{-X}\cos(T - X) + \alpha\varepsilon_N\left[\frac{1}{2} + \frac{\sqrt{2}}{2}e^{-\sqrt{2}X}\cos(2T - \sqrt{2}X + \frac{\pi}{4})\right], \tag{14}$$

where $\varepsilon_N = A\cot(\beta)/L = \alpha\varepsilon S$.

Note that Nielsen (1990) uses ε_N as the perturbation parameter. This parameter (ε_N) can be greater than unity when the slope is small. Then this approximation cannot be used and the solution is invalid for certain value of slopes. Also, $\varepsilon_N = 0$ when $\beta = \pi/2$ for a vertical beach, so (13) reverts to the linear solution. In this analysis ε_N will not be used as a small parameter, but it will be assumed to be an O(1) parameter.

3.2 New solution

To have a basic understanding of the problem, we only consider the zero-order and first-order (in ε) solutions in this paper.

Solution of O(1)

As (11a) is a non-linear equation, it is convenient to introduce α as a second small parameter. In general, the tidal wave amplitude (A) is small compared with the mean tide level (D) and hence $\alpha \ll 1$. Then H_0 is expanded as

$$H_0 = 1 + \sum_{n=1}^{\infty}\alpha^n H_{0n}. \tag{15}$$

Substituting (15) into (11a), the zero-order boundary value problem can be sorted in order of α:

$$O(\alpha):\quad 2H_{01T_1} = H_{01X_1X_1}, \tag{16a}$$

and

$$O(\alpha^2):\quad 2H_{02T_1} = H_{02X_1X_1} + (H_{01}H_{01X_1})_{X_1}. \tag{16b}$$

Equation (16a) and (16b) are the linear equations for a vertical beach. The solutions are (Parlange et al., 1984)

$$H_{01} = e^{-X_1}\cos(T_1 - X_1). \tag{17}$$

$$H_{02} = \frac{1}{4}(1 - e^{-2X_1}) + \frac{1}{2}[e^{-\sqrt{2}X_1}\cos(2T_1 - \sqrt{2}X_1) - e^{-2X_1}\cos 2(T_1 - X_1)] \tag{18}$$

Solution of O(ε)

Following the O(1) procedure, H_1 is expanded as

$$H_1 = \sum_{n=1}^{\infty}\alpha^n H_{1n}. \tag{19}$$

Substituting (19) into (11b), the first-order boundary value problems are

$$O(\varepsilon\alpha):\quad 2H_{11T_1} = H_{11X_1X_1} \tag{20a}$$

$$O(\varepsilon\alpha^2):\quad 2H_{12T_1} + 2\sin(T_1)SH_{01X_1} = H_{12X_1X_1} + (H_{01}H_{11})_{X_1X_1} \tag{20b}$$

The solution of (20a), subject to (12) and (13), is $H_{11} = 0$, and the solution of (20b) is

$$H_{12} = \frac{1}{\sqrt{2}}S\{\frac{1}{\sqrt{2}} - e^{-X_1}\cos(X_1 - \frac{\pi}{4}) + e^{-\sqrt{2}X_1}\cos(2T_1 - \sqrt{2}X_1 + \frac{\pi}{4}) - e^{-X_1}\cos(2T_1 - X_1 + \frac{\pi}{4})\}. \tag{21}$$

In summary, the present solution can be written as

$$H = 1 + \alpha e^{-X_1}\cos(\theta_1) + \alpha^2\{\frac{1}{4}(1 - e^{-2X_1}) + \frac{1}{2}[e^{-\sqrt{2}X_1}\cos(\theta_2) - e^{-2X_1}\cos(2\theta_1)]\} + \frac{1}{\sqrt{2}}\varepsilon\alpha^2 S\{\frac{1}{\sqrt{2}} - e^{-X_1}\cos(X_1 - \frac{\pi}{4}) + e^{-\sqrt{2}X_1}\cos(\theta_2 + \frac{\pi}{4}) - e^{-X_1}\cos(\theta_3 + \frac{\pi}{4})\}. \tag{22}$$

where $\theta_1 = T_1 - X_1$, $\theta_2 = 2T_1 - \sqrt{2}X_1$ and $\theta_3 = 2T_1 - X_1$.

Comparing (14) and (22), it is clear that Nielsen's solution only contains part of (22), and his solution does not satisfy the boundary condition at $x = x_0$, i.e., $h(x_0(t), t) = D + A\cos(\omega t)$.

3.3 Solution for a vertical beach

As mentioned in 3.1, when β = π/2 Nielsen's (1990) solution reverts to the linear solution. If we substitute β = π/2 into (22), i.e., S=0, X=X_1, we have

$$H_{0v}(X,T) = 1 + \alpha e^{-X}\cos(T - X)$$
$$+\alpha^2\{\frac{1}{4}(1-e^{-X}) - \frac{1}{2}e^{-2X}\cos 2(T-X)$$
$$+\frac{1}{2}e^{-\sqrt{2}X}\cos(2T-\sqrt{2}X)\}\,. \qquad (23)$$

This is identical to the previous solution for a vertical beach (Plarange et al., 1984).

4. RESULTS AND DISCUSSIONS

The objectives of this paper are to derive a completed first-order solution (ε) and to investigate the effects of (1) higher-order components and (2) the slope of the beach, on the tide-induced water table fluctuations in a coastal aquifer.

4.1 Comparisons with Nielsen (1990)

By comparing (22) and (14), it is clear that Nielsen's (1990) solution excluded the terms for cos $(\theta_2 + \pi/4)$ and the non-oscillating term, $e^{-X_1}\cos(X_1 - \pi/4)$ and the O(α^2). This indicates that the first-order solution in Nielsen is not complete.

Numerical comparisons between Nielsen's solution (14) and the present solution (22) are plotted in Figures 2 and 3 for various slopes of beach.

As shown in Figure 2, the solution proposed by Nielsen (1990) does not match the boundary condition at X=X_0, which can be seen in the figure. A significant difference between two solutions near the intersection of ocean and inland is found in the figure, especially for small slope (Figure 2d).

Figure 3 further illustrates the difference between Nielsen (1990) solution and the present solution. It is clear both solutions are close to each other for a larger slope, but significant difference between two solutions is observed for a smaller slope, such as 15°. Also, there are phase differences among the three solutions, which can be attributed to the effects of higher-order components.

4.2 Effects of higher-order components

One objective of this study is to investigate the effects of the higher-order component. Figure 3 also demonstrate the influence of higher-order components on the water table level in a sloping beach. Significant differences between linear and second order solutions have been observed. This difference increases as the slope decreases.

4.3 Effects of slope

The second objective of this paper is to examine the effects of slope. Figure 3 also illustrates the tide-induced water table level (H) versus time (T) for various values of the slopes. Among these, β=90° represents the case of a vertical beach. The figure clearly indicates that the water table level decreases as the slope decreases.

CONCLUSIONS

In this study, a new perturbation approximation for the tide-induced water table fluctuations in a sloping beach is proposed. The shortcoming of the previous work has been overcome in the new approximation. The special case (i.e., for a vertical beach) can be obtained directly from the new solution. Numerical examples demonstrate the significant influences of higher-order components and the beach slope.

The solution presented here is up to the zero-order boundary value problem, which is based on Boussinesq equation and first-order correction. The second-order solution and more detailed discussions will be reported in Jeng et al. (2003).

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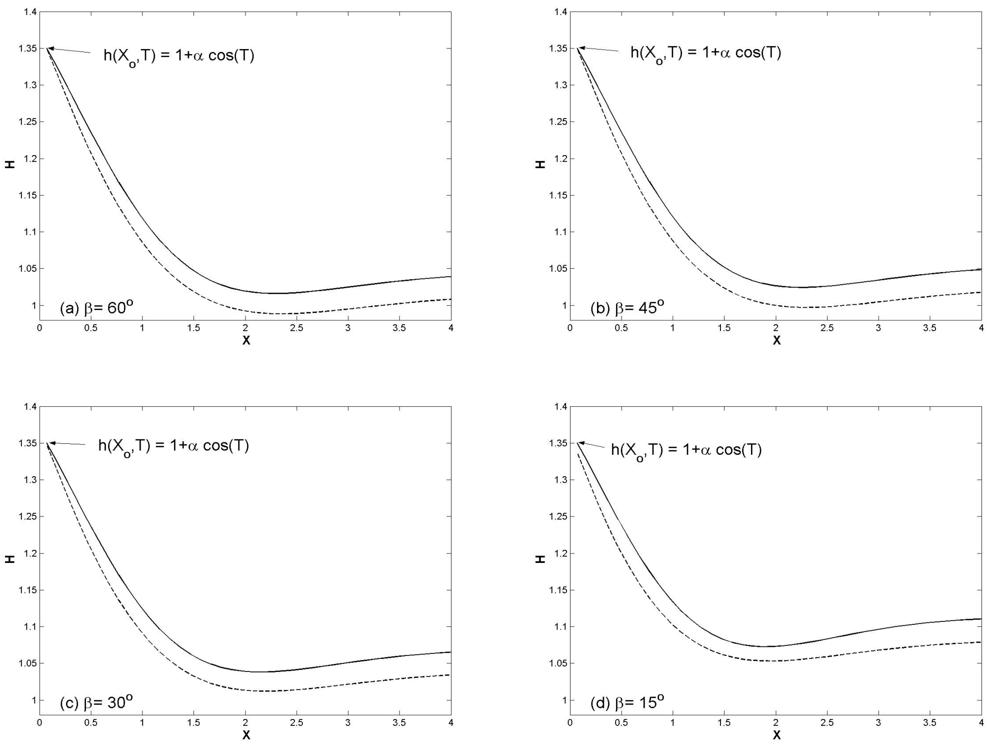

Figure 2. Comparisons between Nielsen (1990)'s solution, dashed line, and the present solution, solid line for α=ε=0.35, T=0.

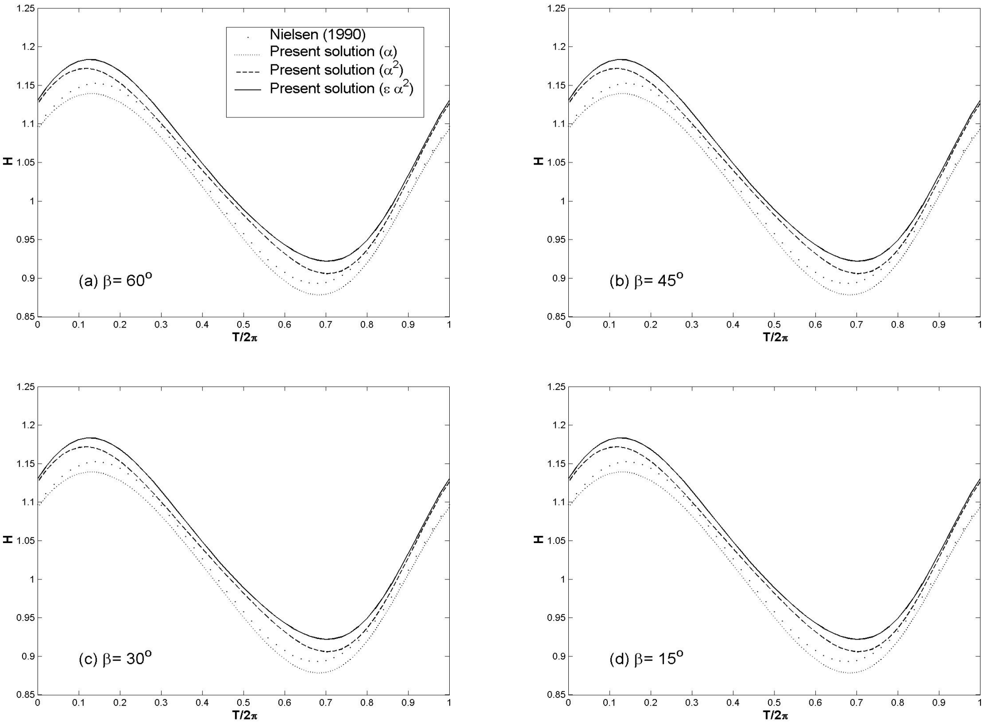

Figure 3. Comparisons of Nielsen (1990) and the present solutions (α=ε=0.35, X=1)