Steady recharge-induced groundwater flow over a plane bed: nonlinear and linear solutions

T.G. Chapman

UNSW School of Civil and Environmental Engineering

Abstract: The form of the free surface in two-dimensional groundwater flow over a plane impermeable bed, induced by percolating rainfall or irrigation, has been the subject of considerable study since the 1960's. The earlier analyses were related to drainage design, but more recently, and particularly for the case of a sloping bed, the motivation has been the development of models of hillslope hydrology, and there has been particular interest in linearisation of the Boussinesq equation. This paper first develops a quasi-analytical solution for the maximum water-table height in the case of a horizontal bed, and compares this with the Boussinesq and linearised Boussinesq solutions. In the case of a sloping bed, it is shown that the Boussinesq equation can be reduced to a single dimensionless form, dependent on the system of coordinates and the direction of the recharge. Numerical solutions are given for the steady flow case, indicating the conditions under which the linearised solutions depart markedly from the Boussinesq equation.

Keywords: Groundwater; Hillslope hydrology; Drainage; Boussinesq equation; Linearisation

1. INTRODUCTION

Figure 1 illustrates the two-dimensional groundwater system that is the subject of this paper. Water flows over a plane impermeable bed AB under the influence of a uniform recharge flux velocity P. The position of the free surface is defined in terms of horizontal and vertical coordinates x and h, or in terms of coordinates parallel and perpendicular to the base, x' and h'. Some authors have defined the flux velocity in terms of its component P' normal to the bed.

This system can be analysed by an extended form of the Dupuit-Forchheimer assumptions. Dupuit (1863) assumed that (a) for small inclinations of the free surface the streamlines can be taken as horizontal, and (b) the hydraulic gradient is equal to the slope of the free surface and does not vary with depth. Boussinesq (1877) modified assumption (a) for appreciable slopes by assuming the streamlines will be nearly parallel to the bed, and this was shown by Wooding and Chapman (1966) to lead to the differential equation

$$\frac{\partial}{\partial x'}\left(h' \frac{\partial h'}{\partial x'}\right) - (1\text{-}p) \tan\alpha \frac{\partial h'}{\partial x'} + p = \frac{\varepsilon}{K \cos\alpha} \frac{\partial h'}{\partial t} \tag{1}$$

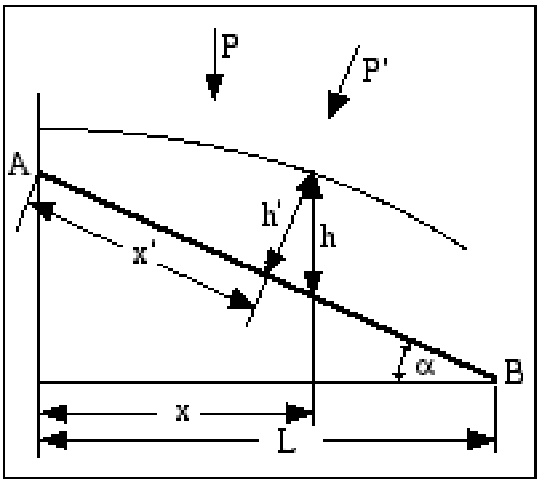

Figure 1. Coordinate definition diagram for free surface flow over a sloping bed

where K is the hydraulic conductivity, p = P/K, and ε is the effective porosity (dependent on P), i.e. the ratio of the unfilled pore space above the free surface to the total volume.

Use of the same assumptions in horizontal and vertical coordinates (x,h) results in a cubic second order differential equation, but Towner (1975) obtained an analytical solution for steady flow which agreed closely with results from a Hele-

Shaw viscous flow model (Guitjens and Luthin, 1965). By assuming $\partial h/\partial x \ \sin\alpha \ \cos\alpha \ll 1$, Chapman (1980) developed the approximation

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)\cos\alpha - \frac{\partial h}{\partial x}\sin\alpha + p = \frac{\varepsilon}{K\cos\alpha}\frac{\partial h}{\partial t} \qquad (2)$$

Chapman and Dressler (1984) showed that (1) also results from the Darcy law and the assumption that the flow is shallow. Their derivation satisfies the boundary condition at the free surface, which the Boussinesq assumptions do not, and indicates that (1) may be a better approximation than might have been supposed from its original derivation.

It is intuitively obvious that the assumption of 'shallowness' will be least valid for high values of p and low values of α. In the next section, (1) will be compared with a quasi-analytical solution for steady flow with $\alpha = 0$.

2. STEADY FLOW OVER A HORIZONTAL BED

2.1 A Quasi-analytical Solution

The analytical approach adopted here is the use of Green's second identity, which states that, for any two harmonic functions f, ϕ in a region bounded by a closed surface S

$$\oint f\,\frac{\partial\phi}{\partial n}\,dS = \oint \phi\,\frac{\partial f}{\partial n}\,dS \qquad (3)$$

where the derivatives $\partial f/\partial n$, $\partial\phi$,∂n are in the direction of the external normal to the boundary. This technique has been used to demonstrate that the Dupuit equation gives the analytically correct result for flow through a dam with vertical faces (Chapman, 1957).

For the configuration shown in Figure 2, and taking $\phi = Kh$ and $f = x$, the values of the variables in (3) are shown in Table 1 for each segment of the boundary.

It will be seen that an assumption is required for the value of ϕ on EA before all the terms in (3) are defined. As EA and AB are streamlines, the field in the neighbourhood of EAB will be close to the family of streamlines xy = constant, and the family of potentials $x^2 - y^2$ = constant. Thus the velocity on EA will vary linearly from 0 at A to -P at E, and the value of ϕ is given by

$$\phi = Kh_0 - (h_0^2 - y^2)P/2h_0 \qquad (4)$$

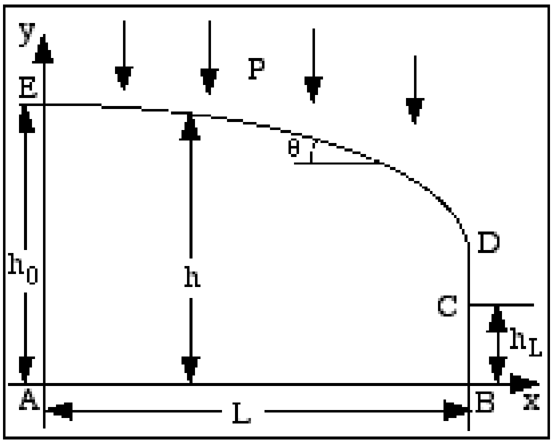

Figure 2. Definition diagram for flow over a horizontal bed

Table 1. Boundary conditions for flow system shown in Figure 2.
(v_x is the velocity component in the x-direction)

Line	f	$d\phi/dn$	ϕ	df/dn
AB	x	0	?	0
BC	L	$-v_x$	Kh_L	1
CD	L	$-v_x$	Ky	1
DE	x	$P\cos\theta$	Ky	$-\sin\theta$
EA	0	0	?	-1

Substituting the values from Table 1 and (4) in (3), and performing the integrations, the final result for the depth at E is

$$H_o^2 = \frac{H_L^2 + p}{1 - 2p/3} \qquad (5)$$

where $H = h / L$ and $p = P/K$.

It can be readily verified that this satisfies Youngs' [1965] inequality, which with the present symbols can be written

$$H_o^2 > H_L^2 + p > H_o^2\,(1 - p)$$

2.2 Comparison with Dupuit and Linearised Solutions

For steady flow with α=0, (1) reduces to

$$\frac{\partial}{\partial x}\left(h\,\frac{\partial h}{\partial x}\right) + p = 0 \qquad (6)$$

which is linear in h^2 and has the solution

$$H_o^2 = H_L^2 + p \qquad (7)$$

The linearised form of (6) is

$$\bar{h}\frac{\partial^2 h}{\partial x^2} + p = 0 \qquad (8)$$

where $\bar{h}$ is an average value of h. This has the solution

$$\mathcal{H}_o = \mathcal{H}_L + p / 2\bar{\mathcal{H}}$$

If $\bar{h}$ is calculated as $\frac{1}{L}\int_0^L h\,dx$ this becomes

$$\mathcal{H}_o = \mathcal{H}_L + p/(\mathcal{H}_L + \sqrt{\mathcal{H}_L^2 + 4p/3}\,) \qquad (9)$$

Figure 3 shows the values of $\mathcal{H}_o$ calculated from (5), (7), and (9) for 3 values of $\mathcal{H}_L$.

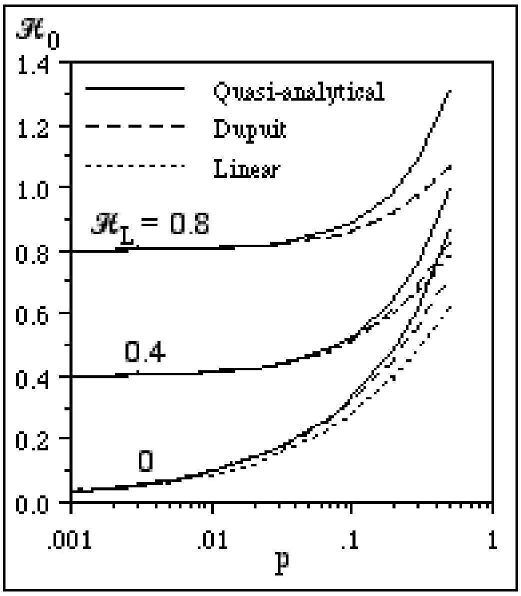

Figure 3. Relation of dimensionless upstream head to dimensionless recharge rate

As the Dupuit solution and its linear approximation both ignore the seepage surface (CD in Figure 2), it is not surprising that they both underestimate the value of $\mathcal{H}_o$. It will be seen that the departure of the Dupuit equation from the quasi-analytical solution increases with increasing values of p and $\mathcal{H}_L$. For $\mathcal{H}_o$ < 0.3, the error is less than 3%, and this could be taken as a practical criterion of 'shallowness'.

The linear solution underestimates the Dupuit solution by 13% when $\mathcal{H}_L$ = 0, but this error decreases with increasing $\mathcal{H}_L$ and is negligible when $\mathcal{H}_L$ = 0.8. It may be noted however that if the average value of h is defined as $(h_o + h_L)/2$, the linear solution gives exactly the same result as the Dupuit equation.

3. FLOW OVER A SLOPING BED

3.1 Dimensionless Equations

Equations (1) and (2) can be expressed in the same dimensionless form

$$\frac{\partial}{\partial X}\left(H\frac{\partial H}{\partial X}\right) - 2\frac{\partial H}{\partial X} + \lambda = \frac{\partial H}{\partial \tau} \qquad (10)$$

where the values of the dimensionless variables given in Table 2 relate in order to (1), (2) and Equation 18 in Henderson and Wooding (1964), with p' = P'/K.

Table 2. Relations between dimensionless variables and symbols shown in Figure 1.

X	H	λ	τ
$\frac{x'}{L'}$	$\frac{2h'}{(1-p)L'\tan\alpha}$	$\frac{4p}{(1-p)^2\tan^2\alpha}$	$\frac{K(1-p)t\sin\alpha}{2\varepsilon L'}$
$\frac{x}{L}$	$\frac{2h}{L\tan\alpha}$	$\frac{4p}{\sin^2\alpha}$	$\frac{Kt\sin 2\alpha}{4\varepsilon L}$
$\frac{x'}{L'}$	$\frac{2h'}{L'\tan\alpha}$	$\frac{4p'}{\sin^2\alpha}$	$\frac{Kt\sin\alpha}{2\varepsilon L'}$
$\frac{x}{L}$	$\frac{2(h-px\tan\alpha)}{(1-p)L\tan\alpha}$	$\frac{4p}{(1-p)^2\tan^2\alpha}$	–

3.2 Steady Flow Solution

For steady flow, (10) can be integrated to give

$$H\frac{\partial H}{\partial X} - 2H + \lambda X = A \qquad (11)$$

Towner's (1975) equation for steady flow in horizontal and vertical coordinates can be expressed in this form by the transformations shown in the last line of Table 2.

The integration constant A in (11) will be zero if the boundary condition at X=0 is either H = 0 or ∂H/∂x = 2 . For all but the first line in Table 2, the latter condition implies that ∂h/∂x = ∂h'/∂x' = tan α, so that the water surface is horizontal, i.e. there is a groundwater divide. For the first line, the condition ∂H/∂X = 2 implies ∂h'/∂x' = (1-p) tan α, so that the surface is not quite horizontal.

With A = 0, (11) is homogeneous and can be readily solved. It has two forms, depending on whether λ<1 or λ>1. Henderson and Wooding (1964) gave solutions in terms of a scaling constant and, for λ>1, a parameter. Equivalent, though apparently dissimilar, solutions can be obtained by following the integration technique of Schmid and Luthin (1964), with the following results:

For $\lambda<1$

$$H(X) = W_2X - (W_2-H(1))\left(\frac{W_1X - H(X)}{W_1 - H(1)}\right)^{W_1/W_2} \qquad (12)$$

where $W_1 = 1 + \sqrt{1-\lambda}$ and $W_2 = 1 - \sqrt{1-\lambda}$

For $\lambda>1$

$$\ln \frac{(H(X)-X)^2+(\lambda-1)X^2}{(H(1)-1)^2+\lambda-1} = \frac{2}{\sqrt{\lambda-1}}\left(\tan^{-1}\frac{H(1)-1}{\sqrt{\lambda-1}} - \tan^{-1}\frac{H(X)-X}{X\sqrt{\lambda-1}}\right) \qquad (13)$$

Both (12) and (13) can be solved recursively. From the free surface profiles obtained with downstream depth H(1) = 0 for low (5°) and high (30°) slopes, three characteristics of the depth H = h/L, in the original coordinates, have been calculated for each of the transformations shown in Table 2. These are the depth $H(0)$ at the upstream boundary, the mean depth H_{av}, and the maximum depth H_m. The general form of the variation of $H(0)$ with recharge p is shown in Figure 4, while Figure 5 shows that the position of maximum depth moves up the slope with increasing recharge and decreasing bed slope. Both graphs conform with the Dupuit solution for a horizontal bed.

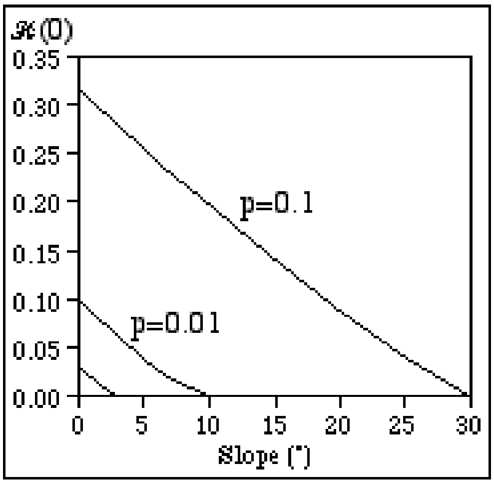

Figure 4. Relation of upstream depth to bed slope for recharge rates 0.1, 0.01 and 0.001.

Figures 6 and 7 show that the upstream depth from the Wooding and Chapman (1966) solution agrees with the Towner (1975) solution at low bed slopes and lower recharge rates, but overestimates it at high slopes. The Chapman (1980) solution overestimates the upstream depth at low slopes and higher recharge rates, and at all recharge rates for high slopes. The Henderson and Wooding (1964) solution underestimates the upstream depth at low slopes and high recharge rates, and seriously underestimates it under all conditions on high slopes. Figures 8 and 9 show similar results for the maximum depth, and are very close to the graphs for the mean flow depth (not shown).

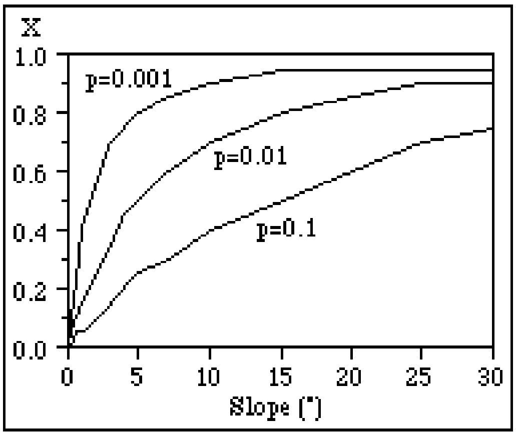

Figure 5. Value of X for maximum flow depth, related to bed slope and recharge.

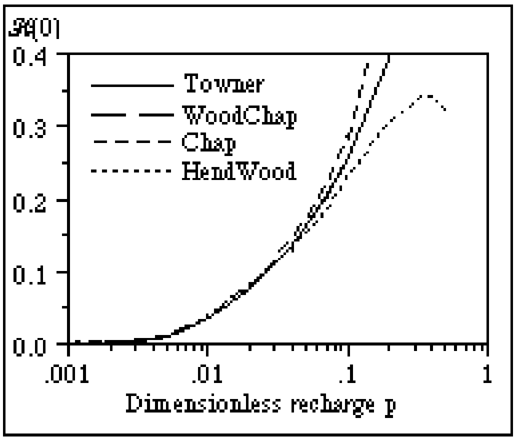

Figure 6. Upstream depth for bed slope 5°

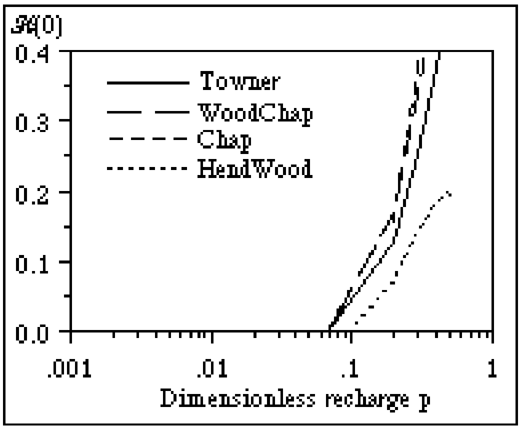

Figure 7. Upstream depth for bed slope 30°

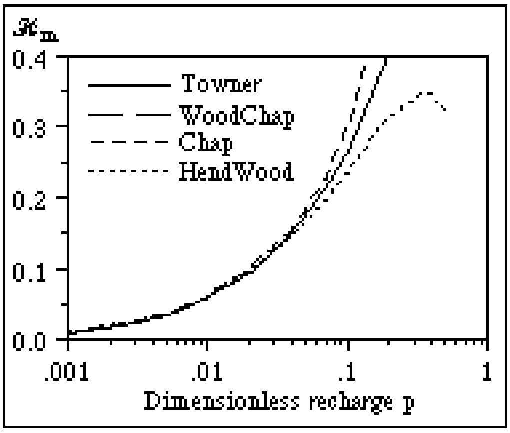

Figure 8. Maximum depth for bed slope 5°

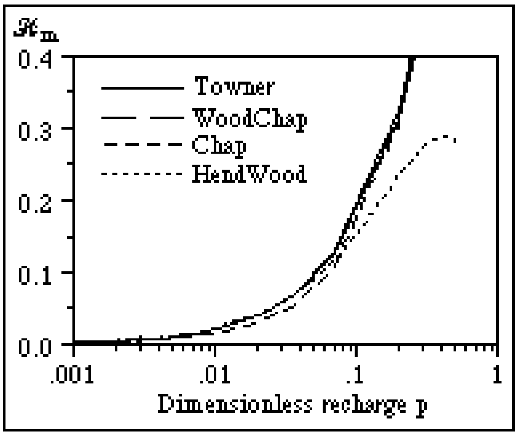

Figure 9. Maximum depth for bed slope 30°

3.2 Linearised Solutions

Two forms of linearisation have been studied, in terms of linearising H and H^2 in (10), and will be referred to as Types 1 and 2 respectively.

For Type 1, (11) becomes

$$\bar{H}\frac{\partial H}{\partial X} - 2H + \lambda X = A \qquad (14)$$

which has the general solution

$$H(X) = (\lambda\bar{H} - 2A)/4 + \lambda X/2 + Be^{2X/\bar{H}} \qquad (15)$$

If it is assumed that H(0) = 0, this results in

$$H(X) = \frac{1}{2}\lambda\Big[X - (1--2H(1))/\lambda)(e^{2X/\bar{H}}-1)/(e^{2/\bar{H}}-1)\Big] \qquad (16)$$

which is an extension of the result obtained by Koussis and Lien (1982) for H(1)=0.

The mean depth $\bar{H}$ is obtained by setting

$\bar{H} = \int_0^1 H(X)\,dX$ in (16), and this gives rise to an equation which can be solved recursively for H(X).

However, if this equation results in $dH/dX \geq 2$ at X = 0, it follows that H(0) > 0, and the appropriate boundary condition for a no flow boundary is dH/dX = 2, from which it is readily shown that

$$A = 2(\bar{H} - H(0)) \text{ and } B = \bar{H}(1 - \lambda/4) \qquad (17)$$

and a recursive solution can be obtained. By setting to zero a linearised expression for the flow at the upstream boundary, Koussis (1992) obtained solutions in which H(0)>0 for all values of p, and therefore do not accord with the behaviour of the nonlinear solution.

For linearisation Type 2 (Werner,1957), we put Y = H^2 in (10), which for steady flow results in

$$\frac{\partial^2 Y}{\partial X^2} - \frac{2}{\sqrt{Y}}\frac{\partial Y}{\partial X} + 2\lambda = 0 \qquad (18)$$

This is linearised by putting $\sqrt{Y} = \bar{H}$, and integrated to give

$$\partial Y/\partial X - 2\,Y/\bar{H} + 2\,\lambda X = A \qquad (19)$$

which has the solution

$$H(X)^2 = Y = \bar{H}(\lambda\bar{H} - A)/2 + \lambda\bar{H}X + Be^{2X/\bar{H}} \qquad (20)$$

Equations which can be solved recursively are then developed in the same way as for Type 1.

Typical comparisons of the two linearisations with the Dupuit solution are shown in Figures 10 and 11. In general, Type 1 fits better at low flow depths and Type 2 at higher depths. Both types retain zero upstream depth at higher recharge rates than the Dupuit solution.

4. CONCLUSIONS

For a horizontal bed, the quasi-analytical solution leads to the conclusion that the Dupuit equation can be used with negligible error when the ratio of flow depth to length is less than 0.3, conditions which would be valid in most natural situations.

The Boussinesq equation for steady flow over sloping beds is best modelled by Towner's (1975) solution, but this has no extension into unsteady flow situations. Chapman's (1980) approximation does not suffer from this limitation, with errors only under conditions of low slopes and high

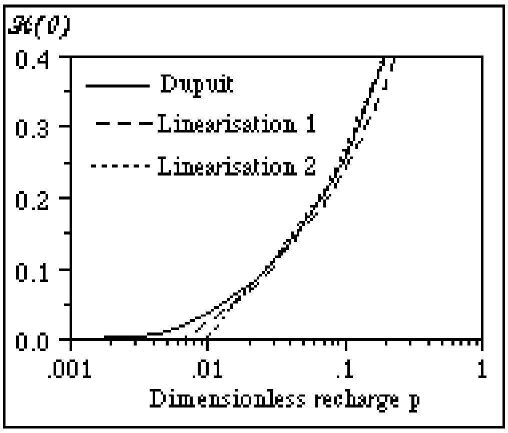

Figure 10. Upstream depth for bed slope 5°

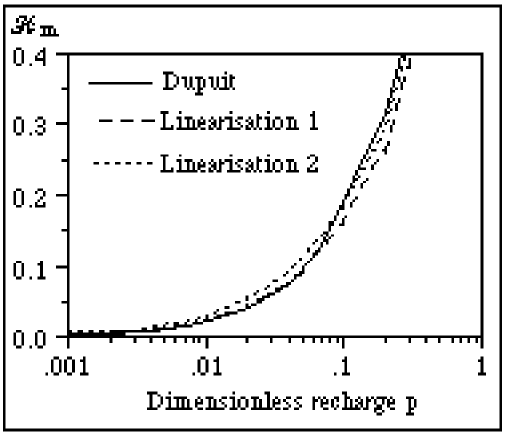

Figure 11. Maximum depth for bed slope 30°

recharge rates. Wooding and Chapman's (1966) solution is almost identical with Towner's, except for the upstream head with high slopes, but has the disadvantage of an inconvenient coordinate system. Henderson and Wooding's (1964) solution is satisfactory at low slopes, but at high slopes demonstrates the problems associated with assuming the recharge is at right-angles to the bed.

Both types of linearisation give close approximations to the nonlinear model, with Type 1 slightly better at small flow depths and Type 2 better at moderate to high depths. However, the solutions are not easy, requiring iterative calculations of the mean flow depth which appear to have been ignored in recent unsteady flow models (Sloan, 2000; Verhoest and Troch, 2000).

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