A Monte Carlo Study of the Probability Weighting Function

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Abstract: The cumulative prospect theory suggests that individuals overweight low probability events and underweight high probability events. This tendency will create an S-shaped probability transformation. This paper is particularly interested in exploring the dynamic nature of the probability weighting functions in the context of lotteries. When individuals change their perceptions about the probabilities of various possible outcomes, they are likely to vary their probability weights over time. This paper poses the question: Would the probability weighting functions converge to the linear probability weighting function over time? On one hand, we assume that individuals would learn from their errors and adjust their probability weights according to a mean-reverting process. On the other hand, we assume that individuals would inflate the probability weights if encouraged by winning any prize. Our Monte Carlo simulation reveals that individuals' probability weighting functions are likely to converge to the linear probability weighting function, therefore confirming the ability, on the part of individuals, to learn slowly about their cognitive biases.

Keywords: Probability weighting function; Mean-reverting process; Jump process; Monte Carlo simulation; Lotteries

1. INTRODUCTION

It is recognised that the von Neumann-Morgenstern expected utility theory is not able to provide an adequate description of choice phenomenon such as framing effects, non-linear preference, source dependence, risk seeking, and lose aversion. The non-expected utility models emerged since the late 1970s have their sight on closing the gap between the theory and the reality. Among the various new paradigms, the rank dependent utility theory [Quiggin, 1982] and the cumulative prospect theory [Kahneman and Tversky, 1979; Tversky and Kahneman, 1992], have received much attention.

A significant implication of the cumulative prospect theory is the fourfold pattern of risk attitudes. For non-mixed prospects, the shapes of the value and the probability weighting functions (PWFs) imply (a) risk aversion for gains of high probability, (b) risk seeking for gains of low probabilities, (c) risk seeking for losses of high probabilities, and (d) risk aversion for losses of low probabilities. This characterisation is observed by several experiments [see, e.g., Tversky and Fox, 1995; Lattimore et al., 1992; Wu and Gonzalez, 1996; Jullien and Salanié, 2000].

The PWFs portrayed by these non-expected utility theories are static PWFs and there is no attempt on the part of the authors to provide clues about the dynamic nature of the PWFs. In the context of repeated bets, it is difficult to imagine that individuals would not change their perception about the probabilities of the possible outcomes over time. A literature search could not disclose any work done on the dynamics of PWFs. This paper aims to study the dynamic nature of PWFs, in particular the PWF for gains, in the context of lotteries [see Fennema and Wakker, 1994]. We pose the question: Would individuals' PWF for gains converge to the linear PWF given enough time has elapsed? A Monte Carlo study of a mean-reverting model with a jump process is used to simulate the adjustment process for the PWF for gains. The results reveal that individuals' PWF for gains do have a tendency to converge to the linear PWF.

2. PROBABILITY WEIGHTING

Edwards [1962] suggests that individuals when confronted with risky outcomes tends to inflate the probabilities of low-probability events (e.g., winning the jackpot of a lotto game) and to deflate the probabilities of high-probability events (e.g., being caught speeding by a speed camera). This tendency would manifest an S-shaped probability transformation. Psychological research attributes this phenomenon to cognitive factors such as (a) illusion of control [Langer, 1975] whereby individuals perceive chance activities as if they were determined by skill, and (b) retrievability of instances [Tversky and Kahneman, 1974] whereby the subjective probability of an event is raised by familiarity, salience, and recentness.

In their laboratory experiment on (graduate students') risk attitudes, Tversky and Kahneman [1992] estimate the relationship between the certainty-equivalent-outcome (c/x) ratio to the probability of outcome (p). Their experimental data shows a fourfold pattern of risk attitudes: risk aversion for gains of high probabilities, risk seeking for gains of low probabilities, risk aversion for losses of low probabilities, and risk seeking for losses of high probabilities. They also find close to constant relative risk aversion. According to the cumulative prospect theory, this relationship between c/x and p can be approximated by a two-part function:

$$w^{+}(p)=\frac{p^{\gamma}}{\left[p^{\gamma}+(1-p)^{\gamma}\right]^{1/\gamma}} \quad (1)$$

and $$w^{-}(p)=\frac{p^{\delta}}{\left[p^{\delta}+\left(1-p^{\delta}\right)\right]^{1/\delta}} \quad (2)$$

where $w^{+}(p)$ is a PWF for gains with a parameter of gains (γ = 0.61), $w^{-}(p)$ is the PWF for losses with a parameter of losses (δ = 0.65).

Although the PWFs for gains and for losses resemble each other, the difference in values of γ and δ shows that their characteristic curvatures are not symmetric and therefore, there is no reflection effect over a reference point. Experiments conducted by Cohen et al. [1987] (on 134 college students) and Wehrung [1989] (on 127 oil executives) find that there is independence of risk attitudes on the gain and on the loss side. And these experiments consistently show that the PWF for gains is slightly more curved than the PWF for losses because risk aversion is more pronounced for gains than risk seeking for losses.

Note that if the probability weighing is absent, then the parameters γ, $\delta = 1$ would equal to unity and we have a linear PWF where $w^{+}(p)=w^{-}(p)=p$.

Figure 1 depicts the PWFs for both gains and losses, specified in Tversky and Kahneman [1992]. The PWFs in Figure 1 show that individuals do overweight low probabilities and underweight moderate and high probabilities with probabilities in the middle of range relatively unchanged. Thus, producing the S-transformation predicted by Edwards [1962]. The crossover probabilities, $\hat{p}$, occur at approximately 0.34 and 0.36 for PWF for gains and for losses, respectively.

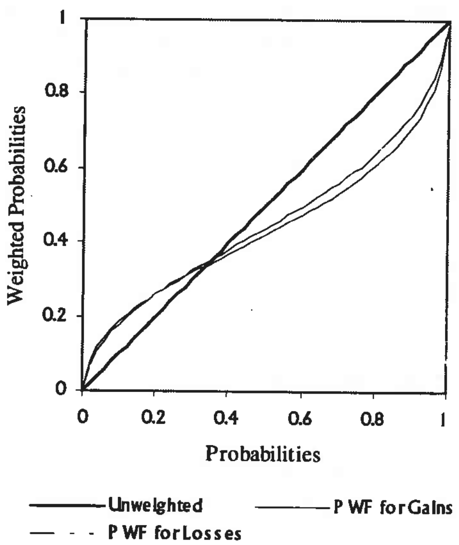

Figure 1: The PWFs for gains and losses from Tversky and Kahneman [1992].

3. THE DISCRETE-EVENT MODEL

In order to study the dynamic nature of the PWFs, we have to construct a discrete-event model to capture the dynamics of weights applied to the probabilities. In this paper, we focus on the PWF for gains in the context of lotteries. The analysis for losses is essentially identical, and will not be discussed.

Consider the lotto game <P, D + S, F> with $S < D \leq P \leq F$, where individuals select P numbers from a field of F numbers, D are the winning numbers drawn by the operator, and S are the supplementary numbers drawn by the operator. For example, the format of the Australian Lotto Bloc's (ALB) Saturday and Oz Lotto is <*P*, 6 + 2, 45>.

Suppose individuals begin with an S-shaped PWF for gains with its initial position prescribed by eq (1) with $\gamma = 0.61$. Suppose each individual purchases one lotto game per draw (or event time) with $P = D = 6$. Individuals' perception of the probabilities of winning the various prizes of the lotto game at any event time will be influenced by the outcomes happened in the previous event time. If individuals failed to win any prize over a stretch of time, they would feel discouraged and would adjust their weighted probabilities for winning the prizes downward, thus, leading to less and less curved PWFs. This can be regarded as similar to the anchoring adjustment phenomenon described in the psychology literature [see Wagenaar, 1988].

More specifically, the mean-reverting process described in the previous paragraph is specified as follows:

$$\Delta w_t^+(p) = \alpha_t \left(\overline{w}(p) - w_{t-1}^+(p) \right) \qquad (3)$$

where $\alpha_t > 0$ is an adjustment rate at time *t*, $\overline{w}(p)$ is the long-term trend of *p* and are the unweighted probabilities. Furthermore, $w_t^+(p) \gtreqless \overline{w}(p) \Leftrightarrow p \lesseqgtr \hat{p}$. The value of the adjustment rate depends on the speed of individuals could learn from their cognitive biases (overweighing the probabilities of rare events and underweighting the probabilities of frequent events) by processing incoming information and cues. Note that α_t can be pre-determined for each individual. Casual conversation with 28 lotto players reveals that they feel disappointed if they did not win any prize in each draw and would lower their expectation of winning. However, the players are not able to ascertain their rates of adjustment.

Reid [1986] suggests that near misses (or failures that are close to being successful) are believed to encourage further play. Individuals often report that they feel their luck has arrived after experiencing near misses. Near misses also enhance the feeling of control, therefore, exaggerate the probability distortion. As a result, PWF for gains are pushed further away from the linear PWF. A jump process is employed to describe the impact of near misses on the PWF for gains:

$$\Delta w_t^+ = \beta_t \qquad (4)$$

where $\beta_t > 0$ is a jump parameter.

Even though individuals may adjust upward or downward their subjective probabilities, there is only one force at work at any point of time; they either win a prize or they do not. The encouragement and discouragement factor is mutually exclusive. Putting the two mutually exclusive processes together yields:

$$w_t^+(p) = \begin{cases} w_{t-1}^+(p) + \alpha_t \left[\overline{w}(p) - w_{t-1}^+(p) \right] \\ \qquad when \quad \beta_t = 0, 0 < \alpha_t \le a < 1 \\ w_{t-1}^+(p)(1 + \beta_t) \\ \qquad when \quad 0 < \beta_t, \alpha_t = 0 \end{cases} \qquad (5)$$

The excursion of the process $w_t^+(p)$ away from the long-term trend $\overline{w}(p)$ may take considerable time depending on the dominance of either the mean-reverting process or the jump process. Since winning a prize, especially the upper divisional prizes, is a rare event, it is expected that the mean-reverting process will dominate the jump process in the long haul.

Simulation is preferred to laboratory experiments in the study of the dynamics of PWF for gains in the context of lotteries. Firstly, it is almost impossible to elicit the values of α_t and β_t. The probabilities of winning lotto jackpots are almost incomprehensible for the untrained mind (1.2774E-7 in ALB's Saturday and 1.81887E-8 in Oz Lotto), let alone comprehending the changes in these probabilities due to encouraging or discouraging factors.

Secondly, there may be cross-pollination of the real world and laboratory PWFs. Without completely isolating the subjects in real time, there may be cross-pollination of their real world PWFs and their experimental PWFs. This can be partially prevented by screening out subjects who purchase lotteries in the real world. The cross-pollination of the subjects' real time experience in the real world and the experimental setting may distort the results obtained in the experiment. This makes simulation an ideal candidate in the study of the evolution of PWFs.

4. NUMERICAL VALUES

The prizes of ALB's Saturday or Oz Lotto are distributed as a Bin(5, 0.0064563), obtained by using the software @RISK (Palisade Corp). Table 1 shows the relationship of the prizes and the distribution of Bin(5, 0.0064563).

Table 1: The Probabilities of occurrence of the various prizes in the ALB's Saturday and Oz Lotto and the distribution of Bin(5, 0.0064825).

x_i	Prizes	p_i	$F(x_i)$
0	-	0.99510614327	0.99510614327
1	Fifth	0.003362	0.99846814327
2	Fourth	0.001503	0.99997114327
3	Third	2.72557E-5	0.99999839897
4	Second	1.47329E-6	0.99999987226
5	First	1.2774E-7	1

The next step is to generate the numerical values for α_t and β_t. To avoid assigning *ad hoc* numbers to α_t and β_t, we select to randomise them. The adjustment rate α_t per event time if no prize was won is assumed to be a random variable between 0 and 0.05. This range of values may be very conservative given the observation by Cohen et al. [1987] in their experiment where their subjects exhibit steep learning curve in the 10-week repeated experiment. As for the jump parameter β_t, it is argued that β_t will assume different values if individuals won different divisional prizes. It is postulated that the bigger the prizes the bigger the jump. Again, β_t is randomised for each fractile or prize. Because of the difficulties of and the costs associated with eliciting the values of the upper bound and lower bound of the ranges, the ranges are subjective estimates. The relationship between the $F(x_t)$, α_t, and β_t are depicted in Table 2.

Table 2: Relationship $F(x_t)$, α_t, and β_t.

$F(x_t)$	α_t	β_t
$F(x_t) \leq F(x_0)$	0 – 0.05	NA
$F(x_0) < F(x_t) \leq F(x_1)$	NA	0 – 0.05
$F(x_1) < F(x_t) \leq F(x_2)$	NA	0.05 – 0.10
$F(x_2) < F(x_t) \leq F(x_3)$	NA	0.10 – 0.15
$F(x_3) < F(x_t) \leq F(x_4)$	NA	0.15 – 0.30
$F(x_4) < F(x_t) \leq F(x_5)$	NA	0.30 – 1.00

5. SIMULATION RESULTS

In this section, we examine the dynamic process obtained by the Monte Carlo simulation technique. The initial PWF for gains is taken from Figure 1. To see exactly how the PWF for gains evolves,

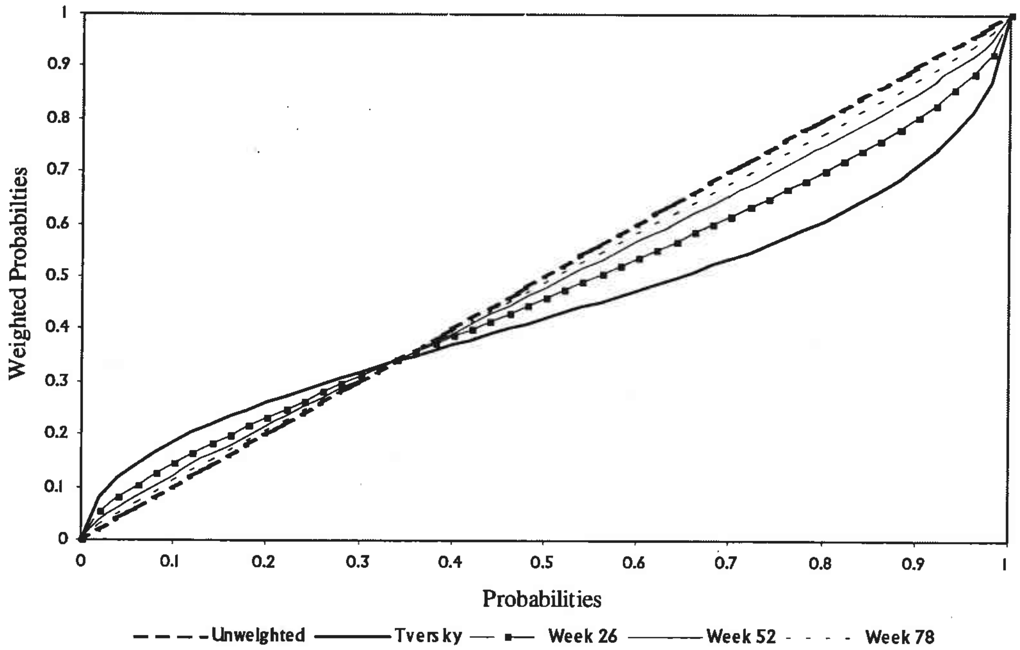

Figure 2: Simulation results for weeks 26, 52, 78, and 104.

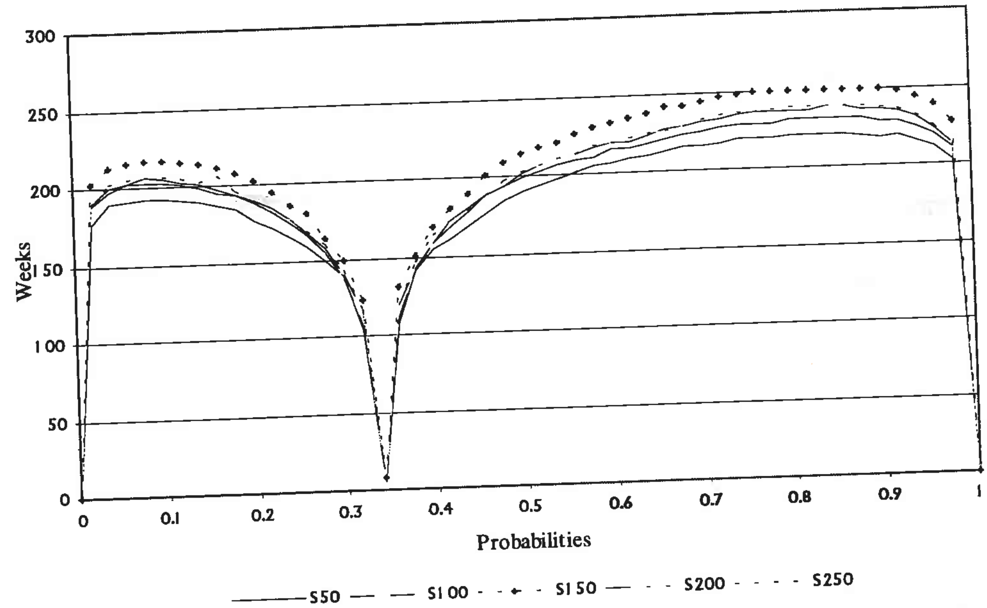

Figure 3: Convergence of weighted to objective probabilities.

250 simulations are done for 260 weeks. For each simulation, a random number is generated from U(0, 1) for each week of the 260 weeks and then converted into random variates by the inverse transform method. The simulation results for weeks 26, 52, 78, and 104 are presented in Figure 2. The Week 104 and 156 plots are omitted

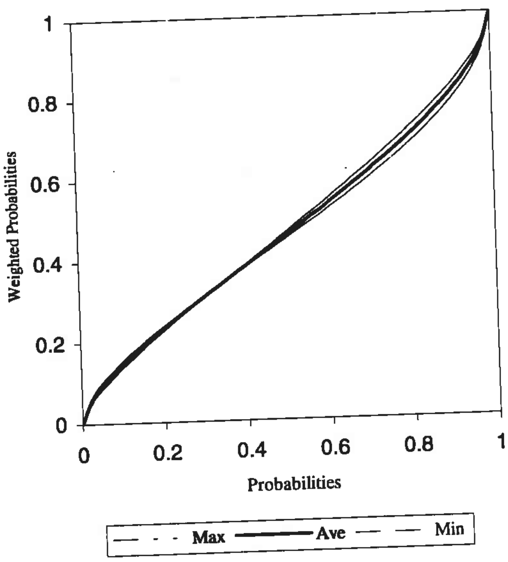

Figure 4: Simulation result for week 26.

because they are almost identical to the linear PWF and are very difficult to discern in a diagram of this size.

Figure 2 shows that there is indeed a tendency for the PWF for gains to converge to the linear PWF over time. Figure 3 shows the speed of convergence of the weighted probabilities to the objective probabilities. It seems that individuals take less time to revise downward their weighted probabilities of rare events and much longer time to revise upward to revise upward their weighted probabilities of frequent event.

Figure 4 shows the upper bound and lower bound of the simulation results for Week 26. The upper bounds and the lower bounds for other weeks resemble what is presented in Figure 3.

6. DISCUSSIONS AND CONCLUSIONS

In this paper, we are interested in the dynamic nature of probability weighting functions. We analyse the reasons that the probability weighting function should be dynamic rather than static as traditionally assumed in cumulative prospect theory. A mean-reverting model with a jump process is constructed to emulate the evolution of the probability weighting function. Our Monte Carlo simulation results show that individuals' probability weighting functions are likely to converge to the linear probability weighting function over time. Although probability weighting function for losses is not symmetric to probability weighting function for gains, we anticipate similar pattern of convergence.

The discrete-event model presented here could be further improved by (a) differentiate the impact of early winning may have had more influence on behaviour than late winning, (b) including long-term (more than one event time) impact of winning any prize, and (c) obtaining the ranges for the adjustment rate and the jump rate in an experimental setting.

7. REFERENCES

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